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SLANT LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD

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Abstract. In this paper, we introduce the notion of a slant lightlike submanifold of an indefinite Sasakian manifold. We provide a non-trivial example and obtain necessary and sufficient conditions for the existence of a slant lightlike submanifold. Also, we prove some characterization theorems.

0. Introduction

The study of lightlike submanifolds of semi-Riemannian geometry appears to fill a gap in the general theory of submanifolds. The main difference between the lightlike submanifolds and non-degenerate submanifolds comes to the fact that the normal vector bundle has non-trivial intersection with the tangent vector bundle. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented in a book by Duggal and Bejancu [4]. Chen has introduced the notion of slant immersions by generalizing the concept of holomorphic and totally real immersions [2]. To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two (radical and screen) distributions. The radical distribution is totally lightlike and therefore it is impossible to define an angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Using these facts, the notion of slant lightlike submanifold was introduced by Sahin, Gupta and Sharfuddin [8, 9, 11].

The purpose of this paper is to introduce the notion of slant lightlike submanifold of an indefinite Sasakian manifold. In Section 1, we have collected the formulae and information which are useful in our subsequent sections. In Section 2, we introduce the concept of a slant lightlike submanifold of an indefinite Sasakian manifold and

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provide a non-trivial example. After then, we provide a characterization theorem for the existence of slant lightlike submanifolds and show that co-isotropic CR-lightlike submanifolds are slant lightlike submanifolds. Finally, we consider minimal slant ligthlike submanifolds and prove two characterization thoerems.

1. Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be a submanifold of dimension m of $\bar{M}$. We follow Duggal-Jin [5] for notations and results used in this paper. Throughout this paper we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. We say that M is a *lightlike submanifold* of $\bar{M}$ if it admits a degenerate metric g induced from $\bar{g}$. In this case the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, called the *screen* and *co-screen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike* if $1 \leq r < \min\{m, n\}$;
(2) *co-isotropic* if $1 \leq r = n < m$;
(3) *isotropic* if $1 \leq r = m < n$;
(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)$\sim$(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of r-lightlike submanifolds is more general than that of the other three type submanifolds. For this reason, in this paper we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$.

For the rest of this paper, by a lightlike submanifold we shall mean an r-lightlike submanifold, unless specified.

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively. Then we have

$$(1.2) \qquad tr(TM) = ltr(TM) \oplus S(TM^{\perp}),$$

$$(1.3) \qquad T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \oplus S(TM) \oplus S(TM^{\perp}).$$

We call $tr(TM)$ and $ltr(TM)$ *transversal* and *lightlike transversal vector bundle* of M. Consider the following local quasi-orthonormal field of frames of $\bar{M}$ along M:

$$(1.4) \qquad \{\xi_1, \ldots, \xi_r, \ N_1, \ldots, N_r, \ X_{r+1}, \ldots, X_m, \ W_{r+1}, \ldots, W_n\}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$, $\{N_1, \ldots, N_r\}$ a lightlike basis of $\Gamma(ltr(TM))$, $\{X_{r+1}, \ldots, X_m\}$ and $\{W_{r+1}, \ldots, W_n\}$ orthonormal basis of $\Gamma(S(TM)|\mathcal{U})$ and $\Gamma(S(TM^{\perp})|\mathcal{U})$ respectively. Then we have

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Due to (1.3) we put

$$(1.5) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \forall X, Y \in \Gamma(TM),$$

$$(1.6) \qquad \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \ \forall X \in \Gamma(TM), \ V \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_V\}$ and $\{h(X,Y), \nabla^{\perp}_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. ∇ and $\nabla^{\perp}$ are linear connections on M and $tr(TM)$ respectively. Besides ∇ is torsion-free linear connection. Also, h is a $\Gamma(tr(TM))$-value symmetric $F(M)$-bilinear form on $\Gamma(TM)$ and A_V is a shape operator on $\Gamma(TM)$. We call ∇ and $\nabla^{\perp}$ the induced connection and the transversal connection on M respectively. Also h is called the second fundamental form of M with respect to $tr(TM)$. Using (1.2) and (1.3), (1.5) and (1.6) become

$$(1.7) \qquad \bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^s(X, Y),$$

$$(1.8) \qquad \bar{\nabla}_X N = -A_N X + \nabla^{\ell}_X N + D^s(X, N),$$

$$(1.9) \qquad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^{\ell}(X, W),$$

for any $X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (1.5) $\sim$ (1.9) and the fact that $\bar{\nabla}$ is metric, we obtain

