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FALLING SUBALGEBRAS AND IDEALS IN BH-ALGEBRAS

Eun Mi Kim [a] and Sun Shin Ahn [b,*]

Abstract. Based on the theory of a falling shadow which was first formulated by Wang([14]), a theoretical approach of the ideal structure in BH-algebras is established. The notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n-fold strong ideal and a falling translation ideal of a BH-algebra are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal and a falling strong ideal, a falling n-fold strong ideal are stated. A relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal is provided.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3,4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [8] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [9] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [5] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [11] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. S. S. Ahn and J. H. Lee ([2]) defined the notion of strong ideals in BH-algebra and studied some properties of it. They considered the notion of a rough set in BH-algebras. S. S. Ahn and E. M. Kim ([1]) introduced the notion of n-fold strong ideal in BH-algebra and gave some related properties of it.

In this paper we introduced the notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n-fold strong ideal and a falling translation ideal of a

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*Corresponding author.

BH-algebra. We investigate some fundamental properties. Also we give relations among a falling subalgebra, a falling ideal and a falling strong ideal, a falling n-fold strong ideal. We study a relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal.

2. Preliminaries

By a *BH-algebra* ([5]), we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following conditions:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call X a BH-algebra. In X we can define an order relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x, y \in S$, $x * y \in S$, i.e., S is a closed under binary operation.

Definition 2.1 ([5]). A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,

(I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x * y \in A$ and $y * x \in A$ imply $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and X are ideals of X. For any elements x and y of a BH-algebra X, $x * y^n$ denotes $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times.

Definition 2.2. A non-empty subset A of a BH-algebra X is called a *strong ideal* ([2]) of X if it satisfies (I1) and

(I4) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$ for all $x, y, z \in X$.

A non-empty subset A of a BH-algebra X is called an *n-fold strong ideal* ([1]) of X if it satisfies (I1) and

(I5) for every $x, y, z \in X$ there exists a natural number n such that $x * z^n \in A$ whenever $(x * y) * z^n \in A$ and $y \in A$.

Definition 2.3 ([11]). A BH-algebra X is called a *BH^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Definition 2.4. A BH-algebra $(X;*,0)$ is said to be *transitive* if $x*y=0$ and $y*z=0$ imply $x*z=0$ for all $x,y,z\in X$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $t\in[0,1]$, define $U(\mu;t)$ to be the set $U(\mu;t)=\{x\in X|\mu(x)\geq t\}$, which is called a *level subset* of μ.

Definition 2.5. A fuzzy set μ in a BH-algebra X is called a *fuzzy BH-ideal* (here call it a *fuzzy ideal*) ([6]) of X if

(FI1) $\mu(0)\geq\mu(x), \forall x\in X$,
(FI2) $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}, \forall x,y\in X$.

A fuzzy set μ in a BH-algebra X is called a *fuzzy translation BH-ideal*([6]) of X if it satisfies (FI1), (FI2) and

(FI3) $\min\{\mu((x*z)*(y*z)),\mu((z*x)*(z*y))\}\geq\min\{\mu(x*y),\mu(y*x)\}, \forall x,y,z\in X$.

A fuzzy set μ in a BH-algebra X is called a *fuzzy strong ideal*([7]) of X if it satisfies (FI1) and

(FI4) $\mu(x*z)\geq\min\{\mu((x*y)*z),\mu(y)\}, \forall x,y,z\in X$.

A fuzzy set μ in a BH-algebra X is called a *fuzzy n-fold strong ideal*([7]) of X if it satisfies (FI1) and

(FI5) $\mu(x*z^n)\geq\min\{\mu((x*y)*z^n),\mu(y)\}, \forall x,y,z\in X$.

