SHORTFALL RISK MINIMIZATION: THE DUAL APPROACH

Ju Hong Kim

ABSTRACT. We find the solution minimizing the shortfall risk by using the Lagrange-multiplier method. The conventional duality method in the expected utility maximization problem is used and we get the same results as in the paper [21].

1. INTRODUCTION

We consider an agent or an investor who sell a contingent claim and want to get rid of the associated shortfall risk by means of a dynamic hedging strategy. The shortfall risk is the difference between the payoff of the contingent claim and the value of the agent's or the investor's hedging strategy at maturity.

It is known that there is a dynamic self-financing hedging strategy with arbitrage-free hedging price to super-replicate a contingent claim in complete or incomplete markets. The super-hedging price is the minimal initial capital that an agent or an investor has to invest to find a strategy which dominates the claim payoff with certainty [15]. The super-hedging price of a contingent claim is given by the supremum of the expected values over all equivalent martingale measures. If an agent or an investor sells the claim for the super-hedging price, then he/she could eliminate the shortfall risk completely by choosing a suitable hedging strategy. The corresponding value process is a supermartingale under equivalent martingale measures. The super-hedging strategy is determined by the optional decomposition [18]. But the prices derived by super-replication are too high and not acceptable in practice. Then the claim should be sold for a price less than the super-hedging price. With the initial capital less than the super-hedging price, i.e., under the capital constraint

an agent or an investor is unable to eliminate all exposed risk associated to the contingent claim completely and so wants to find optimal strategies which minimize the shortfall risk.

Föllmer and Leukert [11] constructed a quantile hedging strategy which maximizes the probability of a successful hedge under the objective measure P under the capital constraint. In the quantile hedging approach, the size of the shortfall is not taken into account but only the probability of its occurrence. Föllmer and Leukert [12] also introduced optimal hedging strategies which minimize the shortfall risk under the capital constraint by using the expected loss functions as risk measures. In these papers the Neyman-Pearson lemma approach is used to find the solution to the static problem. In [12], the risk measure ρ is the form of $\rho(X) = E_P[\ell(X^+)]$, where X is a random variable on $(\Omega, \mathcal{F})$, P is a fixed probability measure on Ω, and $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly convex function. See the papers [6, 7, 23, 20] for the related works. Nakano [19] uses coherent risk measures [3, 8] as risk measures in the $L^1(\Omega, \mathcal{F}, P)$ random variable spaces instead of the loss function. Arai [1] obtained robust representation results of shortfall risk measures on Orlicz hearts under the continuous time setting. The Orlicz hearts setting allows us to treat various loss functions and various claims in a unified framework.

In this paper, we find the solution minimizing the shortfall risk by the dual approach [22]. The conventional duality method used in the expected utility maximization problem is adopted and we get the same results as in [21]. This paper is constructed as follows. The definition of a superhedging price and mathematical settings are given in section 2. The optimal solution of shortfall risk is found in complete market case and in incomplete market case in section 3 and 4, respectively.

2. Mathematical Settings and Superhedging

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ be a complete filtered probability space. Let

$$S = (S_t)_{0 \le t \le T}$$

be an adapted positive process which is a semimartingale. It is assumed that the riskless interest rate is zero for simplicity.

Definition 2.1. A *self-financing* strategy with initial capital $x \ge 0$ is defined as a predictable process ξ_t such that the value process (value of the current holdings)

$$X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T]$$

is P-a.s. well-defined.

Definition 2.2. A self-financing strategy (x, ξ_t) is called *admissible* if there exists some constant $c > 0$ such that

$$\forall t \in [0, T] \quad x + \int_0^t \xi_u dS_u \geq -c \quad P - a.s.$$

Here c is a credit line of an agent or an investor.

Definition 2.3. A contingent claim H is called *attainable* (or *replicable, redundant*) if there exists admissible strategy such that

$$H = x_0 + \int_0^T \xi_u dS_u.$$

Lemma 2.4. *Let $H \geq 0$ be a $\mathcal{F}_T$-measurable contingent claim. Then there exists admissible strategy (x_0, ξ) such that*

$$(2.1) \qquad H \leq x_0 + \int_0^T \xi_u dS_u \quad P - a.s.$$

if and only if

$$(2.2) \qquad H \in \left\{ X \geq 0 \,\middle|\, X \text{ is } \mathcal{F}_T - measurable, \quad \sup_{Q \sim P} E_Q[X] \leq x_0 \right\}.$$

Proof. See the proof in [17]. $\qquad \square$

Lemma (2.4) means that the pricing rule of H, i.e., $E_Q[H]$ is less than or equal to x_0 which is the initial capital of the admissible superhedging strategy (x_0, ξ) for H.