$$(1.10) \qquad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^{\ell}(X, W)) = g(A_W X, Y),$$

$$(1.11) \qquad \bar{g}(h^{\ell}(X, Y), \xi) + \bar{g}(Y, h^{\ell}(X, \xi)) + g(Y, \nabla_X \xi) = 0,$$

$$(1.12) \qquad \bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X),$$

$$\bar{g}\,(A_N X,\, N') + \bar{g}\,(A_{N'} X,\, N)) = 0, \tag{1.13}$$

$$\bar{g}\,(A_N X,\, PY) = \bar{g}\,(N,\, \bar{\nabla}_X PY), \tag{1.14}$$

for any $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^{\perp})$ and $N,\, N' \in \Gamma(ltr(TM))$.

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X, Y)\, \eta_i(Z) + h_i^{\ell}(X, Z)\, \eta_i(Y)\}, \tag{1.15}$$

for all $X,\, Y \in \Gamma(TM)$, where η_is are the r differential 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i),\ \forall\, X \in \Gamma(TM). \tag{1.16}$$

But the connection ∇^* on $S(TM)$ is metric. Denote by P the projection morphism of TM on $S(TM)$ with respect to (1.1). According to (1.1) we set

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{1.17}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{1.18}$$

for any $X,\, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where the sets $\{\nabla^*_X PY,\, A^*_{\xi} X\}$ and $\{h^*(X, PY),\, \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$ respectively. On the other hand, h^* is $\Gamma(Rad(TM))$-valued $F(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and A^*_{ξ} is a linear operator on $\Gamma(TM)$. Call h^* the second fundamental form of $S(TM)$ and A^* the shape operator of $S(TM)$ with respect to ξ. Also, call ∇^* and ∇^{*t} the induced connections on $S(TM)$ and $Rad(TM)$ respectively. It is important to note that both ∇^* and ∇^{*t} are metric connections. The second fundamental form and the shape operator of a non-degenerate submanifold of a semi-Riemannian manifold are related by means of the metric tensor field (see Chen[1]). Contrary to this, in the lightlike case there are interrelations between geometric objects induced by $tr(TM)$ and $S(TM)$. More precisely, by using (1.7), (1.17) and (1.18) we obtain

$$\bar{g}(h^{\ell}(X, PY),\, \xi) = \bar{g}(A^*_{\xi} X,\, PY), \tag{1.19}$$

$$\bar{g}\,(h^*(X,\, PY),\, N) = \bar{g}\,(A_N X,\, PY), \tag{1.20}$$

for any $X,\, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. As h^{ℓ} is symmetric, from (1.19) it follows that the shape operator of $S(TM)$ is a self-adjoint operator on $S(TM)$, i.e., we have

$$g(A^*_{\xi} PX,\, PY) = g(PX,\, A^*_{\xi} PY),\ \forall\, X,\, Y \in \Gamma(TM).$$

Replace Y by ξ in (1.11) we deduce

$$\bar{g}(h^{\ell}(X,\,\xi),\,\xi) \,=\, 0,\; \forall\, X \,\in\, \Gamma(TM). \tag{1.21}$$

Then replace X by ξ in (1.19) and by using (1.21), we obtain

$$A^*_{\xi}\xi \,=\, 0.$$

By using the linear connections introduced by (1.9)∼(1.11), we use the following covariant derivatives:

$$(\nabla_X h^{\ell})(Y,\,Z) = \nabla^{\ell}_X(h^{\ell}(Y,\,Z)) \,-\, h^{\ell}(\nabla_X Y,\,Z) \,-\, h^{\ell}(Y,\,\nabla_X Z), \tag{1.22}$$

$$(\nabla_X h^{s})(Y,\,Z) = \nabla^{s}_X(h^{s}(Y,\,Z)) \,-\, h^{s}(\nabla_X Y,\,Z) \,-\, h^{s}(Y,\,\nabla_X Z), \tag{1.23}$$

for $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM)))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [6, 7] if there exists a $(1, 1)$-type tensor field φ, a vector field V, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned}
&\varphi^2 X = -X + \eta(X)V,\ \varphi V = 0,\ \eta\circ\varphi = 0,\ \eta(V) = 1,\\
&\bar{g}(V,V) = \epsilon, \quad \bar{g}(\varphi X, \varphi Y) = \bar{g}(X,Y) - \epsilon\,\eta(X)\eta(Y),\\
&\eta(X) = \epsilon\bar{g}(V,X), \quad d\eta(X,Y) = \bar{g}(\varphi X, Y),\ \epsilon = \pm 1,
\end{aligned} \tag{1.24}$$

