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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A CLASS OF p-LAPLACIAN EQUATIONS

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Abstract. The existence and uniqueness of T-periodic solutions for the following p-Laplacian equations:

$$(\phi_p(x'))' + \alpha(t)x' + g(t,x) = e(t), \quad x(0) = x(T), x'(0) = x'(T)$$

are investigated, where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and $\alpha \in C^1$, $e \in C$ are T-periodic and g is continuous and T-periodic in t. By using coincidence degree theory, some existence and uniqueness results are obtained.

1. Introduction

We consider the solvability and uniqueness of the following periodic boundary value problem:

$$(\phi_p(x'))' + \alpha(t)x' + g(t,x) = e(t) \tag{1}$$

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{2}$$

where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and $\alpha \in C^1$, $e \in C$ are T-periodic and g is continuous and T-periodic in t. Moreover, we assume that $\int_0^T e(t)dt = 0$.

By a solution of the problem (1)-(2) we mean a function $x \in C^1([0,T],\mathbb{R})$ with $\phi_p(x')$ absolutely continuous, which satisfies (1)-(2) a.e. on $[0,T]$.

Note that if $p = 2$, (1) reduces to the following second order forced Rayleigh equation:

$$x'' + \alpha(t)x' + g(t,x) = e(t). \tag{3}$$

The existence and uniqueness of periodic solutions of (1) and (3) when $\alpha(t) = C$ with C a constant, have been an important research focus for the study of dynamic behaviors of nonlinear second order differential equations. See, for example, research

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papers [1-9] and the references therein. Recently, Zhang and Li[8] have obtained the following results:

Consider the following p-Laplacian equation:

$$(\phi_p(x'))' + Cx' + g(t,\, x) = e(t), \tag{4}$$

where C is a constant and

$$\int_0^T e(t)dt = 0.$$

Theorem A. *Assume that there exist $K > 0$ and $M > 0$ such that*

(A_1) $(g(t,u_1) - g(t,u_2))(u_1 - u_2) < 0$ *for all* $u_1, u_2, t \in \mathbb{R}$ *with* $u_1 \neq u_2$;

(A_2) $xg(t,x) < 0$ *for all* $x \neq 0$ *and* $t \in \mathbb{R}$;

(A_3) $2^{2-p}MT^p < 1$ *and* $g(t,x) \geq -M|x|^{p-1} - K$ *for all* $x \geq 0$ *and* $t \in \mathbb{R}$.

Then (4) has a unique T-periodic solution.

Theorem B. *Assume that there exist $K > 0$ and $M > 0$ such that*

(A_1') $(g(t,u_1) - g(t,u_2))(u_1 - u_2) < 0$ *for all* $u_1, u_2, t \in \mathbb{R}$ *with* $u_1 \neq u_2$;

(A_2') $xg(t,x) < 0$ *for all* $x \neq 0$ *and* $t \in \mathbb{R}$;

(A_3') $2^{2-p}MT^p < 1$ *and* $g(t,x) \leq M|x|^{p-1} + K$ *for all* $x \leq 0$ *and* $t \in \mathbb{R}$.

Then (4) has a unique T-periodic solution.

More recently, Wang [7] has improved Theorem A and Theorem B, and has obtained the following results:

Theorem C. *Assume that there exists $d \geq 0$ such that*

(B_1) $[g(t,u_1) - g(t,u_2)](u_1 - u_2) < 0 \quad \forall\, u_1,\, u_2,$ *with* $u_1 \neq u_2$, *and* $t \in \mathbb{R}$.

(B_2) $xg(t,x) < 0 \quad \forall\, |x| > d$ *and* $t \in \mathbb{R}$.

Then (4) has a unique T-periodic solution.

In this paper, we discuss the existence and uniqueness of T-periodic solutions of the periodic boundary value problem (1)-(2) under some general conditions. The main results of this paper are the following:

Theorem 1. *Consider problem (1)-(2). Assume that*

(H_1) *there exist constants* $d > 0$ *such that for* $|x| > d$, $xg(t,x) < -x^2 \;\forall\, t \in [0,\, T]$;

(H_2) $\alpha'(t) \geq -1 \;\forall\; t \in [0,\, T]$.

Then the problem (1)-(2) has at least one T-periodic solution.