We now display the basic theory on falling shadows. We refer the reader to the papers [12, 13, 14] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u\in U$, let

$$\dot{u} := \{E \mid u\in E \text{ and } E\subseteq U\}, \tag{2.1}$$

and for each $E\in\mathcal{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u\in E\}. \tag{2.2}$$

An ordered pair $(\mathcal{P}(U),\mathcal{B})$ is said to be a hyper-measurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}(U)$ and $\dot{U}\subseteq\mathcal{B}$. Given a probability space $(\Omega,\mathcal{A},P)$ and a hyper-measurable structure $(\mathcal{P}(U),\mathcal{B})$ on U, a random set on U is defined to be a mapping $\xi:\Omega\to\mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$(\forall C\in\mathcal{B})\,(\xi^{-1}(C)=\{\omega \mid \omega\in\Omega \text{ and } \xi(\omega)\in C\}\in\mathcal{A}). \tag{2.3}$$

Suppose that ξ is a random set on U. Let

$$\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\tilde{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then

$$\xi : [0,1] \to \mathcal{P}(U),\ t \mapsto \tilde{H}_t$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the *cut-cloud* of $\tilde{H}$.

3. Falling Subalgebras/Ideals in BH-algebras

In what follows let X denote a BH-algebra unless otherwise specified.

Definition 3.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi : \Omega \to \mathcal{P}(X),$$

be a random set. If $\xi(\omega)$ is a subalgebra(resp., ideal, strong ideal, n-fold strong ideal and translation ideal) of a BH-algebra X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set ξ, i.e.,

$$\tilde{H}(x) = P(\omega | x \in \xi(\omega))$$

is called a *falling subalgebra*(resp.*, falling ideal, falling strong ideal, falling n-fold ideal and falling translation ideal*) of X.

Example 3.2. (1) Let $X := \{0, 1, 2, 3\}$ be a BH-algebra([5]) with the following table:

$*$	0	1	2	3
0	0	1	3	0
1	1	0	2	0
2	2	2	0	3
3	3	3	3	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \emptyset & \text{if } t \in [0, 0.3), \\ \{0,1,2\} & \text{if } t \in [0.3, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then $\xi(t)$ is an ideal of X for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling ideal of X. If we take $t \in [0.3, 0.8)$, then $\xi(t) = \{0,1,2\}$ is neither a subalgebra nor a translation ideal of X since $0 * 2 = 3 \notin \{0,1,2\}$ and $1 * 2 = 2, 2 * 1 = 2 \in \{0,1,2\}$, $(1*1)*(2*1) = 0*2 = 3 \notin \{0,1,2\}$. Hence $\tilde{H}$ is neither a falling subalgebra nor a falling translation ideal of X.

(2) Let $X := \{0,1,2\}$ be a BH-algebra([5]) with the following table:

*	0	1	2
0	0	0	1
1	1	0	0
2	2	1	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \emptyset & \text{if } t \in [0, 0.4), \\ \{0,1\} & \text{if } t \in [0.4, 0.7), \\ X & \text{if } t \in [0.7, 1]. \end{cases}$$

Then $\xi(t)$ is a subalgebra of X for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling subalgebra of X. If we take $t \in [0.4, 0.7)$, then $\xi(t) = \{0,1\}$ is not an ideal of X since $2*1 = 1, 1 \in \{0,1\}$ and $2 \notin \{0,1\}$. Hence $\tilde{H}$ is not a falling ideal of X.

(3) Let $X := \{0,1,2,3\}$ be a BH-algebra([5]) with the following table:

*	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \emptyset & \text{if } t \in [0, 0.2), \\ \{0,1\} & \text{if } t \in [0.2, 0.7), \\ X & \text{if } t \in [0.7, 1]. \end{cases}$$

Then $\xi(t)$ is both a subalgebra and a translation ideal of X for all $t \in [0,1]$. Hence $\tilde{H}$ is both a falling subalgebra and a falling translation ideal of X.