for any $X,\,Y \in \Gamma(T\bar{M})$. Then $(\varphi,\,\eta,\,V,\,\bar{g})$ is called a contact metric structure on $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_{\varphi} + d\theta \otimes V = 0$, where N_{φ} is the Nijenhuis tensor field of φ [6, 7]. A normal contact metric manifold is called a Sasakian manifold [12] for which we have

$$\bar{\nabla}_X V = \varphi X, \tag{1.25}$$

$$(\bar{\nabla}_X \varphi)Y = \epsilon\,\eta(Y)X - \bar{g}(X,Y)V. \tag{1.26}$$

The next ingredient we consider is a semi-Riemannian metric $\bar{g}$ of index $\mu(>0)$ on the Sasakian manifold $\bar{M} = (\bar{M},\,\varphi,\,V,\,\eta,\,\bar{g})$. Then we say that $\bar{M}$ is an *indefinite Sasakian manifold*. In an indefinite Sasakian manifold $\bar{M}$, the characteristic vector field V is a spacelike vector field on $\bar{M}$ [10].

A general notion of a minimal lightlike submanifold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal [4], is as follows:

Definition 1.1. A lightlike submanifold $(\bar{M}, \bar{g}, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *minimal* if

(1) $h^s = 0$ on $Rad(TM)$;

(2) $traceh = 0$, when *trace* is written with respect to g restricted to $S(TM)$.

Similar to the definition for a contact CR-lightlike submanifold of indefinite Sasakian manifold [7], we state the following:

Definition 1.2. Let $(M, g, S(TM), S(TM^{\perp})$ be a lightlike submanifold and immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We say hat M is *a contact CR-lightlike submanifold* of $\bar{M}$ if the following conditions are satisfied:
(1) $\varphi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \varphi(Rad(TM)) = \{0\}$;
(2) There exist vector bundles $\mathcal{D}_0$ and $\mathcal{D}'$ over M such that

$$\begin{cases} S(TM) = \{\varphi(Rad(TM)) \oplus \mathcal{D}'\} \perp \mathcal{D}_0 \perp \{\zeta\} \\ \varphi\mathcal{D}_0 = \mathcal{D}_0 \\ \varphi\mathcal{D}' = L_1 \perp ltr(TM), \end{cases}$$

where $\perp$ is the orthogonal direct sum, $\mathcal{D}_0$ is nondegenerate and L_1 is a vector subbundle of $S(TM^{\perp})$. A contact CR-lightlike submanifold is *proper* if $\mathcal{D}_0 \neq \{0\}$ and $L_1 \neq \{0\}$.

Example 1.3 ([7])**.** Let M be a lightlike hypersurface of $\bar{M}$. Then M is a contact CR-lightlike hypersurface.

2. Slant Lightlike Submanifolds

Lemma 2.1. *Let M be an r-lightlike submanifoldof an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that $\varphi(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \varphi(Rad(TM)) = \{0\}$. Then $\varphi(Rad(TM))$ is a subbundle of the screen distribution $S(TM)$ and $\varphi(ltr(TM)) \cap \varphi(Rad(TM)) = \{0\}$.*

Proof. By hypothesis, since $\varphi(Rad(TM))$ is a distribution on M such that

$$Rad(TM) \cap \varphi(Rad(TM)) = \{0\},$$

we have $\varphi(Rad(TM)) \in S(TM)$. Choose $N \in ltr(TM)$, $\xi \in Rad(TM)$, $X \in S(TM)$, and $W \in S(TM^{\perp})$ such that $\bar{g}(N, \xi) = \bar{g}(X, X) = \bar{g}(W, W) = 1$, we can write that

$$\varphi N = k_1 N + k_2 \xi + k_3 X + k_4 W, \tag{2.1}$$

where k_1, k_2, k_3, and k_4 are smooth functions on M. Taking the scalar product of (2.1) with N and ξ, we get $k_2 = 0$ and $k_1 = 0$, respectively. Thus we have

$$\varphi N = k_3 X + k_4 W, \tag{2.2}$$

Let us suppose that φN belongs to $S(TM^{\perp})$. Then we have $1 = \bar{g}(\xi, N) = \bar{g}(\varphi\xi, \varphi N) = 0$ due to $\varphi N \in \Gamma(S(TM^{\perp})))$ and $\varphi\xi \in \Gamma(S(TM))$, which is a contradiction. Therefore, from (2.2) we conclude φN belongs to $S(TM)$ and $\varphi(ltr(TM))$ is a distribution on M.