Theorem 2. *Assume that (H_1) and (H_2) in Theorem 1 and (B_1) in Theorem C hold.*

Then (1)-(2) has a unique T-periodic solution.

Remark. In [10], the authors considered a general term $f(t, x')$ in (1) instead of a specific term $\alpha(t)x'$ as in this paper. However, the assumptions (H1) and (H2) of Theorem 1 in this paper do not follow from the assumptions of the Theorem 1 in [10] and hence Theorem 1 in this paper is different from and independent of that in [10].

2. Proofs of Theorems.

We first introduce some well-known results for p-Laplacian-like operators, which will be used in the proof of Theorem 1.

Let $X = C^1_T[0, T]$ be the space of all T-periodic C^1-functions, i.e.,

$$X = C^1_T[0, T] = \left\{x(t) \in C^1([0, T], \mathbb{R}) : x(0) = x(T),\ x'(0) = x'(T)\right\}.$$

The norm of a function $x \in C^1_T[0, T]$ is defined by

$$\|x\| := |x|_\infty + |x'|_\infty,$$

where $|x|_\infty := \max_{t\in[0,T]} |x(t)|$ and $|x'|_\infty := \max_{t\in[0,T]} |x'(t)|$.

Lemma 1 ([5, Theorem 3.1]**).** *Consider the following problem*

$$(\phi_p(u'))' = h(t, u, u'), \qquad u(0) = u(T), \quad u'(0) = u'(T), \tag{5}$$

where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and h is a Caratheodory function and is T-periodic in t. Let $\Omega_r = \{x \in C^1_T[0, T] :\ \|x\| < r\}$ for some $r > 0$. Suppose that the following conditions hold:

(i) For each $\lambda \in (0, 1)$, the problem

$$(\phi_p(u'))' = \lambda h(t, u, u'), \quad u \in C^1_T[0, T] \tag{6}$$

has no solution on $\partial\Omega_r$.

(ii) The function $H(a)$ defined by

$$H(a) := \frac{1}{T}\int_0^T h(t, a, 0)dt$$

satisfies $H(-r)H(r) < 0$.

Then the problem (5) has at least one solution in Ω_r.

Proof of Theorem 1. Let $h(t, x, x') = e(t) - \alpha(t)x' - g(t, x)$. Then (6) reduces to

$$(\phi_p(x'))' + \lambda\alpha(t)x' + \lambda g(t, x) = \lambda e(t), \quad \lambda \in (0, 1). \tag{7}$$

We first show that the set of all possible T-periodic solutions of (7) is bounded. Let $x(t)$ be an arbitrary T-periodic solution of (7). Integrating both sides of (7) from $t = 0$ to $t = T$, and using (2) and integration by parts, we obtain

$$\int_0^T [g(t, x(t)) - \alpha'(t)x(t)]dt = 0.$$

Therefore, there exists $s \in [0,\, T]$ such that $g(s, x(s)) - \alpha'(s)x(s) = 0$, which implies that $x(s)g(s, x(s)) = \alpha'(s)x^2(s)$. Since $\alpha'(s) \geq -1$, we obtain $x(s)g(s, x(s)) \geq -x^2(s)$. Now (H_1) implies that $|x(s)| < d$. Then for $t \in [0,\, T]$, we have

$$|x(t)| = \left| x(s) + \int_s^t x'(\tau)d\tau \right| \leq d + \int_0^T |x'(t)|dt.$$

Thus we obtain

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \leq d + |x'|_1, \tag{8}$$

where $|x'|_1 := \int_0^T |x'(t)|dt$.

To show that $|x'|$ is bounded for all $x \in C_T^1[0, T]$, let $I_1 = \{t \in [0,\, T] : |x(t)| \leq d\}$ and $I_2 = \{t \in [0,\, T] : |x(t)| > d\}$. Multiplying both sides of (7) by $x(t)$ and integrating from $t = 0$ to $t = T$, and noting that $g(t, x(t))x(t) - \frac{1}{2}\alpha'(t)x^2(t) \leq -\frac{1}{2}x^2(t) < 0$ for $t \in I_2$ from (H_1) and (H_2), we obtain

