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AN EXTENSION WHICH IS A WEAKLY LINDELÖFF SPACE

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Abstract. In this paper, we construct an extension (kX, k_X) of a space X such that kX is a weakly Lindelöff space and for any continuous map $f : X \longrightarrow Y$, there is a continuous map $g : kX \longrightarrow kY$ such that $g\mid_X = f$. Moreover, we show that υX is Lindelöff if and only if $kX = \upsilon X$ and that for any P'-space X which is weakly Lindelöff, $kX = \upsilon X$.

1. Introduction

All spaces in this paper are assumed to be Tychonoff spaces and $\beta X(\upsilon X$, resp.) denotes the Stone-Čech compactification(the Hewitt realcompactification, resp.) of a space X.

One of the many charaterizations of $(\beta X,\ \beta_X)$ is following :

(1) βX is a compact space, and

(2) for any continuous map $f : X \longrightarrow Y$, there is a continuous map $f^\beta : \beta X \longrightarrow \beta Y$ such that $f^\beta \mid_X = f$ ([5]).

There have been many ramifications from the Stone-Čech compactifications of spaces. In fact, realcompactifications of spaces and zero-dimesional compactifisations of zero-dimensional spaces have been studied by various authors ([3], [5]).

The purpose to write this paper is to construct an extension of a space which has similar properties to the above extensions. We first construct an extension (kX, k_X) of a space X such that $\upsilon X \subseteq kX \subseteq \beta X$ and kX is a weakly Lindelöff space. We show that for any continuous map $f : X \longrightarrow Y$, there is a continuous map $g : kX \longrightarrow kY$ such that $g\mid_X = f$. Blasco ([1], [2]) showed that for a paracompact (or separable) space X, υX is a Lindelöff space if and only if every separating nest generated intersection ring on X is complete. We show that υX is Lindelöff if and

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only if $kX = \upsilon X$. Using these, we then show that $kX = X$ if and only if X is Lindelöff. Finally, we will show that for any P'-space X which is weakly Lindelöff, $kX = \upsilon X$.

For the terminology, we refer to [3] and [5].

2. An Extnsion Which is a Weakly Lindelöff Space

For any space X, let $Z(X)$ be the set of all zero-sets in X. A $Z(X)$-filter is called *a z-filter on X*.

Definition 2.1. Let X be a space and $\mathcal{F}$ a z-filter on X. Then $\mathcal{F}$ is called

(1) *real* if it has the countable intersection property, and

(2) *free(fixed, resp.)* if $\cap\{F \mid F \in \mathcal{F}\} = \emptyset(\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$, resp.).

A space X is called *a realcompact space* if every real z-ultrafilter on X is fixed. It is known that for any real z-ultrafilter $\mathcal{F}$ on a space X, $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} \neq \emptyset$ ([3]).

Let X be a space and $kX = \upsilon X \cup \{p \in \beta X - \upsilon X \mid$ there is a real z-filter $\mathcal{F}$ on X such that $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}\}$.

Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. For any $A \subseteq X$, let $\mathcal{F}_A$ denote the set $\{F \cap A \mid F \in \mathcal{F}\}$.

Proposition 2.2. *Let X be a space. Then we have the following :*

(1) $\upsilon X \subseteq kX \subseteq \beta X$,

(2) $k(\upsilon X) = kX$, *and*

(3) kX *is realcompact if for any non-empty zero-set* Z *in* kX, $Z \cap X \neq \emptyset$.

Proof. (1) It is trivial.

(2) Let $p \in kX - \upsilon X$. Then there is a real z-filter $\mathcal{F}$ on X such that $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}$. Let $\mathcal{F}_\upsilon = \{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\}$. Note that for any zero-set Z in X, $cl_{\upsilon X}(Z)$ is a zero-set in υX and for any sequence (Z_n) in $Z(X)$, $cl_{\upsilon X}(\cap\{Z_n \mid n \in N\}) = \cap\{cl_{\upsilon X}(Z_n) \mid n \in N\}$([3]). Hence $\mathcal{F}_\upsilon$ is a real z-filter $\mathcal{F}$ on υX. Note that $\cap\{cl_{\upsilon X}(H) \mid H \in \mathcal{F}_\upsilon\} = \{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(H) \mid H \in \mathcal{F}_\upsilon\} = \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}$. Since $\upsilon(\upsilon X) = \upsilon X$ and $\beta(\upsilon X) = \beta X$, $p \in k(\upsilon X)$. Hence $kX \subseteq k(\upsilon X)$.