Definition 2.5. The superhedge price H_0 for H is defined as

$$H_0 = \inf \left\{ x \,\middle|\, \exists \text{ admissible strategy } (x, \xi) \text{ s.t. } H \leq x + \int_0^T \xi_u dS_u \quad P - a.s. \right\}.$$

By the Lemma (2.4) we can see the superhedge price is $H_0 = \sup_{Q \sim P} E_Q[H]$. That is, H_0 is the smallest initial capital eliminating all shortfall risk. The seller of H can cover almost any possible obligation from the sale of H and thus eliminate completely the corresponding risk. However, the super-hedging price of the seller is too high and can't be used in practice.

When the seller is unwilling to invest the superhedge price in a hedging strategy, the seller is seeking for the optimal partial hedging strategy minimizing the problem [13]

(2.3)
$$\min_{(x,\xi)\in\mathcal{X}(\alpha)}\left[\rho\left(\left(H-x-\int_0^T \xi_u dS_u\right)^+\right)\right]$$

with the initial capital constraint

(2.4)
$$0 < \alpha < H_0 = \sup_{Q\sim P} E_Q[H].$$

The admissible set $\mathcal{X}(\alpha)$ is defined as

$$\mathcal{X}(\alpha) := \{(x,\xi)\,|\,x \le \alpha < H_0,\ (x,\xi) \text{ is admissible strategy}\}.$$

Hereafter the risk measure ρ is taken as $\rho(X) = E_P[\ell(X)]$ as in the traditional literature, where X is a random variable on $(\Omega, \mathcal{F})$, P is a fixed probability measure on Ω, and $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly convex function as in [12]. We assume that the function $\ell \in C^1(0,\infty)$, the derivative ℓ' is strictly increasing with $\ell'(0+) = 0$ and $\ell'(+\infty) = +\infty$.

We will often use the short notation $(\xi.S)_T$ as the same expression as $\int_0^T \xi_u dS_u$. We consider the general set $K(x)$ of the terminal wealths at T with initial wealth x, and the set $K(x)$ is defined as

$$K(x) := \{X_T\,|\,X_T = x + (\xi.S)_T \text{ is } \mathcal{F}_T - \text{measurable}\}.$$

We can rewrite the minimizing shortfall problem (2.3) as the primal problem

(2.5)
$$\mathcal{P}(x) := \inf_{X_T \in K(x)} E[\ell(H - X_T)^+].$$

Define $K_Q(x)$ as

$$K_Q(x) = \{X \in L^1(Q)\,|\,E_Q[X] \le x\},$$

which contains the set $K(x)$ and the norm-closure of $K(x) - L^1_+(Q)$ in $L^1(Q)$.

The pricing measure Q is unique in complete market but not unique in incomplete market. The set of pricing measures is

$$\mathcal{M} = \{Q\,|\,Q \sim P,\ S \text{ is a local martingale under } Q\}$$

as stated in [9].

Assume that $\mathcal{M} \ne \emptyset$ for the no-arbitrage condition of the markets [9, 10].

Note that if $x \ge \sup_{Q\mathcal{M}} E_Q[H] := H_0$, then there exist admissible strategies (x,ξ) such that $H \le H_0 + (\xi.S)_T \le x + (\xi.S)_T$ by the Lemma (2.4) and hence it is hedged completely with the superhedge price.

Assume that $x < \sup_{Q\mathcal{M}} E_Q[H]$ throughout this paper.

2.1. Complete market case. Assume the market is complete and then the pricing measure $Q \in \mathcal{M}$ is unique. We try to solve the primal problem (2.5) under the constraint set $K_Q(x)$ which is larger than the set $K(x)$.

Fenchel-Legendre transform or conjugate functional ℓ^* of the convex function ℓ is defined by

$$(2.6) \qquad \ell^*(z) := \sup_{y \in \mathbb{R}} \{yz - \ell(y)\}.$$

Note that ℓ^* is a proper convex function, i.e. it is convex and takes some finite value. Denote $J := (\ell^*)'_+$ its right-continuous derivative. Form (2.6), for all $y, z \in \mathbb{R}$

$$(2.7) \qquad yz \leq \ell(y) + \ell^*(z),$$

and the equality holds if $y = J(z)$.

