

## ESTIMATIONS FOR THE ORDER OF SOLUTIONS OF LINEAR COMPLEX DIFFERENTIAL EQUATIONS

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ABSTRACT. We research the properties of solutions of general higher order non-homogeneous linear differential equations and apply the hyper order to obtain more precise estimation for the growth of solutions of infinite order.

### 1. INTRODUCTION AND RESULTS

The growth order of solutions of the complex differential equations has been an important problem. For the growth of solutions of the complex linear differential equations in  $\mathbb{C}$ , one obtained many precise estimations of the order of solutions. We see that the Wiman-Valiron theory was shown to be a powerful tool to estimation of the order of solutions of linear differential equations in  $\mathbb{C}$ . But, since it does not hold in the unit disc  $\Delta$ , few results of precise estimation of the order of solutions are obtained for the linear differential equations in  $\Delta$ , up to now. The estimation to the hyper order of solutions is still a blank so far. One applied many other method and obtained some upper bounds or lower bounds for the growth of solutions of linear differential equations in  $\Delta$ . We [1, 2] researched the properties of the growth of solutions of a class of higher order differential equations and the growth of solutions of differential equations with coefficients of small growth in  $\Delta$ . Also we [3] obtained some results on the growth and fixed points of solutions of second order differential equations with meromorphic coefficients.

In this paper, we obtain some precise estimations of the order and the hyper order of solutions for some linear differential equations in  $\Delta$ .

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We use the standard notations of the Nevanlinna's value distribution theory of meromorphic functions in  $\mathbb{C}$  and in  $\Delta$ . (e.g., see [7, 11]). In addition, the order of a meromorphic function  $f$  in  $\Delta$  is defined by

$$\sigma(f) = \limsup_{r \rightarrow 1^-} \frac{\log^+ T(r, f)}{\log \frac{1}{1-r}},$$

and we also define

$$\sigma_M(f) = \limsup_{r \rightarrow 1^-} \frac{\log^+ \log^+ M(r, f)}{\log \frac{1}{1-r}}$$

for an analytic function  $f$  in  $\Delta$ . M. Tsuji [8, p.205] gives that

$$\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1,$$

and, as an example, a function

$$\psi(z) = \exp \left\{ \frac{1}{(1-z)^\mu} \right\}$$

satisfies that  $\sigma(\psi) = \mu - 1$  and  $\sigma_M(\psi) = \mu$ .

And we also define the hyper-order of a meromorphic function  $f$  in  $\Delta$  similarly to the plane case

$$\sigma_2(f) = \limsup_{r \rightarrow 1^-} \frac{\log^+ \log^+ T(r, f)}{\log \frac{1}{1-r}}.$$

If  $f$  is an analytic function in  $\Delta$ , we also define

$$\sigma_{M2}(f) = \limsup_{r \rightarrow 1^-} \frac{\log^+ \log^+ \log^+ M(r, f)}{\log \frac{1}{1-r}}.$$

We can easily obtain that

$$\sigma_{M2}(f) = \sigma_2(f).$$

## 2. DEFINITIONS AND LEMMA

**Definition 1.** A meromorphic function  $f$  in  $\Delta$  is called *admissible*, if

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = \infty.$$

And  $f$  is called *non-admissible*, if

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} < \infty.$$

**Definition 2.** Let  $f$  be analytic in  $\Delta$  and let  $q \in [0, \infty)$ . Then  $f$  is said to *belong to the weighted Hardy space  $H_q^\infty$*  provided that

$$\sup_{z \in \Delta} (1 - |z|^2)^q |f(z)| < \infty.$$

We say that  $f$  is an  $\mathcal{H}$ -function when  $f \in H_q^\infty$  for some  $q$ . Heittokangas [8] proved the following theorem in  $\Delta$ .