Lemma 3.3 ([6, 7])**.** *A fuzzy set μ in a BH-algebra X is a fuzzy subalgebra(resp., fuzzy ideal, fuzzy strong ideal, fuzzy n-fold strong ideal, and fuzzy translation ideal) of X if and only if for every $t \in [0,1]$, μ_t is either empty or a subalgebra(resp., ideal, strong ideal, n-fold strong ideal, and translation ideal) of X.*

Theorem 3.4. *Let X be a BH-algebra. Then every fuzzy ideal(resp., fuzzy subalgebra, fuzzy strong ideal, fuzzy n-fold strong ideal, and fuzzy translation ideal) of X is a falling ideal(resp., falling subalgebra, falling strong ideal, falling n-fold strong ideal, and falling translation ideal) of X.*

Proof. Let $\tilde{H}$ be any fuzzy ideal(resp., fuzzy subalgebra, fuzzy strong ideal, fuzzy n-fold strong ideal, and fuzzy translation ideal) of X. By Lemma 3.3, $\tilde{H}_t$ is an ideal(resp., subalgebra, strong ideal, n-fold strong ideal, and translation ideal) of X for all $t \in [0,1]$. Let $\xi(t) : [0,1] \to \mathcal{P}(X)$ be a random set and $\xi(t) = \tilde{H}_t$. Then $\tilde{H}$ is a falling ideal(resp., falling subalgebra, falling strong ideal, falling n-fold strong ideal, and falling translation ideal) of X. □

The converse of Theorem 3.4 is not true in general as seen in general as seen in the following example.

Example 3.5. Let $X := \{0,1,2,3,4\}$ be a BH-algebra([2]) with the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0,0.2), \\ \{0,2\} & \text{if } t \in [0.2,0.5), \\ \{0,3,4\} & \text{if} t \in [0.5,0.8) \\ X & \text{if } t \in [0.8,1]. \end{cases}$$

Then $\xi(t)$ is a subalgebra of X for all $t \in [0,1]$ and

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.4 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2, \\ 0.5 & \text{if} x = 3, \\ 0.5 & \text{if} x = 4. \end{cases}$$

Hence $\tilde{H}$ is a falling subalgebra of X, but not a fuzzy subalgebra of X since $\tilde{H}(3*2) = \tilde{H}(1) = 0.4 \ngeq 0.5 = \min\{\tilde{H}(3), \tilde{H}(2)\}$.

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\eta : [0,1] \to \mathcal{P}(X)$ as follows:

$$\eta : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \{0\} & \text{if} t \in [0, 0.2) \\ \emptyset & \text{if} t \in [0.2, 0.3) \\ \{0,1\} & \text{if } t \in [0.3, 0.5), \\ \{0,2\} & \text{if } t \in [0.5, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then $\eta(t)$ is an ideal and a subalgebra of X for all $t \in [0,1]$ and

$$\tilde{H}(x) = \begin{cases} 0.9 & \text{if } x = 0, \\ 0.4 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2, \\ 0.2 & \text{if} x = 3, \\ 0.2 & \text{if} x = 4. \end{cases}$$

Hence $\tilde{H}$ is a falling ideal and a falling subalgebra of X, but not a fuzzy ideal of X since $\tilde{H}(3) = 0.2 \ngeq 0.4 = \min\{\tilde{H}(3 * 2), \tilde{H}(2)\}$.

Proposition 3.6. *In a BH^*-algebra X, every falling ideal of X is a falling subalgebra of X.*

Proof. Let $\tilde{H}$ be a falling ideal of a BH^*-algebra X. Then $\xi(\omega)$ is an ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y \in X$ be such that $x, y \in \xi(\omega)$. Since $(x*y)*x = 0$ for any $x, y \in X$, we have $(x*y)*x = 0 \in \xi(\omega)$. It follows from (I2) that $x*y \in \xi(\omega)$. Hence $\xi(\omega)$ is a subalgebra of X. Thus $\tilde{H}$ is a falling subalgebra of X. □

In a BH-algebra X, Proposition 3.6 is not true in general(see Example 3.2(1)).