Moreover, φN dose not belong to $\varphi(Rad(TM))$. Indeed if $\varphi N \in \Gamma(Rad(TM))$, we would have $\varphi^2 N = -N + \theta(N)V = -N \in \Gamma(Rad(TM))$, but this is impossible. Thus, we conclude $\varphi(ltr(TM)) \subset S(TM)$ and $\varphi(ltr(TM)) \cap \varphi(Rad(TM)) = \{0\}$. □

Lemma 2.2. *Let M be q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ with the characteristic field tangent to M. Suppose that $\varphi(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \varphi(Rad(TM)) = \{0\}$. Then any complementary distribution to $\varphi(ltr(TM)) \oplus \varphi(Rad(TM))$ in the screen distribution $S(TM)$ is Reimannian.*

Proof. Let D' be the complemantary distribution to $\varphi(ltr(TM)) \oplus \varphi(Rad(TM)) \subset S(TM)$ and let $dim(\bar{M}) = m + n$ and $dim(M) = m$. We can choose a local quasi orthornomal frame on $\bar{M}$ along M as follows:

$$\{\xi_i, N_i, \varphi\xi_i, \varphi N_i, X_\alpha, V, W_\beta\}, \qquad i \in \{1, \ldots, q\},$$
$$\alpha \in \{3q+1, \ldots, m-1\}, \quad \beta \in \{q+1, \ldots, n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $Rad(TM)$ and $ltr(TM)$, respectively, and $\{\varphi\xi_i, \varphi N_i, X_\alpha, V\}$ is an orhonormal basis of $S(TM)$ and $\{W_\beta\}$ is an orthonormal basis of $S(TM^{\perp})$. Now, we construct an orthonormal basis $\{U_1, \ldots, U_{2q}, V_1, \ldots, V_{2q}\}$ as follows:

$$U_1 = \frac{1}{\sqrt{2}}\{\xi_1 + N_1\} \qquad U_2 = \frac{1}{\sqrt{2}}\{\xi_2 - N_2\},$$
$$U_3 = \frac{1}{\sqrt{2}}\{\xi_3 + N_3\} \qquad U_4 = \frac{1}{\sqrt{2}}\{\xi_4 - N_4\},$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$U_{2q-1} = \frac{1}{\sqrt{2}}\{\xi_q + N_q\} \qquad U_{2q} = \frac{1}{\sqrt{2}}\{\xi_q - N_q\},$$
$$V_1 = \frac{1}{\sqrt{2}}\{\varphi\xi_1 + \varphi N_1\} \qquad V_2 = \frac{1}{\sqrt{2}}\{\varphi\xi_2 - \varphi N_2\},$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$

$$V_{2q-1}1 = \frac{1}{\sqrt{2}}\{\varphi\xi_q + \varphi N_q\} \qquad V_{2q} = \frac{1}{\sqrt{2}}\{\varphi\xi_q - \varphi N_q\}.$$

Hence, $\{\xi_i, N_i, \varphi\xi_i, \varphi N_i\}$ gives a non-degenerate space of constant index $2q$ which implies that $Rad(TM) \oplus ltr(TM) \oplus \varphi(Rad(TM)) \oplus \varphi(ltr(TM))$ is nod-degenerate and of constant index $2q$ on $\bar{M}$. As

$$\begin{aligned} index(T\bar{M}) &= index(Rad(TM) \oplus ltr(TM)) \\ &\quad + index(\varphi(Rad(TM)) \oplus \varphi(ltr(TM))) \\ &\quad + index(D' \oplus S(TM^{\perp})), \end{aligned}$$

we have

$$2q = 2q + index(D' \oplus S(TM^{\perp})),$$

which implies that $index(D' \perp S(TM^{\perp})) = 0$. Hence D' is Riemannian. □

To define slant lightlike submanifolds of indefinite Sasakian manifolds, one needs to consider an angle between two vector fields. We shown from Section 1 that a lightlike submanifold has two(radical and screen) distributions. The radical distribution is totally lightlike and therefore it is not impossilbe to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Thus one way to define slant lightlike submanifolds is to choose a Riemannian screen distribution on lightlike submanifolds, for which we use Lemma 2.2.