**Theorem 1** ([8]). *Let  $A_0(z), \dots, A_{k-1}(z)$  be the sequence of analytic coefficients of the equation*

$$(2.1) \quad f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_d(z)f^{(d)} + \dots + A_0(z)f = 0$$

*in  $\Delta$ . Let  $A_d(z)$  be the last coefficient not being an  $\mathcal{H}$ -function while the coefficients  $A_{d+1}(z), \dots, A_{k-1}(z)$  are  $\mathcal{H}$ -functions. Then (2.1) possesses at most  $d$  linearly independent analytic solutions of finite order of growth in  $\Delta$ .*

We [5] researched the properties of the hyper order of solutions of second order differential equations and subnormal solutions of periodic equations. Hence we have a problem : How do more precisely estimate the growth of these  $k - j$  linearly independent solutions of infinite order?

We use the concept of the hyper order and new methods to answer this problem. We [4] know the following theorem 2.

**Theorem 2** ([4]). *Let  $A_j$  ( $j = 0, \dots, k - 1$ ) be analytic in  $\Delta$ . Suppose that there exists some  $d \in \{0, \dots, k - 1\}$  such that  $A_d$  is admissible and  $\sigma(A_d) = \sigma_M(A_d) = \mu$ , while  $\sigma_M(A_j) < \mu$  for  $j = d + 1, \dots, k - 1$ ; (or  $\mu = 0$ , while  $A_j$  are  $\mathcal{H}$ -functions for  $j = d + 1, \dots, k - 1$ ;) and  $\sigma_M(A_s) \leq \mu$  for  $s = 0, \dots, d - 1$ . Then the equation (2.1) possesses at least  $k - d$  linearly independent analytic solutions of the hyper-order  $\sigma_2(f) = \mu$  and the order  $\sigma(f) = \infty$  (at most  $d$  linearly independent analytic solutions of the hyper-order  $\sigma_2(f) < \mu$ ).*

By Theorems 1 and 2, we obtain the following proposition.

**Proposition 3.** *Let  $A_j$  ( $j = 0, \dots, k - 1$ ) be analytic in  $\Delta$ . Suppose that there exists some  $d \in \{0, \dots, k - 1\}$  such that  $A_d$  is admissible and  $\sigma(A_d) = \sigma_M(A_d) = \mu$ ,  $\sigma_M(A_s) \leq \mu$  for  $s = 0, \dots, d - 1$ . Then the following properties hold:*

(1) *If  $A_j$  are all  $\mathcal{H}$ -function for  $j = d + 1, \dots, k - 1$ , then the equation (2.1) possesses at least  $k - d$  linearly independent analytic solutions of  $\sigma_2(f) = \mu$  and  $\sigma(f) = \infty$ .*

(2) *If  $\sigma_M(A_j) < \mu$  for  $j = d + 1, \dots, k - 1$  and there is  $A_\delta$  ( $\delta \in \{d + 1, \dots, k - 1\}$ ) being the last coefficient being admissible, while the coefficients  $A_{\delta+1}, \dots, A_{k-1}$  are  $\mathcal{H}$ -functions, then the equation (2.1) possesses at least  $k - d$  linearly independent*

analytic solutions of  $\sigma_2(f) = \mu$ , and at least  $k - \delta$  linearly independent analytic solutions of  $\sigma(f) = \infty$ .

*Proof.* By Theorem 1, we see that the equation (2.1) in  $\Delta$  possesses at least  $k - j$  linearly independent solutions of  $\sigma(f) = \infty$ . By Theorem 2, we have the results.  $\square$

For homogeneous linear differential equations in  $\mathbb{C}$ , if the coefficient of  $f(z)$  dominates the growth of solutions of the equation, one obtained many precise estimations of the order of solutions. For homogeneous linear differential equations in  $\Delta$ , Pommerenke [9] proved the following theorem.

