J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 19, Number 1 (February 2012), Pages 37–42

TOTAL ANGULAR DEFECT AND EULER'S THEOREM FOR POLYHEDRA

Dong-Soo Kim [a,*] and Young Ho Kim [b]

Abstract. We give an elementary proof of Descartes' theorem for polyhedra. Since Descartes' theorem is equivalent to Euler's theorem for polyhedra, this also gives an elementary proof of Euler's theorem.

1. Introduction

Let us begin with a convex polyhedron Σ. Euclid proved that the sum of the face angles at any vertex P of Σ is less than 2π ; the difference between this sum and 2π is called the angular defect at P and denoted by $\Delta_\Sigma(P)$. If we sum the angular defects over all the vertices of Σ, we obtain the total angular defect $\Delta(\Sigma)$ ([3] or [4]). René Descartes used spherical trigonometry to prove that $\Delta(\Sigma) = 4\pi$ for every convex polyhedron Σ. Descartes' theorem for polyhedra in space is analogous to the exterior angle theorem for polygons in a plane ([5]). A number of proofs of the exterior angle theorem are given in [2].

For a convex polygon σ we denote by $\delta_\sigma(P)$ and $\delta(\sigma)$ the exterior angle of σ at a vertex P of σ and the sum of exterior angles of σ, respectively. Then we give a proof of the exterior angle theorem for convex polygons as follows.

A proof of the exterior angle theorem. First note the following which can be checked easily:

(L_1) Every triangle σ has total exterior angle $\delta(\sigma) = 2\pi$.

(L_2) If a convex polygon σ is dissected by a straight line l into two polygons σ_1 and σ_2, then we have the following:

Received by the editors September 1, 2011. Accepted February 10, 2012.
2000 *Mathematics Subject Classification.* 52B05, 53B45.
Key words and phrases. polygon, polyhedron, exterior angle theorem, Euler's theorem, Descartes' theorem.

*Corresponding author.

Let P_1P_2 denote the common edge of σ_1 and σ_2. Then for each $i = 1, 2$ we have

$$\delta_{\sigma_1}(P_i) + \delta_{\sigma_2}(P_i) = \delta_{\sigma}(P_i) + \pi, \tag{1}$$

and hence we get

$$\delta(\sigma_1) + \delta(\sigma_2) = \delta(\sigma) + 2\pi. \tag{2}$$

(L_3) Let σ be a convex polygon which is not a triangle. Then σ can be dissected by a straight line into two polygons σ_1 and σ_2 which have less number of edges than that of the polygon σ.

Then, together with $(L_1), (L_2)$, and (L_3), a mathematical induction argument on the number of edges of convex polygons shows the exterior angle theorem for convex polygons.

Hence it is quite natural to consider whether a similar argument can be made on the total angular defect of convex polyhedra.

In this short note, we give an affirmative answer, that is, we give a proof of Descartes' theorem for convex polyhedra as that of an exterior angle theorem for polygons.

First of all, we give a proposition which shows that Descartes' theorem is equivalent to Euler's theorem for convex polyhedra ([3] or [4]).

Proposition 1. *For a convex polyhedron Σ, we have $\Delta(\Sigma) = 2\pi(V - E + F)$, where V, E, and F denote the number of vertices, edges, and faces, respectively.*

Proof. We give a proof for completeness. For each vertex P of Σ, we put $r = r(P)$ as the number of edges which are incident with P. If we denote by $\alpha_1, \cdots, \alpha_r$ the face angles at P, then the angular defect $\Delta(P)$ at P is

$$\begin{aligned}\Delta(P) &= 2\pi - (\alpha_1 + \cdots + \alpha_r)\\ &= (\pi - \alpha_1) + \cdots + (\pi - \alpha_r) - r\pi + 2\pi.\end{aligned}$$