Definition 2.3. Let M be a q-lightlike submaifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ with V tangent to M. Then we say that M is *a slant lightlike submanifold* of $\bar{M}$ if the following conditions are satisfied:

(i) $\varphi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \varphi(Rad(TM)) = \{0\}$.

(ii) For all $x \in \mathcal{U} \subset M$ and for each non-zero vector field X tangent to $\bar{D} = D \perp \{V\}$, if X and V are linearly independent , then the angle $\theta(X)$ between φX and the vector space $\bar{D}_x$ is constant, where D is complementary distribution to $\varphi(ltr(TM)) \oplus \varphi(Rad(TM))$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called *the slant angle* of $\bar{D}$. A slant lightlike submanifold M is said to be *proper* if $D \neq \{0\}$ and $\theta \neq 0, \frac{\phi}{2}$.

If M is totally lightlike submanifold of $\bar{M}$, then we have $TM = Rad(TM)$, and hence $D = \{0\}$. Therefore we have the following:

Proposition 2.4. *There exist no proper slant totally lightlike or isotropic submanifold M in indefinite Sasakian manfiold $\bar{M}$ with the characteristic vector field V tangent to M.*

From now on, $\left(\mathbb{R}_q^{2m+1}, \varphi_0, V, \eta, \bar{g}\right)$ will denote the manifold $\mathbb{R}_q^{2m+1}$ with its usual Sasakian structure given by

$$\begin{aligned}
&\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \quad V = 2\partial z \\
&\bar{g} = \eta \otimes \eta \\
&\quad + \frac{1}{4}\{-\sum_{i=1}^{q}(dx^i \otimes dx^i + dy^i \otimes dy^i) + \sum_{i=q+1}^{m}(dx^i \otimes dx^i + dy^i \otimes dy^i)\}, \\
&\varphi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x^i - X_i \partial y^i) + \sum_{i=1}^{m} Y_i y^i \partial z,
\end{aligned} \tag{2.3}$$

where (x^i, y^i, z) are the Cartesian coordinates.

Example 1. Let $\bar{M} = (\mathbb{R}_2^9, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-, +, +, -, +, +, +, +)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$$

Suppose M is a submanifold of $\mathbb{R}_2^9$ defined by

$$x^1 = y^4, \qquad x^2 = \sqrt{1 - (y^2)^2}, \qquad y^2 \neq \pm 1$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned}
&Z_1 = 2(\partial x_1 + \partial y_4 + y^1 \partial z), \qquad Z_2 = 2(\partial x_4 - \partial y_1 + y^4 \partial z) \\
&Z_3 = \partial x_3 + \partial y_2 + y^3 \partial z, \quad Z_4 = \partial y_1 + 2\partial y_3, \\
&Z_5 = -\frac{y^2}{x^2}\partial x_2 + \partial y_2 - \frac{(y^2)^2}{x_2}\partial z, \; Z_6 = \partial x_4 + \partial y_1, \; \partial Z_7 = V = 2\partial z.
\end{aligned} \tag{2.4}$$

Hence, we show that $Rad(TM) = span\{Z_1\}$, $\varphi_0(Rad(TM)) = span\{Z_2\}$, and $Rad(TM) \cap \varphi_0(Rad(TM)) = \{0\}$, hence (i) holds. Next, $\bar{\mathcal{D}} = \mathcal{D} \perp \{V\} = \{Z_3, Z_4\} \perp \{V\}$ is Riemannian, where $\perp$ is the orthogonal direct sum. By direct calculations, we get

$$S(TM^\perp) = span\{W = \partial x_2 + \frac{y^2}{x^2}\partial y_2 + y^2 \partial z\} \text{ such that } \varphi_0(W) = -Z_5,$$

and $ltr(TM) = span\{N = -\partial x_1 + \partial y_4 - y\partial z\}$ such that $\varphi_0(N) = Z_6$. Next, we have $\bar{D} = D \perp V = \{Z_3, Z_4, Z_5\} \perp \{V\}$ is Riemannian, where $\perp$ is the orthogonal direct sum. Then M is proper slant lightlike.