We consider the primal problem

$$(2.8) \qquad \mathcal{P}(x) := \inf_{X_T \in K_Q(x)} E[\ell(H - X_T)^+].$$

When we use the Lagrange-multiplier method [2], we can express the dual problem of the primal one as

$$(2.9) \qquad \mathcal{D}(x) := \sup_{\lambda > 0} \inf_{X_T \in L^1(Q)} \{E[\ell(H - X_T)^+] + \lambda(x - E_Q[X_T])\}.$$

We will show that there is no duality gap, i.e.

$$\mathcal{P}(x) = \mathcal{D}(x).$$

First we show that for X_T satisfying $E_Q[X_T] \leq x$

$$\inf_{X_T \in L^1(Q)} \{E[\ell(H - X_T)^+] + \lambda(x - E_Q[X_T])\}$$

$$(2.10) \qquad \geq \lambda(E_Q[H] - x) - E\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right].$$

If we take $y = (H - X_T)^+$ and $z = \lambda \frac{dQ}{dP}$ in the equation (2.7), and then take the expectation and add $\lambda(x - E_Q[X_T])$ to both sides of the inequality, then we have

$$(2.11) \quad E[\ell(H - X_T)^+] - \lambda(x - E_Q[X_T]) \geq \lambda(E_Q[H] - x) - E\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right],$$

since $E_Q[H] > x \geq E_Q[X_T]$ and so $E_Q[(H - X_T)^+] = (E_Q[H] - E_Q[X_T])^+ = E_Q[H] - E_Q[X_T]$. So we have

$$(2.12) \quad \mathcal{P}(x) := \inf_{X_T \in K_Q(x)} E[\ell(H - X_T)^+] \geq \lambda(E_Q[H] - x) - E\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right].$$

The inequality (2.10) holds from (2.11) and the equality in (2.10) holds if for each $\lambda > 0$ the relation

$$(2.13) \qquad (H - X_T)^+ = (\ell^*)' \left(\lambda \frac{dQ}{dP} \right)$$

satisfies. Hence the dual problem (2.9) becomes

$$\mathcal{D}(x) \quad := \quad \sup_{\lambda > 0} \inf_{X_T \in L^1(Q)} \{ E[\ell(H - X_T)^+] + \lambda(x - E_Q(X_T)) \}$$

$$(2.14) \qquad \geq \sup_{\lambda > 0} \left\{ \lambda(E_Q[H] - x) - E \left[\ell^* \left(\lambda \frac{dQ}{dP} \right) \right] \right\}.$$

If the equation (2.13) satisfies, then the equality in (2.14) holds. If $E_Q[X_T] \leq x$ satisfies in addition to the equation (2.13), then the following relation between the primal and the dual problem holds:

$$\mathcal{P}(x) := \inf_{X_T \in K_Q(x)} E[\ell(H - X_T)^+] \geq \mathcal{D}(x).$$

For each $\lambda > 0$ define

$$g(\lambda) = \lambda(E_Q[H] - x) - E \left[\ell^* \left(\lambda \frac{dQ}{dP} \right) \right].$$

Note that for $\lambda \in (0, +\infty)$ the function $g(\lambda)$ is concave function. Under the assumption of $(\ell^*)' \left(\lambda \frac{dQ}{dP} \right) \leq h$ for some $h \in L^1(P)$, g is differentiable by the Fubini's theorem. Let's find the critical point of g.

$$g'(\lambda) = 0 \quad \text{if and only if} \quad E_Q[H] - x - E_Q \left[(\ell^*)' \left(\lambda \frac{dQ}{dP} \right) \right] = 0.$$

It is said that the function $f : \mathbb{R} \to \mathbb{R}$ has or admits a supporting line at $x \in \mathbb{R}$ if there exists $a \in \mathbb{R}$ such that

$$f(y) \geq f(x) + a(y - x)$$

for all $y \in \mathbb{R}$.