Hence the total angular defect of Σ is given by

$$\Delta(\Sigma) = \sum_P \{(\pi - \alpha_1) + \cdots + (\pi - \alpha_r)\} - \pi \sum_P r(P) + 2\pi \sum_P 1, \tag{3}$$

where the summations are taken over all vertices P of Σ. Note that the first sum is nothing but the sum of exterior angles of all plane face angles of the polyhedron Σ, and hence it simply becomes $2\pi F$ by the exterior angle theorem. Obviously, we have

$$\sum_{P} 1 = V. \tag{4}$$

Furthermore, since every edge has exactly two vertices, we also obtain

$$\sum_{P} r(P) = 2E. \tag{5}$$

Thus, together with (4) and (5), (3) completes the proof. □

2. Proof of the Main Theorem

Now we establish formulas for total angular defect of polyhedra which are analogous to the formulas (1) and (2) for the total exterior angle of polygons.

Lemma 2. *Let Σ be a convex polyhedron which can be dissected by a plane φ into two convex polyhedra Σ_1 and Σ_2. Then we have the following:*
Let σ be a convex r-sided polygon which is the common face of Σ_1 and Σ_2 with vertices $P_1, \cdots, P_r$. Then for each $i = 1, 2, \cdots, r$ we have

$$\Delta_{\Sigma_1}(P_i) + \Delta_{\Sigma_2}(P_i) = \Delta_{\Sigma}(P_i) + 2\delta_\sigma(P_i), \tag{6}$$

and we get

$$\Delta(\Sigma_1) + \Delta(\Sigma_2) = \Delta(\Sigma) + 4\pi. \tag{7}$$

Proof. Note that if P_i is not a vertex of Σ, then we have $\Delta_\Sigma(P_i) = 0$. For simplicity, we prove (6) when P_i is a vertex of Σ and every edge of σ is an edge of Σ. The remaining cases can be treated similarly. We denote by γ_i the interior angle of σ at $P_i, i = 1, \cdots, r$.

Let α_{ij} and β_{ik} denote the face angles of Σ at P_i which are face angles of Σ_1 and Σ_2, respectively. Then, at each P_i, we have the following:

$$\Delta_\Sigma(P_i) = 2\pi - \sum_j \alpha_{ij} - \sum_k \beta_{ik}, \tag{8}$$

$$\Delta_{\Sigma_1}(P_i) = 2\pi - \sum_j \alpha_{ij} - \gamma_i, \tag{9}$$

and

$$\Delta_{\Sigma_2}(P_i) = 2\pi - \sum_k \beta_{ik} - \gamma_i. \tag{10}$$

Thus it follows from (8), (9) and (10) directly that (6) holds.

Let's denote by Δ_1 and Δ_2 the sum of angular defects at vertices of Σ_1 and Σ_2, respectively, which do not belong to the plane φ.

Then we obtain from (6) and the exterior angle theorem that

$$\begin{aligned}\Delta(\Sigma_1) + \Delta(\Sigma_2) &= \Delta_1 + \Delta_2 + \sum_i \{\Delta_{\Sigma_1}(P_i) + \Delta_{\Sigma_2}(P_i)\} \\ &= \Delta_1 + \Delta_2 + \sum_i \Delta_\Sigma(P_i) + 2\sum_i \delta_\sigma(P_i) \\ &= \Delta(\Sigma) + 4\pi,\end{aligned}$$

which completes the proof. □

Next, we prove a lemma for convex polyhedra which is analogous to (L_3) for convex polygons.

Lemma 3. *Let Σ be a convex polyhedron which is not a tetrahedron. Then Σ can be dissected by planes successively into a finite number of polyhedra $\Sigma_1, \cdots, \Sigma_m$ such that each Σ_i has less faces than those of Σ.*

Proof. Recall that for a vertex P of Σ, $r(P)$ (≥ 3) denotes the number of edges of Σ which are incident with P. First of all, we treat two cases.

Case 1. Suppose $r(P) \geq 4$ for some vertex P of Σ. Then choose two vertices P_1 and P_2 such that the segments PP_1 and PP_2 are the edges of Σ which are not those of a common face of Σ. Then the plane through P, P_1, and P_2 dissects Σ into two polyhedra Σ_1 and Σ_2 which have the desired property.