Proposition 2.5. *Slant lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$ with the characteristic vector field V tangent to M do not include invariant and screen real lightlike submanifolds.*

Proposition 2.6. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then any coisotropic CR-lightlike submanifold is a slant lightlike submanifold with $\theta = 0$. In particular, a lightlike real hypersurface of an indefinite Sasakian manifold $\bar{M}$ of index 2 is a slant lightlike submanifold with $\theta = 0$. Moreover, any CR-lightlike submanifold of $\bar{M}$ with $D_0 = \{0\}$ is a slant lightlike submanifold with $\theta = \frac{\pi}{2}$.*

Proof. Let M be a q-lightlike CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\varphi(Rad(TM))$ is a distribution on M such that $Rad(TM) \cap \varphi(Rad(TM)) = \{0\}$. If M is coisotropic, then $S(TM^{\perp}) = \{0\}$. Then the complementary distribution to $\varphi(ltr(TM)) \cap \varphi(Rad(TM))$ is the screen distribution $S(TM)$ is $\bar{\mathcal{D}} = \mathcal{D}_0 \perp \{V\}$ where $\mathcal{D}_0$ is Riemannian by Lemma 2.2. Since $\mathcal{D}_0$ is invariant with respect to φ, it follows that $V = 0$. The second assertion is obvious as a lightlike real hypersurface on $\bar{M}$ is coisotropic. Now, if M is CR-lightlike submanifold with $\mathcal{D}_0 = \{0\}$, then the complementary distribution to $\varphi(ltr(TM)) \cap \varphi(Rad(TM))$ in the screen distribution $S(TM)$ is $\bar{\mathcal{D}} = \mathcal{D}' \perp \{V\}$. Since $\mathcal{D}'$ is anti-invariant with respect to φ, it follows that $\theta = \frac{\pi}{2}$, which completes the proof. □

We know that for any $X \in TM$ and $W \in tr(TM)$,

$$\varphi X = TX + FX, \qquad \varphi W = BW + CW, \tag{2.5}$$

TX and FX are the tangential and transversal components of ϕX, respectively and BW and CW are tangential and transversal components of ϕW, respectively. Morveover, for a slant lightlike submanifold, we denote by P_1, P_2, Q_1, and Q_2 and $\bar{Q}_2$ the projections on the distributions $Rad(TM)$, $\varphi(Rad(TM))$, $\varphi(ltr(TM))$, $\mathcal{D}$ and $\bar{\mathcal{D}} = \mathcal{D} \perp \{V\}$, respectively. Then for any $X \in TM$, we can write

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{2.6}$$

where $\bar{Q}_2X = Q_2X + \theta(X)V$. Using (2.5) in the above equation, we obtain

$$\varphi X = \varphi P_1X + \varphi P_2X + TQ_2X + FQ_1X + FQ_2X, \quad \forall X \in TM. \tag{2.7}$$

Then the tangential components are

$$TX = TP_1X + TP_2X + TQ_2X. \tag{2.8}$$

We now prove two characterization theorems for slant lightlike submanifolds.

Theorem 2.7. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ with the characteristic vector field tangent to M. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) *$\varphi(ltr(TM))$ is a distribution on M.*

(b) *There exist a constant $\lambda \in [-1, 0]$ such that*

$$T^2\bar{Q}_2X = \lambda(\bar{Q}_2X - \theta(\bar{Q}_2X)V),$$

for all $X \in \Gamma(TM)$ linearly independent of the characteristic vector field V. Moreover, in such a case, $\lambda = -\cos^2\theta$ when θ is the slant angle of M.

Proof. Let M be a q-lightlike submanifold of an indefinite Saskian manifold $\bar{M}$ of index $2q$. If M is a slant lightlike submanifold of $\bar{M}$, then $\varphi(Rad(TM))$ is a distribution on $S(TM)$, and hence from Lemma 2.2, it follows that $\varphi(ltr(TM))$ is also a distribution on M and $\varphi(ltr(TM)) \subset S(TM)$. Thus (a) is complete. For $X \in \Gamma(TM)$, $Q_2X \in \bar{\mathcal{D}} - \{V\}$, we have

$$\cos\theta(Q_2X) = \frac{\bar{g}(\varphi Q_2X, TQ_2X)}{|\varphi Q_2X||TQ_2X|} = -\frac{\bar{g}(Q_2X, \varphi TQ_2X)}{|\varphi Q_2X||TQ_2X|} = -\frac{\bar{g}(Q_2X, T^2Q_2X)}{|Q_2X||TQ_2X|} \tag{2.9}$$

On the other hand, $\cos\theta(X) = \frac{|TX|}{|\varphi X|}$, and so, by using 2.9, we obtain

$$\cos^2\theta(Q_2X) = -\frac{\bar{g}(Q_2X, T^2Q_2X)}{|Q_2X|^2}.$$

Since $\theta(Q_2X)$ is constant on $\bar{\mathcal{D}}$, we conclude that

$$T^2\bar{Q}_2X = \lambda Q_2X = \lambda(\bar{Q}_2X - \theta(\bar{Q}_2X)V), \quad \lambda \in (-1, 0).$$

