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$h-$STABILITY OF NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

Yoon Hoe Goo [a,*] and Seung Bum Yang [b]

Abstract. The main purpose of this paper is to investigate $h-$stability of the nonlinear perturbed differential systems using the notion of t_∞-similarity. As results, we generalize some previous h-stability results on this topic.

1. Introduction

The nonlinear variation of constants formula of Alekseev[1] has been used by several authors [2-4, 6-8] to study h-stability of solutions of nonlinear differential systems. In this paper we use the nonlinear variation of constants formular of Alekseev [1] to study h-stability of solutions of nonlinear perturbed differential systems.

The notion of h-stability (hS) was introduced by Pinto [13,15] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. Also, he obtained some properties about asymptotic behavior of solutions of perturbed h-systems, some general results about asymptotic integration and gave some important examples in [14]. Choi and Ryu [3] investigated the important properties about hS for the various differential systems. Recently, Choi et al. [4] and Goo [6] obtained results for hS of nonlinear differential systems via t_∞-similarity. Furthermore, Goo and Ryu [7], Goo and Yang [8], and Goo [9] have investigated hS for the nonlinear differential systems with different forms of perturbed terms.

In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems with perturbed integral forms using the notion of t_∞-similarity.

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*Corresponding author.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $J = [t_0, \infty)$. Consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0 \tag{2.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{2.3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2).

We recall some notions of h-stability [13] and the notion of t_∞-similarity [5]. The symbol $|.|$ denotes arbitrary vector norm in $\mathbb{R}^n$.

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called *h-stable (hS)* if there exist $\delta > 0$ such that (2.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+ = [0, \infty)$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^2 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 10].

We give some related properties that we need in the sequal.

Lemma 2.4 ([15])**.** *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{2.5}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.5).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{2.7}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.5. *If $y_0 \in \mathbb{R}^n$, for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.6 ([3, 15])**.** *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is hS.*

Theorem 2.7 ([4])**.** *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.2) is hS, then the solution $z = 0$ of (2.3) is hS.*

We need the following lemma for an h-system of perturbed nonlinear system.

Lemma 2.8 ([8])**.** *Let $u, p, q, w,$ and $r \in C[\mathbb{R}^+, \mathbb{R}^+]$ and suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} \left[q(\tau)u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)u(a)da \right] d\tau ds,\ t \geq t_0. \tag{2.8}$$

Then

$$u(t) \leq c \exp\left(\int_{t_0}^{t} p(s) \int_{t_0}^{s} \left[q(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)da \right] d\tau ds \right),\ t \geq t_0. \tag{2.9}$$

3. Main Results

In this section, we investigate hS for the nonlinear perturbed differential systems.

We consider the perturbed system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds,\ y(t_0) = y_0, \tag{3.1}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$ and $g(t, 0) = 0$.

Theorem 3.1. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (2.1) is an h-system such that a positive continuous function h and g in (3.1) satisfies*

$$|g(t, y)| \leq \lambda(t)(|y| + \int_{t_0}^{t} \gamma(s)|y(s)|ds),\ t \geq t_0,\ y \in \mathbb{R}^n,$$

where $\lambda, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} \left[\lambda(\tau) \left(h(\tau) + \int_{t_0}^{\tau} h(r)\gamma(r)dr \right) \right] d\tau ds < \infty, \tag{3.2}$$

for all $t_0 \geq 0$, then the solution $y = 0$ of (3.1) is an h-system.

Proof. Using the nonlinear variation of constants formula of Alekseev[1], any solution $y(t) = y(t, t_0, x_0)$ of (3.1) passing through (t_0, x_0) is given by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(\tau, y(\tau))d\tau ds, \tag{3.3}$$

where $x(t) = x(t, t_0, x_0)$ is a solution of (2.1) passing through (t_0, x_0). By Theorem 2.6, since the solution $x = 0$ of (2.1) is an h-system, the solution $v = 0$ of (2.2) is an h-system. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is an h-system.