**Theorem 4** ([9]). *Let  $q(z)$  be analytic in  $\Delta$ . If*

$$(2.2) \quad \int \int_{\Delta} \sqrt{|q(z)|} d\Omega < \infty \quad (d\Omega \equiv dx dy),$$

*then every solution  $w(z)$  of the equation*

$$(2.3) \quad w''(z) + q(z)w(z) = 0$$

*is of bounded characteristic and*

$$(2.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{it})| dt \leq \log^+ (|w(0)| + |w'(0)|) + K \int \int_{\Delta} \sqrt{|q(z)|} d\Omega$$

*for  $0 \leq r < 1$  where  $K$  is an absolute constant.*

Heittokangas [8] obtained the following two results.

**Theorem 5** ([8]). *Let  $B(z)$  and  $C(z)$  be the analytic coefficients of the equation*

$$(2.5) \quad f'' + B(z)f' + C(z)f = 0$$

*in  $\Delta$ . If either  $\sigma(B) < \sigma(C)$  or  $B(z)$  is non-admissible while  $C(z)$  is admissible, then all solutions  $f(\neq 0)$  of (2.5) are of infinite order of the growth.*

**Theorem 6** ([8]). *Let  $A(z)$  be an admissible analytic coefficients of the equation*

$$(2.6) \quad f^{(k)} + A(z)f = 0$$

*in  $\Delta$ . Then all solutions  $f(\neq 0)$  of (2.6) are of infinite order of the growth.*

In this paper, we consider general higher order homogeneous linear differential equations. For a non-homogeneous linear equation

$$(2.7) \quad f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z),$$

Heittokangas [8] obtained the following estimation.

**Theorem 7** ([8]). *If the coefficients  $A_0(z), \dots, A_{k-1}(z)$  of (2.7) are  $\mathcal{H}$ -functions and if  $F(z)$  is analytic and finite order of growth in  $\Delta$ , then all solutions of (2.7) are analytic and of finite order of the growth in  $\Delta$ .*

It is natural to ask a question: How much are the order of all solutions in Theorem 7? In the following theorem 8, we obtain a precise estimation for the order of all solutions in Theorem 7.

**Theorem 8.** *Let  $A_j$  ( $j = 0, \dots, k-1$ ) and  $F$  be analytic in  $\Delta$ ,  $A_j \in H_\beta^\infty$ ,  $F$  be admissible. Suppose that  $\beta-1 \leq \sigma(F) < \infty$  if  $\beta > 1$ ; or  $0 \leq \sigma(F) < \infty$  if  $0 \leq \beta \leq 1$ . Then all solutions  $f$  of the equation (2.7) are admissible and  $\sigma(f) = \sigma(F)$ .*

For the proof of Theorem 8, we need the following lemma.

**Lemma 1** ([5]). *Let  $A_0, A_1, \dots, A_{k-1}$  be the analytic coefficients of (1.7) in  $\Delta$ . Assume that  $|A_j(z)| \leq \frac{\alpha(j)}{(1-|z|)^{\beta(j)}}$ , where  $\alpha(j) > 0$  and  $\beta(j) \geq 0$ . Denote  $\alpha = \max_{0 \leq j \leq k-1} \{\alpha(j)\}$  and  $\beta = \max_{0 \leq j \leq k-1} \{\beta(j)\}$ . Then we have*

- (1)  $f \in H_0^\infty$ , if  $0 \leq \beta < 1$ ,
- (2)  $f \in H_{\sqrt{1+k\alpha^2}}^\infty$ , if  $\beta = 1$ ,
- (3)  $|f(z)| \leq \exp \left\{ \frac{\alpha}{(1-|z|)^{\beta-1}} \right\}$ , if  $1 < \beta < \infty$ .

### 3. PROOF OF THEOREM 8

Suppose that  $f$  is a solution of the equation (2.7). Comparing the growth of both sides of the equation (2.7), we can know that  $f$  is admissible and satisfies  $\sigma(f) \geq \sigma(F)$ . Now we prove that  $\sigma(f) \leq \sigma(F)$ .