Theorem 3.7. *In a BH-algebra, every falling n-fold strong ideal is a falling ideal.*

Proof. Let $\tilde{H}$ be a falling n-fold strong ideal of a BH-algebra X. Then $\xi(\omega)$ is an n-fold strong ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y, z \in X$ be such that $(x*y)*z^n \in \xi(\omega)$ and $y \in \xi(\omega)$ for any positive integer n. Putting $z := 0$ and $n := 1$ in the above statement, we have $x * y = (x * y) * 0^1$ and $y \in \xi(\omega)$. It follows from (I5) that $x = x * 0^1 \in \xi(\omega)$, i.e., $\xi(\omega)$ is an ideal of X. Therefore $\tilde{H}$ is a falling ideal of X. □

Corollary 3.8. *In a BH-algebra, every falling strong ideal is a falling ideal.*

Proof. Put $n := 1$ in Theorem 3.7. □

The converse of Corollary 3.8 is not true in general as seen in the following example.

Example 3.9. Let $X := \{0, a, b, c, d\}$ be a BH-algebra([2]) with the following table:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \{0, a\} & \text{if } t \in [0, 0.4), \\ X & \text{if } t \in [0.4, 1]. \end{cases}$$

Then $\xi(t)$ is a subalgebra and an ideal of X for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling subalgebra and a falling ideal of X. If we take $t \in [0, 0.4)$, then $\xi(t) = \{0, a\}$ is not a strong ideal of X since $(d * a) * b = a \in \{0, a\}, a \in \{0, a\}$ and $d * b = d \notin \{0, a\}$. Therefore $\tilde{H}$ is not a falling strong ideal of X.

Corollary 3.10. *In a BH^*-algebra, every falling n-fold strong ideal is a falling subalgebra.*

Proof. It follow from Proposition 3.6 and Theorem 3.7. □

The converse of Corollary 3.10 is not true in general as seen in the following example.

Example 3.11. Let $X := \{0, a, b, c\}$ be a BH^*-algebra([1]) with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	b	0

For a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \emptyset & \text{if } t \in [0, 0.3), \\ \{0, a, b\} & \text{if } t \in [0.3, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then $\xi(t)$ is an n-fold strong ideal of X for all $t \in [0,1]$ and for every positive integer n. Hence $\tilde{H}$ is a falling n-fold strong ideal of X for every positive integer n.
Define a random set $\xi : [0,1] \to \mathcal{P}(x)$ as follows:

$$\xi : \Omega \to \mathcal{P}(X),\ t \mapsto \begin{cases} \{0,c\} & \text{if } t \in [0, 0.3), \\ \{0,b\} & \text{if } t \in [0.3, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then $\xi(t)$ is a subalgebra of X for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling subalgebra of X. If we take $t \in [0.3, 0.8)$, then $\xi(t) = \{0,b\}$ is not an n-fold strong ideal of X since $(c*b)*0^n = b*0^n = b \in \{0,b\}$ and $c*0^n = c \notin \{0,b\}$. Thus $\tilde{H}$ is not a falling n-strong ideal of X for every positive integer n

Theorem 3.12. *Let X be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of X. Then $\tilde{H}$ is a falling n-fold strong ideal of X if and only if for each $\omega \in \Omega$, the following is valid:*

$$(\forall x \in \xi(\omega))(\forall y, z \in X)(y * z^n \notin \xi(\omega) \Rightarrow (y * x) * z^n \notin \xi(\omega)). \tag{3.1}$$

Proof. Suppose that $\tilde{H}$ is a falling n-fold strong ideal of a BH-algebra X. Then $\xi(\omega)$ is an n-fold strong ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y, z \in X$ with $x \in \xi(\omega)$ and $y * z^n \notin \xi(\omega)$. If $(y * x) * z^n \in \xi(\omega)$, then $y * z^n \in \xi(\omega)$ since $\xi(\omega)$ is an n-fold strong ideal of X. This is a contradiction. Thus $(y * x) * z^n \notin \xi(\omega)$ for all positive integer n.