Let $q \in k(\upsilon X)$ and $q \notin \upsilon X$. Since $\upsilon(\upsilon X) = \upsilon X$, there is a real z-filter $\mathcal{G}$ on υX such that $\cap\{G \mid G \in \mathcal{G}\} = \emptyset$ and $q \in \cap\{cl_{\beta X}(G) \mid G \in \mathcal{G}\}$. Then $\mathcal{G}_X$ is a real z-filter on X and $\cap\{cl_{\upsilon X}(H) \mid H \in \mathcal{G}_X\} = \cap\{G \mid G \in \mathcal{G}\} = \emptyset$. Since $q \in \cap\{cl_{\beta X}(H) \mid H \in \mathcal{G}_X\} = \cap\{cl_{\beta X}(G) \mid G \in \mathcal{G}\}$, $q \in kX$. Hence $k(\upsilon X) \subseteq kX$.

(3) Take any real z-ultrafilter $\mathcal{F}$ on kX. By the assumption, for any $F \in \mathcal{F}$,

$F \cap X \neq \emptyset$ and so $\mathcal{F}_X$ is a z-filter on X. Let Z be a zero-set in X such that for any $F \in \mathcal{F}$, $Z \cap F \neq \emptyset$. Since $X \subseteq kX \subseteq \beta X$, there is a zero-set B in kX such that $Z = B \cap X$. Then for any $F \in \mathcal{F}$, $F \cap B \neq \emptyset$. Since $\mathcal{F}$ is a z-ultrafilter on kX, $B \in \mathcal{F}$ and $B \cap X = Z \in \mathcal{F}_X$. Hence $\mathcal{F}_X$ is a z-ultrafilter on X. Since $\mathcal{F}_X$ is real, $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} = \{q\}$ for some $q \in \upsilon X$. Note that $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} = \cap\{cl_{\upsilon X}(F \cap \upsilon X) \mid F \in \mathcal{F}\}$ and for any $F \in \mathcal{F}$, $cl_{\upsilon X}(F \cap \upsilon X) \subseteq F$. Hence $q \in \cap\{F \mid F \in \mathcal{F}\}$ and so $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$. Thus kX is a realcompact space. □

Let S be a subspace of a space X. Then S is called *$C(C^*$, resp.)-embedded in* X if for any real-valued (bounded, resp.) continuous function f on S, there is a real-valued (bounded, resp.) continuous function g on X such that $g|_S = f$.

Note that X is a dense C-embedded subspace of Y if and only if $X \subseteq Y \subseteq \upsilon X$, equivalently, $\upsilon X = \upsilon Y$ and that a dense subspace X of a space Y is C^*-embedded in Y if and only if $\beta X = \beta Y$ ([3]). Using these, we have the following :

Proposition 2.3. *Let X be a dense C-embedded subspace of Y. Then $kX = kY$.*

Proof. Since X is a dense C-embedded subspace of Y, $\upsilon X = \upsilon Y$([3]). Let $p \in kX - \upsilon X$. Then there is a real z-filter $\mathcal{F}$ on X such that $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}$. Let $\mathcal{G} = \{G \in Z(Y) \mid G \cap X \in \mathcal{F}\}$. Then $\mathcal{G}_X = \mathcal{F}$ and since $\upsilon X = \upsilon Y$, $\mathcal{G}$ is a real z-filter on Y.