$$\begin{aligned}
\textstyle\int_0^T |x'(t)|^p dt &= -\textstyle\int_0^T (\phi_p(x'(t)))'x(t)dt \\
&= \lambda \textstyle\int_0^T \left[g(t, x(t))x(t) - \frac{1}{2}\alpha'(t)x^2(t)\right] dt - \lambda \int_0^T e(t)x(t)dt \\
&= \lambda \textstyle\int_{I_1} \left[g(t, x(t))x(t) - \frac{1}{2}\alpha'(t)x(t)\right] dt \\
&\quad + \lambda \textstyle\int_{I_2} \left[g(t, x(t))x(t) - \frac{1}{2}\alpha'(t)x^2(t)\right] dt - \lambda \int_0^T e(t)x(t)dt \\
&\leq \lambda \textstyle\int_{I_1} \left[g(t, x(t))x(t) - \frac{1}{2}\alpha'(t)x^2(t)\right] dt - \lambda \int_0^T e(t)x(t)dt \\
&\leq \textstyle\int_{I_1} \left[|g(t, x(t))x(t)| + \frac{1}{2}|\alpha'(t)|x^2(t)\right] dt + \int_0^T |e(t)x(t)|dt \\
&\leq G_d T d + \tfrac{1}{2}|\alpha'|_\infty T d^2 + |e|_\infty |x|_1 \\
&:= M_1 + |e|_\infty |x|_1,
\end{aligned}$$

where $G_d := \max_{t\in[0,T],|x|\leq d} |g(t, x)|$ and $M_1 := G_d T d + \frac{1}{2}|\alpha'|_\infty T d^2$. Hence we have

$$|x'|_p^p \leq M_1 + |e|_\infty |x|_1, \tag{9}$$

where $|x'|_p := \left(\int_0^T |x'(t)|^p dt\right)^{1/p}$. But from (8), we have

$$|x|_1 = \int_0^T |x(t)|dt \le \int_0^T \left[d + |x'|_1\right] dt = dT + T|x'|_1. \tag{10}$$

Substituting (10) into (9), we obtain

$$|x'|_p^p \le M_1 + |e|_\infty dT + |e|_\infty T|x'|_1. \tag{11}$$

By Hölder's inequality,

$$|x'|_1 = \int_0^T |x'(t)|dt \le \left(\int_0^T |x'(t)|^p dt\right)^{\frac{1}{p}} \left(\int_0^T 1^q dt\right)^{\frac{1}{q}} = T^{\frac{1}{q}}|x'|_p, \tag{12}$$

where $q = \frac{p}{p-1} > 1$ is the exponent conjugate to p. Substituting (11) into (12), we obtain

$$|x'|_1^p \le T^{p/q}M_1 + |e|_\infty dT^p + |e|_\infty T^p|x'|_1. \tag{13}$$

Since $p > 1$, we see from (13) that there exists a positive constant M_2 such that $|x'|_1 \le M_2$. This, together with (8), implies that $|x|_\infty \le M_3$, where $M_3 := d + M_2$.

Next we show that $|x'(t)|$ is bounded. Since $x(0) = x(T)$, there exists $t_1 \in (0, T)$, such that $x'(t_1) = 0$. It follows from (7) that for $t \in [0, T]$,

$$\begin{aligned} |\phi_p(x'(t))| &= \left|\int_{t_1}^t (\phi_p(x'(s))'ds\right| \\ &= \lambda\left|\int_{t_1}^t \left[\alpha(s)x'(s) + g(s, x(s)) - e(s)\right] ds\right| \\ &\le \int_0^T |\alpha(t)x'(t)|dt + \int_0^T |g(t, x(t))|dt + \int_0^T |e(t)|dt \\ &\le |\alpha|_\infty|x'|_1 + G_M T + |e|_\infty T \\ &\le |\alpha|_\infty M_2 + G_M T + |e|_\infty T, \end{aligned}$$

where $G_M = \max\{|g(t, x)| : t \in [0, T], |x| \le M_3\}$.

Since $|\phi_p(x'(t))| = |x'(t)|^{p-1}$, letting $M_4 := [|\alpha|_\infty M_2 + G_M T + |e|_\infty T]^{1/(p-1)}$, then we have

$$|x'|_\infty = \max_{t\in[0,T]} |x'(t)| \le M_4.$$

Finally let $M = M_3 + M_4 + 1$. Then we have $\|x\| = |x|_\infty + |x'|_\infty < M$. Thus we have shown that the set of all T-periodic solutions $x(t)$ of (7) is bounded, i.e., $\|x(t)\| < M$.