Theorem 2.6. *If f admits a strict supporting line at x_k with slope k, then f^* admits a tangent supporting line at k with slope $(f^*)'(k) = x_k$.*

Proof. From the Legendre-Fenchel transform

$$f^*(k) := \sup_{x \in \mathbb{R}} \{ kx - f(x) \},$$

we have $f^*(k) = kx_k - f(x_k)$ and $(f^*)'(k) = x_k$ where x_k is the solution of $f'(x_k) = k$. $\qquad \square$

Since $\ell : \mathbb{R} \to \mathbb{R}$ is a strict convex function and the tangent slope of ℓ belongs to the range $(0, +\infty)$, the function $(\ell^*)' : (0, +\infty) \to (-\infty, +\infty)$ is bijective by the Theorem (2.6). Hence $\psi(\lambda) := E_Q\left[(\ell^*)'\left(\lambda \frac{dQ}{dP}\right)\right]$ is a bijective function from $(0, +\infty)$ to $(-\infty, +\infty)$. Thus there exists a unique solution λ^* of

$$E_Q\left[H - (\ell^*)'\left(\lambda \frac{dQ}{dP}\right)\right] = x.$$

Therefore, the supremum of the right hand side of (2.14) is taken at λ^*.

Since $x \geq 0$, $H \geq (\ell^*)'\left(\lambda \frac{dQ}{dP}\right)$ Q-a.s.

If we set $X_T^* = H - (\ell^*)'\left(\lambda^* \frac{dQ}{dP}\right)$, then $E_Q[X_T^*] = x$ and so $X_T^* \in K_Q(x)$. By the Lemma (2.4), there exists admissible strategy (x, ξ) satisfying

$$X_T^* = x + \int_0^T \xi_u \, dS_u \in K(x).$$

Since $H - X_T^* = (\ell^*)'\left(\lambda^* \frac{dQ}{dP}\right) \geq 0$ $Q - a.s.$, from Fenchel-Legendre transform we have

(2.15) $$\left(\lambda^* \frac{dQ}{dP}\right) \cdot (H - X_T^*)^+ = \ell(H - X_T^*)^+ + \ell^*\left(\lambda^* \frac{dQ}{dP}\right).$$

By taking expectation to (2.15) with respect to P, we get

$$E[\ell(H - X_T^*)^+] = \lambda^*(E_Q[H] - x) - E\left[\ell^*\left(\lambda^* \frac{dQ}{dP}\right)\right].$$

Hence $X_T^* \in K_Q(x)$ is a solution of the primal problem. Thus we are ended up with

$$\mathcal{P}(x) = \mathcal{D}(x) = \lambda^*(E_Q[H] - x) - E\left[\ell^*\left(\lambda^* \frac{dQ}{dP}\right)\right].$$

2.2. Incomplete market case. In this section we adopt notations and the proof methods from [5, 4] for the more general approaches. Since equivalent martingale measure is not unique in an incomplete market, the main job is to choose economically suitable one in this subsection, and the rest is the same as in a complete market.

Let G be the convex cone which is a subset of L^0. Define

$$\ell_{P,G}(x) := \inf_{X \in G} E_P[\ell(H - x - X)^+].$$

Define the set K as

$$K := \left\{ \int_0^T \xi_u \cdot dS_u \,\bigg|\, \xi \text{ is admissible} \right\},$$

which is the cone of bounded from below claims that are attainable, at zero initial cost, from trading in the d assets with admissible trading strategies.

Define C as

$$C := (K - L^0_+) \cap L^\infty.$$

We know that the norm dual space of L^∞ is $ba = ba(\Omega, \mathcal{F}, \mathbb{P})$, the set of bounded additive set functions on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P.

Let C^0 be the polar cone of C with respect to the dual system (L^∞, ba),

$$C^0 := \{\zeta \in ba \,|\, \zeta(X) \leq 0 \quad \forall X \in C\}.$$

Definition 2.7. $Q << P$ is called a separating measure if $K \subseteq L^1(Q)$ and if $K \subseteq L^1(Q)$ and $E_Q[X] \leq 0 \quad \forall X \in K$.

Define M as

$$M := C^0 \cap L^1(P) = \{z \in L^1_+(P) \,|\, E_P[zX] \leq 0 \quad \forall X \in C\}.$$

Define the set M_1 as

$$M_1 = \{z \in M \,|\, E_p[z] = 1\}.$$

A P-absolute continuous probability measure Q is identified with its Radon-Nikodym derivative $z = \frac{dQ}{dP}$. So we have

$$M_1 = \{Q << P \,|\, E_Q[X] \leq 0 \quad \forall X \in C\}.$$

For all $P \in \mathbb{P}$, $M \subset L^1_+(P)$, M is closed in $L^1(P)$ and if $M_1 \neq \emptyset$, the convex cone M is generated by the convex set M_1.