Moreover, in this case, $\lambda = -\cos^2\theta$. It is clear that the above equation is valid for $\theta = 0$ and $\theta = \frac{\pi}{2}$. Hence for $\bar{Q}_2X \in \bar{\mathcal{D}}$, the proof is complete. Conversely, suppose that (a) and (b) hold. Then (a) implies that $\varphi(Rad(TM))$ is a distribution on M. From Lemma 2.2, it follows that the complementary distribution to $\varphi(ltr(TM)) \oplus \varphi(Rad(TM))$ is a Riemannian distribution. The rest of the proof is clear. □

Corollary 2.8. *Let M be a slant submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ with the characteristic vector field tangent to M. Then, for any*

X, $Y \in \Gamma(TM)$, we have

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = \cos^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \theta(\bar{Q}_2X)\theta(\bar{Q}_2Y)\}, \tag{2.10}$$

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = \sin^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \theta(\bar{Q}_2X)\theta(\bar{Q}_2Y)\}. \tag{2.11}$$

Proof. From $g(TX, Y) = -g(X, TY)$ for all $X \in \Gamma(TM)$ and Theorem 2.7, a direct expension gives (2.10). To prove (2.11), it is enough to take into account (1.19) and (2.5). □

Theorem 2.9. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ with the characteristic vector field tangent to M. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) *$\varphi(ltr(TM))$ is a distribution on M.*

(b) *There exist a constant $\mu \in [-1, 0]$ such that*

$$BF\bar{Q}_2X = \mu(\bar{Q}_2X - \theta(\bar{Q}_2X)V), \quad \forall X \in \Gamma(TM).$$

Moreover, in such a case, $\mu = -\sin^2\theta$ when θ is the slant angle of M.

Proof. It is clear to see that $\varphi(Rad(TM)) \cap \varphi(ltr(TM)) = \{0\}$ and $\varphi(Rad(TM))$ is subbundle of $S(TM)$. Moreover, the complementary distribution to $\varphi(ltr(TM)) \oplus \varphi(Rad(TM))$ in $S(TM))$ is Riemannian. Furthermore, from the proof of Lemma 2.2, $S(TM^\perp)$ is also Riemannian. Thus (i) in the Definition 2.3 of slant lightlike submanifold is satisfied. On the other hand, from (2.5) and (2.7), we obtain

$$-X = -P_1X - P_2X + T^2Q_2X + FTQ_2X + JFQ_1X + BFQ_2X + CFQ_2X.$$

Since $\varphi FQ_1X = -Q_1X \in \Gamma(S(TM))$, takining the tangential parts, we have

$$-X + \theta(X)V = -P_1X - P_2X + T^2Q_2X - Q_1X + BFQ_2X.$$

From (2.6), we obtain

$$-Q_2X = -T^2Q_2X + BFQ_2X. \tag{2.12}$$

Now, if M is slant lightlike, then from Theorem 2.7, we have $T^2Q_2X = -\cos^2\theta Q_2X$, and hence we get $BFQ_2X = -\sin^2\theta Q_2X$. Since $FV = 0$ and $\bar{Q}_2X = Q_2X + \theta(X)V$, we have $BF\bar{Q}_2X = -\sin^2\theta\{\bar{Q}_2X - \theta(\bar{Q}_2X)V\}$.

Conversely, suppose that $BFQ_2X = \mu Q_2X$. Then, from (2.12), we obtain

$$T^2Q_2X = -(1 + \mu)Q_2X.$$

Thus, the proof follows from Theorem 2.7. □

3. Minimal Slant Lightlike Submanifolds

Now we study minimal slant lightlike submanifolds of indefinite Sasakian manifolds. In what follows, we prove two characterization results for minimal slant lightlike submanifolds. First we give the following lemma.