Let $G \in \mathcal{G}$ and $x \in \upsilon X - cl_{\upsilon X}(G \cap X)$. Then there is a zero-set neighborhood Z of x in υX such that $G \cap X \cap Z = \emptyset$. Since $X \subseteq Y \subseteq \upsilon X$, there is a zero-set H in υX such that $G = H \cap Y$. Since $H \cap Z \cap X = \emptyset$ and $H \cap Z$ is a zero-set in υX, $H \cap Z = \emptyset$([5]). Hence $G \cap Z = \emptyset$ and $x \notin cl_{\upsilon X}(G)$. Thus $cl_{\upsilon X}(G) \subseteq cl_{\upsilon X}(G \cap X)$. Clearly, $cl_{\upsilon X}(G \cap X) \subseteq cl_{\upsilon X}(G)$ and so $cl_{\upsilon X}(G \cap X) = cl_{\upsilon X}(G)$.

Since $\cap\{cl_{\upsilon X}(G \cap X) \mid G \in \mathcal{G}\} = \emptyset$, $\cap\{cl_{\upsilon Y}(G) \mid G \in \mathcal{G}\} = \emptyset$. Since X is C^*-embedded in Y, $\beta X = \beta Y$ and $p \in \cap\{cl_{\beta Y}(G) \mid G \in \mathcal{G}\}$. Hence $p \in kY$ and so $kX \subseteq kY$.

Similarly, we have $kY \subseteq kX$. □

For any space X, let $k_X : X \longrightarrow kX$ denote the inclusion map. Then (kX, k_X) is an extension of X.

Note that for any continuous map $f : X \longrightarrow Y$, there is a unique continuous map $f^\upsilon : \upsilon X \longrightarrow \upsilon Y$ such that $f^\upsilon|_X = f$.

Proposition 2.4. *Let $f : X \longrightarrow Y$ be a continuous map. Then there is a unique continuous map $g : kX \longrightarrow kY$ such that $g \circ k_X = k_Y \circ f$.*

Proof. Note that there is a continuous map $h : \beta X \longrightarrow \beta Y$ such that $h \circ \beta_X = \beta_Y \circ f$ and $h(\upsilon X) \subseteq \upsilon Y$. Let $p \in kX - \upsilon X$. Then there is a real z-filter $\mathcal{F}$ on X such that $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(F) \mid F \in \mathcal{F}\}$. Let $\mathcal{G} = \{Z \in Z(Y) \mid h^{-1}(Z) \in \mathcal{F}\}$. Since $\mathcal{F}$ is a real z-filter on X, $\mathcal{G}$ is a real z-filter on Y. Let $G \in \mathcal{G}$. Then $h^{-1}(G) \in \mathcal{F}$. Since $p \in cl_{\beta X}(h^{-1}(G))$, $h(p) \in h(cl_{\beta X}(h^{-1}(G))) \subseteq cl_{\beta Y}(h(h^{-1}(G))) \subseteq cl_{\beta Y}(G)$. Hence $h(p) \in \cap\{cl_{\beta X}(G) \mid G \in \mathcal{G}\}$ and so $h(p) \in kY$. Let $g : kX \longrightarrow kY$ be the restriction and corestriction of h with respect to kX and kY, respectively. Then $g : kX \longrightarrow kY$ is a continuous map and $g \circ k_X = k_Y \circ f$. Since $k_X : X \longrightarrow kX$ is a dense embedding, such an g is unique. □

It is well-known that a space X is Lindelöff if and only if for any real z-filter $\mathcal{F}$ in X, $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$.

Proposition 2.5. *Let X be a space. Then the following are equivalent :*

(1) $\upsilon X = kX$,
(2) *υX is a Lindelöff space,*
(3) *for any free real z-filter $\mathcal{F}$ on X, $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} \neq \emptyset$, and*
(4) *for any free real z-filter $\mathcal{F}$ on X, there is a free real z-ultrafilter $\mathcal{A}$ on X such that $\mathcal{F} \subseteq \mathcal{A}$.*

Proof. (1) ⇒ (2) Take any real z-filter $\mathcal{G}$ on υX. Then $\mathcal{G}_X$ is a real z-filter on X. Suppose that $\cap\{G \cap X \mid G \in \mathcal{G}\} = \emptyset$. Then $\cap\{cl_{\beta X}(G \cap X) \mid G \in \mathcal{G}\} \neq \emptyset$. Pick $p \in \cap\{cl_{\beta X}(G \cap X) \mid G \in \mathcal{G}\}$. Then $p \in kX$ and since $kX = \upsilon X$, $p \in \upsilon X$. Hence $p \in (\cap\{cl_{\beta X}(G) \mid G \in \mathcal{G}\}) \cap \upsilon X = \cap\{G \mid G \in \mathcal{G}\}$ and so $\cap\{G \mid G \in \mathcal{G}\} \neq \emptyset$. Thus υX is a Lindelöff space.