Case 2. Suppose Σ has a triangular face $P_1P_2P_3$. By Case 1, we may assume that $r(P) = 3$ for every vertex P of Σ. Choose a vertex P_4 of Σ which forms an edge P_1P_4 of Σ together with P_1 other than P_1P_2 and P_1P_3. Then by the plane through P_2, P_3 and P_4, Σ is dissected into a tetrahedron Σ_1 and a convex polyhedron Σ_2. Note that Σ_2 has the same number of faces as that of Σ. Furthermore, Σ_2 satisfies $r_{\Sigma_2}(P_4) = 4$. Hence Case 1 shows that Σ_2 can be dissected by a plane into two convex polyhedra Σ_3 and Σ_4. Eventually, Σ can be dissected by planes successively into 3 polyhedra $\Sigma_1, \Sigma_3, \Sigma_4$ all of which have the desired property.

Now we denote by $n(\Sigma)$ the minimum number of edges for all faces of Σ. Then we prove Lemma 3 by a mathematical induction on $n = n(\Sigma)$. If $n(\Sigma) = 3$, then Case 2 implies Lemma 3. Suppose that Lemma 3 holds for all convex polyhedra Σ with $n(\Sigma) \leq k, k \geq 3$. If Σ is a convex polyhedron with $n(\Sigma) = k+1$, then we consider a convex $(k+1)$-gonal face $P_1P_2 \cdots P_{k+1}$ of Σ. By Case 1, we may assume that $r(P) = 3$ for every vertex P of Σ. Choose a vertex Q of Σ which forms an edge P_kQ of Σ together with P_k other than $P_{k-1}P_k$ and P_kP_{k+1}. Then the plane through P_1, P_k, and Q dissects Σ into two polyhedra Σ_1 and Σ_2, which have faces, respectively, less than or equal to the number of faces of Σ. Obviously, we have $n(\Sigma_1) \leq k$, and $n(\Sigma_2) \leq k$. Hence the induction hypothesis completes the proof of Lemma 3. □

Finally, we prove Descartes' theorem for convex polyhedra.

Theorem 4. *Let Σ be a convex polyhedron. Then the total angular defect is given by $\Delta(\Sigma) = 4\pi$.*

Proof. We prove by a mathematical induction on the number of faces $F = F(\Sigma)$. If $F(\Sigma) = 4$, that is, Σ is a tetrahedron, then it is obvious that the total angular defect of Σ is 4π. Suppose that the theorem holds for all convex polyhedra Σ with $F(\Sigma) \leq k, k \geq 4$. If Σ is a convex polyhedron with $F(\Sigma) = k + 1$, then Lemma 3 shows that Σ can be dissected by planes successively into a finite number of convex polyhedra $\Sigma_1, \Sigma_2, \cdots, \Sigma_m$ such that $F(\Sigma_i) \leq k$. Hence from the induction hypothesis we see that each Σ_i satisfies $\Delta(\Sigma_i) = 4\pi$. Thus, by repeatedly using Lemma 2, we conclude that $\Delta(\Sigma) = 4\pi$. □

As a corollary of Lemma 3, we give a dissection theorem for polyhedra as follows (cf. [1, p. 212]).

Corollary 5. *Every polyhedron can be dissected by planes successively into a finite number of tetrahedra.*

Proof. We may use the same induction argument as above to prove Corollary 5 for a convex polyhedron. For an arbitrary polyhedron Σ, let $\varphi_1, \cdots, \varphi_m$ denote the face planes. Then $\varphi_1, \cdots, \varphi_m$ divide space into a set of convex regions, a finite number of which consist of the polyhedron Σ. Hence the polyhedron Σ can be dissected by planes successively into a finite number of convex polyhedron. Thus the corollary follows from the convex case. □

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[a]DEPARTMENT OF MATHEMATICS, CHONNAM NATIONAL UNIVERSITY, KWANGJU 500-757, KOREA
Email address: dosokim@chonnam.ac.kr

[b]DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU, 702-701, KOREA
Email address: yhkim@knu.ac.kr