Conversely, let $\tilde{H}$ be a falling subalgebra of X satisfying (3.1). Then $\xi(\omega)$ is a subalgebra of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence $0 \in \xi(\omega)$. Let $x, y, z \in X$ be such that $(y * x) * z^n \in \xi(\omega)$ and $x \in \xi(\omega)$. If $y * z^n \notin \xi(\omega)$, then $(y * x) * z^n \notin \xi(\omega)$ by (3.1). This is a contradiction and so $\tilde{H}$ is a falling n-fold strong ideal of X. □

Corollary 3.13. *Let X be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of X. Then $\tilde{H}$ is a falling strong ideal of X if and only if for each $\omega \in \Omega$, the following is valid:*

$$(\forall x \in \xi(\omega))(\forall y, z \in X)(y * z \notin \xi(\omega) \Rightarrow (y * x) * z \notin \xi(\omega)).$$

Proof. Put $n := 1$ in Theorem 3.12. □

Corollary 3.14. *Let X be a BH-algebra. Assume that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathcal{P}(X)$ is a falling subalgebra of X. Then $\tilde{H}$ is a falling ideal of X if and only if for each $\omega \in \Omega$, the following is valid:*

$$(\forall x \in \xi(\omega))(\forall y \in X)(y \notin \xi(\omega) \Rightarrow y * x \notin \xi(\omega)).$$

Proof. Put $z := 0$ in Corollary 3.13. □

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \to \mathcal{P}(X)$. For any $x \in X$, let

$$\Omega(x;\xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}. \tag{3.2}$$

Then $\Omega(x;\xi) \in \mathcal{A}$.

Lemma 3.15. *If $\tilde{H}$ is a falling subalgebra of a BH-algebra X, then*

$$(\forall x \in X)\left(\Omega(x;\xi) \subseteq \Omega(0;\xi)\right). \tag{3.3}$$

Proof. If $\Omega(x;\xi) = \emptyset$, then it is clear. Assume that $\Omega(x;\xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x;\xi)$. Then $x \in \xi(\omega)$, and so $0 = x * x \in \xi(\omega)$ since $\xi(\omega)$ is a subalgebra of X. Hence $\omega \in \Omega(0;\xi)$, and therefore $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ for all $x \in X$. □

Combing Proposition 3.6 and Lemma 3.15, we have the following corollary.

Corollary 3.16. *If $\tilde{H}$ is a falling ideal of a BH^*-algebra X, then* (3.3) *is valid.*

Theorem 3.17. *If $\tilde{H}$ is a falling subalgebra of a BH-algebra X, then*

$$(\forall x, y \in X)(\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * y;\xi)).$$

Proof. Let $\omega \in \Omega(x;\xi) \cap \Omega(y;\xi)$ for any $x, y \in X$. Then $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a subalgebra of X, $x * y \in \xi(\omega)$. Hence $\omega \in \Omega(x * y, \xi)$. Thus $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * y;\xi)$. □

Theorem 3.18. *If $\tilde{H}$ is a falling ideal of a BH-algebra X, then*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow \Omega(y;\xi) \subseteq \Omega(x;\xi)$.
(ii) $(\forall x, y \in X)(\Omega(x * y;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi)$.

Proof. (i) Let $x, y \in X$ with $x \leq y$ and $\omega \in \Omega(y;\xi)$. Then $y \in \xi(\omega)$ and $0 = x * y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X, $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x;\xi)$. Hence (i) holds.
(ii) Let $\omega \in \Omega(x * y;\xi) \cap \Omega(y;\xi)$ for any $x, y \in X$. Then $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X, $x \in \xi(\omega)$. Hence $\omega \in \Omega(x;\xi)$. Thus (ii) holds. □

Theorem 3.19. *If $\tilde{H}$ is a falling n-fold strong ideal of a BH-algebra X, then*

(i) $(\forall x, y, z \in X)(x * y \leq z^n \Rightarrow \Omega(y;\xi) \subseteq \Omega(x * z^n;\xi)$,
(ii) $(\forall x, y, z \in X)(\Omega((x * y) * z^n;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * z^n;\xi)$

for any positive integer n.