Lemma 3.1. *Let M be a proper slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $dim(\mathcal{D}) = dim(S(TM^{\perp}))$. If $\{e_1, \dots, e_m\}$ is a local orthonormal basis of $\Gamma(\mathcal{D})$, then $\{\csc\theta Fe_1, \dots, \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$.*

Proof. Since $\{e_1, \dots, e_m\}$ is a local orthonormal basis for $\mathcal{D}$ and $\mathcal{D}$ is Riemannian, from Corollary 2.8, we find

$$\bar{g}(\csc\theta Fe_i, \csc\theta Fe_j) = \delta_{ij},$$

where $i, j = 1, 2, \dots, m$, which proves the assertion. □

Theorem 3.2. *Let M be a proper slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with the characteristic vector field tangent to M. Then M is minimal if and only if*

$$traceA_{W_j}|_{S(TM)} = 0, \quad traceA^*_{\xi_k}|_{S(TM)} = 0, \quad \text{and} \quad \bar{g}(D^l(X, W), Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM)), W \in \Gamma(S(TM^{\perp}))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad(TM)$ and $\{W_j\}_{j=1}^r$ is a basis of $S(TM^{\perp})$.

Proof. From (1.25), we have $\bar{\nabla}_V V = 0$ and thus from (1.3) we get $h^l(V, V) = h^s(V, V) = 0$. Now, take an orthonormal frame basis of $S(TM^{\perp})$ of $\mathcal{D}$.
From (1.16), we know $h^l_i = 0$ on $Rad(TM)$ for all i. Thus, M is minimal if and only if

$$\sum_{k=1}^{r} h(\varphi\xi_k, \varphi\xi_k) + \sum_{k=1}^{r} h(\varphi N_k, \varphi N_k) + \sum_{i=1}^{m} h(e_i, e_i) = 0.$$

Using (1.10) and (1.19), we obtain

$$\sum_{k=1}^{r} h(\varphi\xi_k, \varphi\xi_k) = \sum_{k=1}^{r} \frac{1}{r} \sum_{a=1}^{r} \bar{g}(A^*_{\xi_a}\varphi\xi_k, \varphi\xi_k)N_a + \sum_{k=1}^{r} \frac{1}{m} \sum_{j=1}^{m} \bar{g}(A_{W_j}\varphi\xi_k, \varphi\xi_k)W_j. \tag{3.1}$$

Similarly, we have

$$\sum_{k=1}^{r} h(\varphi N_k, \varphi N_k) \tag{3.2}$$
$$= \sum_{k=1}^{r} \frac{1}{r} \sum_{a=1}^{r} \bar{g}(A^*_{\xi_a} N_k, \varphi N_k) N_a + \sum_{k=1}^{r} \frac{1}{m} \sum_{j=1}^{m} \bar{g}(A_{W_j} \varphi N_k, \varphi N_k) W_j$$

and

$$\sum_{i=1}^{r} h(e_i, e_i) = \sum_{i=1}^{r} \frac{1}{r} \sum_{a=1}^{r} \bar{g}(A^*_{\xi_a} e_i, e_i) N_a + \sum_{i=1}^{r} \frac{1}{m} \sum_{j=1}^{m} \bar{g}(A_{W_j} e_i, e_i) W_j. \tag{3.3}$$

Thus our assertion follows from (3.1) $\sim$ (3.3). □

Theorem 3.3. *Let M be a proper slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with the characteristic vector field tagent to M such that $dim(\mathcal{D}) = dim(S(TM^{\perp}))$. Then M is minimal if and only if*

$$traceA_{Fe_j}|_{S(TM)} = 0, \quad traceA^*_{\xi_k}|_{S(TM)} = 0, \text{and} \quad \bar{g}(D^l(X, Fe_j), Y) = 0,$$

for X, $Y \in \Gamma(Rad(TM))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad(TM)$ and $\{e_j\}_{j=1}^r$ is a basis of $\mathcal{D}$.

Proof. From (1.25), we have $\bar{\nabla}_V V = 0$ and thus from (1.3) we get $h^l(V, V) = h^s(V, V) = 0$. Moreover, from Lemma 3.1, $\{\csc\theta F e_1, \ldots, \csc\theta F e_m\}$ is an orthonormal basis of $S(TM^{\perp})$. Thus,

$$h^s(X, X) = \sum_{i=1}^{m} \csc\theta \bar{g}(A_{Fe_i} X, X),$$

for $X \in \Gamma((\varphi(Rad(TM)) \oplus \varphi(ltr(TM))) \perp \mathcal{D})$. Thus the proof follows from Theorem 3.2. □

Remark 3.4. (a) It is known that a proper slant submanifold of a Sasakian manifold is odd dimensional, but this is not true in case of our definition of slant lightlike submanifold. For instance, see two examples given in this paper.

(b) We notice that the second fundamental forms and their shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (1.7) $\sim$ (1.14) that in case of lightlike submanifold manifolds there are interrelations between these geometric objects and those of its screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $(S(TM), S(TM^{\perp}), ltr(TM))$.

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