Lemma 2.8. *Q is a separating measure if and only if $Q \in M_1$. If S is bounded, then $M = \{Q << P \,|\, X$ is a $Q - martingale\}$. If S is locally bounded, then $M = \{Q << P \,|\, S$ is a $Q - local\ martingale\}$.*

Proof. It is clear that $M \subset M_1$. Conversely, suppose that $Q \in M_1$. Let $X \in K$ and set $X_n = \min\{X, n\}$. Then $X_n = X - (X - X_n) \in C$, and $X_n \uparrow X$ P-a.s., and hence Q-a.s.. By the Lebesgue Dominated Convergence Theorem, $E_Q[X] = \lim_{n\to\infty} E_Q[X_n] \leq 0$. Hence Q is a separating measure. If S is bounded and $S = 1_A(S_t - S_s)$, $A \in \mathcal{F}_s$, $0 \leq s < t \leq T$, then $S \in K$ and $-S \in K$. $Q \in M_1$ implies $E_Q[1_A(S_t - S_s)] = 0$ and Q is a martingale measure. $\square$

Define $\ell_{P,K}(x)$ as

$$\begin{aligned}
\ell_{P,K}(x) &:= \inf_{X \in K} E_P[\ell(H - x - X)^+] \\
&= \inf_{\xi \in \mathcal{X}(x)} E_P[\ell(H - x - (\xi.S)_T)^+] \quad \text{with } E_P[\ell(H + c)] < +\infty,
\end{aligned}$$

where $-c$ is the credit line of the investors.

Note that $0 \le \ell_{P,K}(x) < +\infty$.

Delbaen and Schchermayer [10] showed that NFLVR (no free lunch with vanishing risk) : $\bar{C} \cap L_+^\infty = \{0\}$, which is the weak no-arbitrage condition of the market, is equivalent to $M_1 \cap \mathbb{P} \ne \emptyset$. Here $\bar{C}$ is the L^∞-norm closure of C.

Assume that $M_1 \cap \mathbb{P} \ne \emptyset$ hereafter.

Note that if $x \ge \sup_{Q \in M_1} E_Q[H] := H_0$, then there exists an admissible strategy ξ satisfying $H \le H_0 + (\xi.S)_T \le x + (\xi.S)_T$ by the Lemma (2.4) and hence $\ell_{P,K}(x) = 0$. For each $\zeta \in ba$ define $\zeta(X) = E_P[\zeta X]$ for all $X \in C$.

Lemma 2.9. *If $M_1 \cap \mathbb{P} \ne \emptyset$, then we have*

$$C = M^0.$$

Proof. From the definition of polar cone of C, we have

$$
\begin{aligned}
C^0 &= \{\zeta \in ba \,|\, \zeta(X) \le 0 \quad \forall X \in C\} \\
&= \{\zeta \in ba \,|\, E^P[\zeta X] \le 0 \quad \forall X \in C\} = M.
\end{aligned}
$$

Since $NFLVR$ implies that C is weak*-closed, by the bipolar theorem we have

$$M^0 = C^{00} = C.$$

$\square$

The above lemma is adopted from the paper [4].

Lemma 2.10. *The following equality holds.*

$$\ell_{P,K}(x) = \ell_{P,C}(x) = \ell_{P,M^0}(x).$$

Proof. By (2.9), the second equality hold. For the proof of the first equality, let $X \in K$ and $X_n \in \min\{X, n\}$. Then $X_n \uparrow X$ and $X_n \in L^\infty$. Since $X_n = X - (X - X_n) \in K - L_+^0$, $X_n \in C$. Moreover, since $H - x - X_n \downarrow H - x - X$ and $\ell(H - x - X_n) \downarrow \ell(H - x - X)$, by Lebesgue Dominated Convergence Theorem, $E_P[\ell(H - x - X_n)] \downarrow E_P[\ell(H - x - X)]$. Therefore, we have

$$\ell_{P,C}(x) = \inf_{X \in C} E_P[\ell(H - x - X)^+] \le \inf_{X \in K} E_P[\ell(H - x - X)^+] = \ell_{P,K}(x).$$

On the other hand, we have

$$
\begin{aligned}
\inf_{X \in C} E_P[\ell(H - x - X)^+] &= \inf_{X \in K - L_+^0} E_P[\ell(H - x - X)^+] \\
&\ge \inf_{X \in K} E_P[\ell(H - x - X)^+].
\end{aligned}
$$