Suppose that  $\{g_1, \dots, g_k\}$  is a fundamental solution of the corresponding homogeneous differential equation

$$(3.1) \quad g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_0g = 0$$

of (2.7). Since  $A_j \in H_\beta^\infty$  ( $j = 0, \dots, k-1$ ), we know that there is  $M$  ( $0 < M < \infty$ ) satisfying

$$(3.2) \quad \sup_{z \in \Delta} |A_j(z)| \leq \frac{M}{(1-|z|)^\beta} \quad (j = 0, \dots, k-1).$$

By (3.1), (3.2) and Lemma 1, we get that if  $0 \leq \beta \leq 1$ , then  $g_j \in H_{\sqrt{1+kM}}^\infty$ , i.e.,

$$(3.3) \quad |g_j(z)| \leq \left( \frac{1}{1-r} \right)^C \quad (j = 1, \dots, k),$$

where  $C(> 0)$  is some constant; if  $1 < \beta < \infty$ , then

$$(3.4) \quad |g_j(z)| \leq \exp \left\{ \frac{C_1}{(1-r)^{\beta-1}} \right\} \quad (j = 1, \dots, k),$$

where  $C_1(> 0)$  is some constant.

Now by variation of parameters, we can write

$$(3.5) \quad f(z) = a_1(z)g_1 + \dots + a_k(z)g_k,$$

where  $a_1(z), \dots, a_k(z)$  are determined by

$$\begin{cases} a'_1(z)g_1 + \dots + a'_k(z)g_k = 0 \\ a'_1(z)g'_1 + \dots + a'_k(z)g'_k = 0 \\ \dots \\ a'_1(z)g_1^{(k-1)} + \dots + a'_k(z)g_k^{(k-1)} = F(z). \end{cases}$$

Noting that the Wronskian  $W(z) = W(g_1, \dots, g_k)$  is a differential polynomial in  $g_1, \dots, g_k$  with constant coefficients, since  $\{g_1, \dots, g_k\}$  is a fundamental solution, we see that  $W(z) \neq 0$ . Set

$$W_j = \begin{vmatrix} g_1 & \dots & g_{j-1} & 0 & g_{j+1} & \dots & g_k \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ g_1^{(k-1)} & \dots & g_{j-1}^{(k-1)} & F & g_{j+1}^{(k-1)} & \dots & g_k^{(k-1)} \end{vmatrix} = FG_j \quad (j = 1, \dots, k),$$

where  $G_j$  are differential polynomials in  $g_1, \dots, g_k$  with constant coefficients.

If  $1 < \beta < \infty$ , then by (3.4), we have

$$(3.6) \quad \sigma(W) \leq \max\{\sigma(g_j) : j = 1, \dots, k\} \leq \beta - 1; \quad \sigma(G_j) \leq \beta - 1 \quad (j = 1, \dots, k).$$

By  $a'_j(z) = \frac{W_j(z)}{W(z)} = \frac{F(z)G_j(z)}{W(z)}$  and (3.6), we get that

$$(3.7) \quad \sigma(a_j) = \sigma(a'_j) \leq \sigma(F).$$

By (3.3), (3.5) and (3.7), we get  $\sigma(f) \leq \sigma(F)$ .

If  $0 \leq \beta \leq 1$ , then by (3.3), we have

$$(3.8) \quad |W(z)| \leq \left(\frac{1}{1-r}\right)^{C_2}, \quad |G_j(z)| \leq \left(\frac{1}{1-r}\right)^{C_2}, \quad (j = 1, \dots, k),$$

where  $C_2(> 0)$  is some constant. By  $a'_j(z) = \frac{W_j(z)}{W(z)} = \frac{F(z)G_j(z)}{W(z)}$  and (3.8), we get that (3.7). By (3.4), (3.5) and (3.7), we also get  $\sigma(f) \leq \sigma(F)$ . □

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