By Lemma 2.4, we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau,y(\tau))| d\tau ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} \\
&+ \int_{t_0}^{t} c_2\frac{h(t)}{h(s)}\left[\int_{t_0}^{s} \lambda(\tau)(h(\tau)\frac{|y(\tau)|}{h(\tau)} + \int_{t_0}^{\tau} h(r)\gamma(r)\frac{|y(r)|}{h(r)}dr)d\tau\right]ds.
\end{aligned}$$

Setting $u(t) = |y(t)|h(t)^{-1}$ and using Lemma 2.8, we obtain

$$\begin{aligned}
|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1}\exp\left(c_2\int_{t_0}^{t}\frac{1}{h(s)}\int_{t_0}^{s}\left[\lambda(\tau)(h(\tau)+\int_{t_0}^{\tau}h(r)\gamma(r)dr)\right]d\tau ds\right) \\
&\le c|y_0|h(t)\,h(t_0)^{-1},\ t \ge t_0,
\end{aligned}$$

where $c = c_1\exp(c_2\int_{t_0}^{\infty}\frac{1}{h(s)}\int_{t_0}^{s}[\lambda(\tau)(h(\tau)+\int_{t_0}^{\tau}h(r)\gamma(r)dr)]d\tau ds)$. It follows that $y = 0$ of (3.1) is an h-system. Hence, the proof is complete. □

Remark 3.2. Letting $\gamma(s) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 2.5 in [6].

Remark 3.3. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is an h-system, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s,y(s))ds,\ y(t_0) = y_0,$$

is also an h-system under the same hypotheses in Theorem 3.1 except the condition of t_∞-similarity.

Theorem 3.4. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.1) is hS such that the increasing function h, and g in (3.1) satisfies*

$$\left|\int_{t_0}^{s} g(\tau,y(\tau))d\tau\right| \le a(s)\left(|y(s)| + \int_{t_0}^{s} c(\tau)|y(\tau)|d\tau\right),\ t \ge t_0 \ge 0,$$

where $a, c \in C[\mathbb{R}^+,\mathbb{R}^+]$ and $\int_{t_0}^{\infty}[a(s)(1+\int_{t_0}^{s}c(\tau)d\tau)]ds < \infty$. Then, the solution $y = 0$ of (3.1) is hS.

Proof. It is known that the solution of (3.1) is represented by the integral equation (3.3). By Theorem 2.6, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is hS. By

Lemma 2.4 and the increasing property of the function h, we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau)) d\tau \right| ds$$
$$\le c_1 |y_0| h(t)\, h(t_0)^{-1}$$
$$+ \int_{t_0}^{t} c_2 h(t)\, h(s)^{-1} \left[a(s) \left(|y(s)| + \int_{t_0}^{s} h(\tau) c(\tau) |y(\tau)| h(\tau)^{-1} d\tau \right) \right] ds.$$

Set $u(t) = |y(t)| h(t)^{-1}$. Then, by Gronwall's inequality, we obtain

$$|y(t)| \le c_1 |y_0| h(t)\, h(t_0)^{-1} \exp\left(c_2 \int_{t_0}^{t} \left[a(s) \left(1 + \int_{t_0}^{s} c(\tau) d\tau \right) \right] ds \right)$$
$$\le c |y_0| h(t)\, h(t_0)^{-1}, \quad c = c_1 \exp\left(c_2 \int_{t_0}^{\infty} \left[a(s) \left(1 + \int_{t_0}^{s} c(\tau) d\tau \right) \right] ds \right).$$

It follows that $y = 0$ of (3.1) is hS and so the proof is complete. □

Remark 3.5. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s)) ds, y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.4 except the condition of t_∞-similarity.

Remark 3.6. Letting $c(\tau) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.3 in [7].

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[a]Department of Mathematics, Hanseo University, Seosan, Chungnam, 356-706, Korea
Email address: yhgoo@hanseo.ac.kr

[b]Department of Mathematics, Hanseo University, Seosan, Chungnam, 356-706, Korea
Email address: ysb-good@hanmail.net