Thus the proof is done. $\square$

For $Q << P$, define

$$\ell(x; Q, P) \quad := \quad \inf\{E_P[\ell(H - x - X)^+] \,|\, X \in L^\infty, E_Q[X] \leq 0\}$$
$$= \quad \inf\{E_P[\ell(H - x)^+] \,|\, X \in L^\infty, E_Q[X] \leq x\}.$$

Since $M_1 \neq \emptyset$ by assumption and M is generated by the convex set M_1, the polar cone M^0 can be expressed as

$$M^0 = \{X \in L^\infty \,|\, E_Q[X] \leq 0 \ \forall Q \in M_1\}.$$

Note that

$$\ell_{P,M^0}(x) \leq \ell(x; Q, P),$$

since $\ell_{P,M^0}(x) = \inf\{E_P[\ell(H - X)^+] \,|\, X \in L^\infty, \ E_Q[X] \leq x \ \forall Q \in M_1\}$.

Definition 2.11. $\hat{Q}_x$ is called a minimax measure if

$$\ell_{P,M^0}(x) = \min_{Q \in M_1} \ell(x; Q, P) = \ell(x; \hat{Q}_x, P).$$

If there exists $Q^* \in M_1$ such that $\ell_{P,M^0}(x) \neq \ell(x; Q^*, P)$, then

$$\ell_{P,K}(x) = \ell_{P,M^0}(x) < \ell(x; Q^*, P) \leq \inf\{E_P[\ell(H - x)^+] \,|\, X \in L^\infty, E_Q[X] \leq x\}.$$

That is,

$$\inf_{\xi \in \mathcal{X}(x)} E_P[\ell(H - x - (\xi.S)_T)^+] < \inf\{E_P[\ell(H - X)^+] \,|\, X \in L^\infty, E_{Q^*}[X] \leq x\},$$

which is economically unreasonable.

Define the indicator functional of a convex set $F \subset L^\infty$ with

$$\delta(w|F) := \begin{cases} 0 & w \in F, \\ +\infty & w \notin F. \end{cases}$$

The convex conjugate $\delta_F^* : ba \to \mathbb{R}$ is denoted by

$$\delta_F^*(\zeta) := \sup_{w \in F}\{\zeta(w) - \delta(w)\} = \sup_{w \in F} \zeta(w).$$

Let $g : L^\infty \to \mathbb{R}$ be

$$g(X) = \delta(X|G + H - x).$$

Define the convex integral functional $I_u : L^\infty \to \mathbb{R}$ as

$$I_\ell(X) := E_P[\ell(X)^+].$$

Lemma 2.12. *Suppose that $\ell : \mathbb{R} \to \mathbb{R}$ is a convex function and that G is a convex cone. Then*

$$\ell_{P,G}(x) := \inf_{X \in x + G} E_P[\ell(H - X)^+]\} = \min_{\zeta \in G^0}\{g^*(\zeta) - I_\ell^*(\zeta)\}.$$

Proof. Let's consider the dual expression g^* of g.

$$g^*(\zeta) = \sup_{X \in G + H - x} \zeta(X) = \zeta(H - x) + \sup_{X \in G} \zeta(X) = \begin{cases} \zeta(H - x) & X \in G^0 \subset ba, \\ +\infty & otherwise \end{cases}.$$

$$\begin{aligned} \ell_{P,G}(x) &= \inf_{X \in G} E_P[\ell(H - x - X)^+] = \inf_{X \in G + H - x} E_P[\ell(X)^+] \\ &= \inf_{X \in L^\infty} \{E_P[\ell(X)^+] - \delta(X|G + H - x)\} \\ &= \max_{\zeta \in G^0} \{g^*(\zeta) - I_\ell^*(\zeta)\}, \end{aligned}$$

by the Fenchel duality theorem. $\qquad\square$

Proposition 2.13. *Suppose that $\ell : \mathbb{R} \to \mathbb{R}$ is a convex function and that G is a convex cone with $L_-^\infty \subset G \subset L^\infty$, and $N \subset L_+^1(P)$ is not an empty convex cone $\sigma(ba, L^\infty)$-closed, G is defined as*

$$G := N^0 = \{w \in L^\infty \,|\, E^P[zw] \leq 0 \quad \forall z \in N\}.$$

Then $G^0 = N$ and

$$\ell_{P,G}(x) = \max_{z \in N} E_P[z(H - x) - \ell^*(z)].$$

Proof. Since $N \subset L_+^1(P)$ is $\sigma(ba, L^\infty)$-closed, by the bipolar theorem $G^0 = N^{00} = N$.