(2) ⇒ (3) It is trivial.

(3) ⇒ (4) Let $\mathcal{F}$ be a free real z-filter on X. By the assumption, $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} \neq \emptyset$. Pick $p \in \cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\}$. Let $\mathcal{A}^p = \{A \in Z(X) \mid p \in cl_{\upsilon X}(A)\}$. Then $\mathcal{A}^p$ is a free real z-ultrafilter on X and $\mathcal{F} \subseteq \mathcal{A}^p$.

(4) ⇒ (1) Let $p \in kX - \upsilon X$. Then there is a real z-filter $\mathcal{F}$ on X such that $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{cl_{\beta X}(G \cap X) \mid G \in \mathcal{G}\}$. Since $\mathcal{F}$ is free, by (4), there is a free real z-ultrafilter $\mathcal{A}$ on X such that $\mathcal{F} \subseteq \mathcal{A}$. Since $\cap\{cl_{\upsilon X}(A) \mid A \in \mathcal{A}\} \neq \emptyset$, $\cap\{cl_{\upsilon X}(F) \mid F \in \mathcal{F}\} \neq \emptyset$ and this is a contradiction. □

By Proposition 2.2. and Proposition 2.5., we have the following :

Corollary 2.6. *Let X be a sapce. Then $kX = X$ if and only if X is Lindelöff.*

Recall that a space X is called *a pseudo-compact space* if every real-valued continuous function on X is bounded, equivalently, $\upsilon X = \beta X$.

Corollary 2.7. *If X is a pseudo-compact space, then $kX = \beta X$.*

Let X be a space. The collection $\mathcal{R}(X)$ of all regular closed sets in X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows : For any $A \in \mathcal{R}(X)$ and any $\mathcal{F} \subseteq \mathcal{R}(X)$,

$\bigvee \mathcal{F} = cl_X(\cup\{F \mid F \in \mathcal{F}\})$,

$\bigwedge \mathcal{F} = cl_X(int_X(\cap\{F \mid F \in \mathcal{F}\}))$, and

$A' = cl_X(X - A)$.

A sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and finite meets ([7]).

An $\mathcal{R}(X)$-filter $\mathcal{A}$ is said to have *the countable meet property* if for any sequence (A_n) in $\mathcal{R}(X)$, $\bigwedge\{A_n \mid n \in N\} \neq \emptyset$.

Let $Z(X)^{\#} = \{cl_X(int_X(A)) \mid A \in Z(X)\}$. Then $Z(X)^{\#}$ is a sublattice of $R(X)$.

A space X is called *a weakly Lindelöff space* if for any open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is dense in X.

A space X is a weakly Lindelöff space if and only if for any $Z(X)^{\#}$-filter $\mathcal{A}$ with the countable meet property, $\cap\{A \mid A \in \mathcal{A}\} \neq \emptyset$ ([4]).