Proof. (i) Let $x, y, z \in X$ with $\omega \in \Omega(y;\xi)$ and $x * y \leq z^n$ for any integer n. Then $y \in \xi(\omega)$ and $(x * y) * z^n = 0 \in \xi(\omega)$. Since $\xi(\omega)$ is an n-fold strong ideal of X, we have $x * z^n \in \xi(\omega)$. Hence $\omega \in \Omega(x * z^n;\xi)$. Thus (i) holds.
(ii) Let $x, y, z \in X$ be such that $\omega \in \Omega((x*y)*z^n;\xi) \cap \Omega(y;\xi)$. Then $(x*y)*z^n \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is an n-fold strong ideal of X, we have $x * z^n \in \xi(\omega)$, i.e., $\omega \in \Omega(x * z^n;\xi)$. Thus (ii) is holds. □

Corollary 3.20. *If $\tilde{H}$ is a falling strong ideal of a BH-algebra X, then*

(i) $(\forall x, y, z \in X)(x * y \leq z \Rightarrow \Omega(y;\xi) \subseteq \Omega(x * z;\xi)$.
(ii) $(\forall x, y, z \in X)(\Omega((x * y) * z;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * z;\xi)$.

Proof. Since the 1-fold strong ideal is precisely a strong ideal, these two conditions hold by Theorem 3.19 . □

Theorem 3.21. *If $\tilde{H}$ is a falling translation ideal of a BH-algebra X, then*

(i) $(\forall x, y, z \in X)(x \leq y \Rightarrow \Omega(y*x;\xi) \subseteq \Omega((x*z)*(y*z);\xi) \cap \Omega((z*x)*(z*y);\xi)$.
(ii) $(\forall x, y, z \in X)(\Omega(x*y;\xi) \cap \Omega(y*x;\xi) \subseteq \Omega((x*z)*(y*z);\xi) \cap \Omega((z*x)*(z*y);\xi)$.

Proof. (i) Let $x, y, z \in X$ be such that $\omega \in \Omega(y * x;\xi)$ and $x \leq y$. Then $y * x \in \xi(\omega)$ and $0 = x * y \in \xi(\omega)$. Since $\xi(\omega)$ is a translation ideal of X, we have $(x * z) * (y * z) \in \xi(\omega)$ and $(z * x) * (z * y) \in \xi(\omega)$. Hence $\omega \in \Omega((x * z) * (y * z);\xi) \cap \Omega((z * x) * (z * y);\xi)$. Hence (i) holds.
(ii) Let $x, y, z \in X$ be such that $\omega \in \Omega(x * y;\xi) \cap \Omega(y * x;\xi)$. Then $x * y \in \xi(\omega)$ and $y * x \in \xi(\omega)$. Since $\xi(\omega)$ is a translation ideal of X, we have $(x * z) * (y * z) \in \xi(\omega)$ and $(z * x) * (z * y) \in \xi(\omega)$. Hence $\omega \in \Omega((x * z) * (y * z);\xi) \cap \Omega((z * x) * (z * y);\xi)$. Thus (ii) holds. □

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[a]DEPARTMENT OF MATHEMATICS EDUCATION, DONGGUK UNIVERSITY, SEOUL 100-715, KOREA
Email address: duchil@hanmail.net

[b]DEPARTMENT OF MATHEMATICS EDUCATION, DONGGUK UNIVERSITY, SEOUL 100-715, KOREA
Email address: sunshine@dongguk.edu