If $z \in G^0 \subset L_+^1(P)$, then $g^*(z) = z(H - x) = E_P[z(H - x)]$. Hence we have

$$\begin{aligned} \ell_{P,G}(x) &= \max_{\zeta \in G^0} \{g^*(\zeta) - I_\ell^*(\zeta)\} \\ &= \max_{z \in N} E_P[z(H - x) - \ell^*(z)]. \end{aligned}$$

$\qquad\square$

Corollary 2.14. *Suppose that $\ell : \mathbb{R} \to \mathbb{R}$ is a convex function, $Q << P$, and that $\ell(x; Q, P) > \inf_{y \in \mathbb{R}} \ell(y)$. Then*

$$\begin{aligned} \ell(x; Q, P) &:= \inf\{E_P[\ell(H - x - X)^+] \,|\, X \in L^\infty, E_Q[X] \leq 0\} \\ &= \inf\{E_P[\ell(H - X)^+] \,|\, X \in L^\infty, E_Q[X] \leq x\} \\ &= \max_{\lambda \in (0, +\infty)} \left\{\lambda(E_Q[H] - x) - E_P\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right]\right\}. \end{aligned}$$

Proof. Let Q be given. Set

$$N = \left\{z \in L_+^1(P) \,|\, z = \lambda \frac{dQ}{dP}, \lambda \geq 0\right\}.$$

Then

$$G := N^0 = \{X \in L^\infty \,|\, E[zX] \le 0 \;\forall z \in N\} = \{X \in L^\infty \,|\, E_Q[X] \le 0\},$$

and so by definition

$$\ell(x; Q, P) = \ell_{P,G}(x).$$

Hence we have

$$
\begin{aligned}
\ell(x; Q, P) &= \ell_{P,G}(x) = \max_{z \in N} E_P[z(H - x) - \ell^*(z)] \\
&= \max_{\lambda \in [0,+\infty)} \left\{ \lambda(x - E_Q[H]) - E_P\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right] \right\}.
\end{aligned}
$$

If $\lambda = 0$, then

$$\ell(x; Q, P) = E_P[-\ell^*(0)] = -\ell^*(0) = -\sup_{y \in \mathbb{R}}\{y \cdot 0 - \ell(y)\} = \inf_{y \in \mathbb{R}} \ell(y).$$

So $\lambda = 0$ is excluded.   □

Lemma 2.15.

$$M = \{Q << P \,|\, E_Q[X] \le 0 \,\forall X \in C\}.$$

$$\inf\{E_P[\ell(H - x - X)^+] \,|\, X \in K\} = \inf\{E[\ell(H - x - X)^+] \,|\, X \in C\}.$$

Proof. Let $X \in K$ and

$$X_n := \min\{X, n\} \in X - (X - X_n) \in (K - L_+^0) \cap L^\infty(P) := C.$$

Then $X_n \uparrow X$ P-a.s. and $0 \ge \lim E_Q[X_n] = E_Q[X]$ by Monotone Convergence Theorem. The right hand set includes M. The other inclusion is obvious by the definition of C.   □

Theorem 2.16. $M \ne \emptyset$ *and there exists* $Q_x \in M$ *that satisfies*

$$\inf\{E_P[\ell(H - x - X)^+] \,|\, X \in C\} = \ell_{P,G}(x) = \max_{Q \in M} \ell(x; Q, P) = \ell(x; Q_x, P).$$

Proof. If $X \in C$, then $X \le 0$ and so $E_P[X] \le 0$. Therefore $P \in M \ne \emptyset$.

$$
\begin{aligned}
\ell_{P,G}(x) &= \max_{z \in N, z \ne 0} E_P[z(H - x) - \ell^*(z)] \\
&= \max_{Q \in M_1} \left\{ \max_{\lambda \in (0,+\infty)} \left\{ \lambda(x - E_Q[H]) - E_P\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right] \right\} \right\} \\
&= \max_{Q \in M_1} \ell(x; Q, P).
\end{aligned}
$$

                                                                                    □

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Department of Mathematics, Sungshin Women's University, Seoul 136-742, Korea
Email address: jhkkim@sungshin.ac.kr