Theorem 2.8. *Let X be a space. Then kX is a weakly Lindelöff space.*

Proof. Take any $Z(X)^{\#}$-filter $\mathcal{U}$ on kX with the countable meet property. Let $\mathcal{F} = \{Z \in Z(kX) \mid cl_{kX}(int_{kX}(Z)) \in \mathcal{U}\}$. Clearly, $\emptyset \notin \mathcal{F} \neq \emptyset$. For any $A, B \in \mathcal{F}$, $cl_{kX}(int_{kX}(A \cap B)) = cl_{kX}(int_{kX}(A)) \wedge cl_{kX}(int_{kX}(B)) \in \mathcal{U}$ and hence $A \cap B \in \mathcal{F}$. Thus $\mathcal{F}$ is a z-filter on kX. By the definition of $\mathcal{F}$, for any $F \in \mathcal{F}$, $F \cap X \neq \emptyset$. Hence $\mathcal{F}_X$ is also a z-filter on X. Let (A_n) be a sequence in $\mathcal{F}_X$. For any $n \in N$, there is a $B_n \in \mathcal{F}$ such that $A_n = B_n \cap X$. Since $\mathcal{U}$ has the countable meet property, $cl_{kX}(int_{kX}(\cap\{B_n \mid n \in N\})) \neq \emptyset$ and since X is dense in kX, $cl_{kX}(int_{kX}(\cap\{B_n \mid n \in N\})) \cap X \neq \emptyset$. Since $cl_{kX}(int_{kX}(\cap\{B_n \mid n \in N\})) \cap X$

$=cl_{kX}(int_{kX}(\cap\{B_n \cap X \mid n \in N\}))$

$=cl_{kX}(int_{kX}(\cap\{A_n \mid n \in N\}))$,

$\cap\{A_n \mid n \in N\} \neq \emptyset$ and so $\mathcal{F}_X$ has the countable intersection property. Note that $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} \neq \emptyset$ or $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} = \emptyset$.

Assume that $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} \neq \emptyset$. Pick $x \in \cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\}$. Let $U \in \mathcal{U}$. Suppose that $x \notin U$. Since U is a closed set in kX, there is a zero-set Z in kX such that $x \notin Z$ and $U \subseteq Z$. Then $Z \cap X \in \mathcal{F}_X$ and since $cl_{\upsilon X}(Z \cap X) = Z \cap \upsilon X$, since $cl_{\upsilon X}(Z \cap X) = Z \cap \upsilon X$, $x \in Z$. This is a contradiction and so $x \in U$. Hence $x \in \cap\{U \mid U \in \mathcal{U}\}$.

Assume that $\cap\{cl_{\upsilon X}(F \cap X) \mid F \in \mathcal{F}\} = \emptyset$. Let $p \in \cap\{cl_{\beta X}(F \cap X) \mid F \in \mathcal{F}\}$. Then $p \in kX$. Let $U \in \mathcal{U}$. Suppose that $p \notin U$. Then there is a zero-set B in βX such that $p \notin B$ and $U \subseteq B$. Since $B \cap X \in \mathcal{F}_X$, $p \in cl_{\beta X}(B \cap X) \subseteq B$. This is a contradiction and so $p \in U$. Hence $p \in \cap\{U \mid U \in \mathcal{U}\}$.

Thus $\cap\{U \mid U \in \mathcal{U}\} \neq \emptyset$ and kX is a weakly Lindelöff space. □

A space X is called a P'-space if for any non-empty zero-set Z in X, $int_X(Z) \neq \emptyset$, equivalently, every zero-set in X is a reqular closed set in X. Clearly, a space X is a P'-space if and only if υX is a P'-space. If X is a realcompact and locally compact space, then $\beta X - X$ is a P'-space ([6]).

Proposition 2.9. *Let X be a P'-space. Then X is a weakly Lindelöff space if and only if X is a Lindelöff space.*

Proof. Suppose that X is a weakly Lindelöff space. Let $\mathcal{F}$ be a real z-filter on X. Since X is a P'-space, $Z(X) = Z(X)^{\#}$ and since $Z(X)$ is closed under countable intersections, for any sequence (A_n) in $Z(X)$,

$\bigwedge\{A_n \mid n \in N\} = cl_X(int_X(\cap\{A_n \mid n \in N\})) = \cap\{A_n \mid n \in N\}$.

Hence $\mathcal{F}$ is a $Z(X)^{\#}$-filter with the countable meet property. Since X is a weakly Lindelöff space, $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$ and hence X is a Lindelöff space.

The converse is trivial. □

A space with a dense weakly Lindelöff space is also a weakly Lindelöff space. Using this, Proposition 2.9. and Proposition 2.5., we have the following :

Corollary 2.10. *For any P'-space X which is weakly Lindelöff, υX is a Lindelöff space and $\upsilon X = kX$.*

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