Now set $\Omega_M = \{x \in X : \|x\| = |x|_\infty + |x'|_\infty < M\}$. Then the equation (7) has no solution on $\partial\Omega_M$ for $\lambda \in (0, 1)$, which implies that the condition (i) of Lemma 1 is satisfied. Also, by the definition of $H(a)$, we see that

$$H(a) = \frac{1}{T}\int_0^T h(t,a,0)dt = \frac{1}{T}\int_0^T [e(t) - g(t,a)]\,dt = -\frac{1}{T}\int_0^T g(t,a)dt.$$

Moreover, for $x = \pm M \in \mathbb{R}$, we have $x \in \partial\Omega_M$ and since $M > d$, from the assumption (H_1), we see that $H(-M)H(M) < 0$. This implies that the condition (ii) of Lemma 1 is satisfied. Now Lemma 1 implies that problem (1)-(2) has at least one solution in Ω_M. □

Proof of Theorem 2. We need only to show that under the additional condition (B_1), the problem (1)-(2) has at most one solution.

Suppose on the contrary that (1)-(2) has two distinct solutions $x(t)$ and $y(t)$. Let $u(t) = x(t) - y(t)$. Since $u \in C^1_T[0,T]$, there exists a $t^* \in [0,\,T]$ such that $u(t^*) = \max_{t\in[0,T]} u(t)$. Suppose $u(t^*) > 0$. Then $u'(t^*) = x'(t^*) - y'(t^*) = 0$ and $u''(t^*) = x''(t^*) - y''(t^*) \le 0,\ a.e.$. Since $x(t)$ and $y(t)$ are solutions of (1) and (2), we get from (1) and the above equality that

$$0 = (p-1)\left[|x'(t^*)|^{p-2}u''(t^*)\right] + [g(t^*,x(t^*)) - g(t^*,y(t^*))] < 0, a.e. \tag{14}$$

because the first part of the right side of (14) is non-positive a.e. and the second part of the right side (14) is negative by (B_1). This contraction shows that $x(t) \le y(t)\ \forall\ t \in [0,\,T]$. Exchanging the role of x and y, we can show that $x(t) \ge y(t)\ \forall\, t \in [0,\,T]$. This shows that $x(t) \equiv y(t)$. Hence (1)-(2) has a unique solution. □

REFERENCES

1. A. Capietto & Z. Wang: Periodic solutions of Liénard equations with asymmetric nonlinearities at resonance. *J. London Math.* Soc. **68** (2003), no. 2, 119-132.
2. Y. Li & L. Huang: New results of periodic solutions for forced rayleigh-type equations. *J. Comput. Appl. Math.* **221** (2008), 98-105.
3. S. Lu & W. Ge: Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument. *Nonlinear analysis: TAM* **56** (2004), 501-514.
4. S. Lu & Z. Gui: On the existence of periodic solutions to p-Laplacian rayleigh differential equations with a delay. *J. Math. Anal. Appl.* **325** (2007), 685-702.
5. R. Manasevich & J. Mawhin: Periodic solutions for nonlinear systems with p-Laplacian-like operators. *J. Diff. Equations* **145** (1998), 367-393.
6. L. Wang & J. Shao: New results of periodic solutions for a kind of forced rayleigh-type equations. *Nonlinear Analysis : RWA* **11** (2010), 99-105.

7. Y. Wang: Novel existence and uniqueness criteria for periodic solutions of a Duffing type p-Laplacian equation. *Appl. Math. Lett.* **23** (2010), 436-439.
8. F. Zhang & Y. Li: Existence and uniqueness of periodic solutions for a kind of Duffing type $p-$-Laplacian equation. *Nonlinear Anal. RWA* **9** (2008), 985-989.
9. M. Zong & H. Liang: Periodic solutions for Rayleigh type p-Laplacian equation with deviating arguments. *Appl. Math. Lett.* **12** (1999), 41-44.
10. X. Yang, Y. Kim & K. Lo: Periodic solutions for a generalized p-Laplacian equation. *Appl. Math. Lett.* **25** (2011), 586-589.

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