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A NOTE ON SPECIAL LAGRANGIANS OF COTANGENT BUNDLES OF SPHERES

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Abstract. For each submanifold X in the sphere S^n, we show that the corresponding conormal bundle N^*X is Lagrangian for the Stenzel form on T^*S^n. Furthermore, we correspond an austere submanifold X to a special Lagrangian submanifold N^*X in T^*S^n. We also discuss austere submanifolds in S^n from isoparametric geometry.

1. Introduction

A Calabi-Yau manifold is a Kähler manifold with a parallel holomorphic volume form Ω, and a submanifold calibrated by $\mathrm{Re}\,\Omega$ is called a *special Lagrangian* submanifold. Special Lagrangians were studied as a type of calibrated submanifolds [6], which are absolutely minimal submanifolds, and recently these have attracted a lot of attentions thanks to their roles in mirror symmetry on Calabi-Yau manifolds.

For a submanifold X of a smooth manifold M, its conormal bundle N^*X is a Lagrangian submanifold of the cotangent bundle T^*M with the canonical symplectic structure. In particular, Harvey and Lawson showed if M is the Euclidean space $\mathbb{R}^n$, the conormal bundle N^*X is a special Lagrangian in $T^*\mathbb{R}^n$ if and only if X is *austere*, namely the shape operator of X in $\mathbb{R}^n$ for each normal vector has the set of eigenvalues invariant under multiplication of -1.

On the other hand, Stenzel[14] showed that the cotangent bundle of a sphere has a complete Ricci-flat metric, i.e. it is a Calabi-Yau manifold. Therefore a natural question is whether the austere submanifolds X in the sphere S^n correspond to special Lagrangians N^*X in T^*S^n. However, one issue is that the Kähler form ω_{sz} corresponding to the Ricci flat metric is not the canonical one. In this paper, we show that the conormal bundle N^*X is also a Lagrangian submanifold in T^*S^n for ω_{sz}, and moreover, it is a special Lagrangian if and only if X is austere in S^n.

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This result is also achieved in [11] independently. Note that examples of special Lagrangian submanifolds in T^*S^n with Stenzel metric were constructed in [2] and [9].

For the examples of austere submanifolds in spheres, we consider the minimal isoparametric hypersurface and the focal submanifolds in spheres. The *isoparametric hypersurfaces* in a sphere are hypersurfaces with constant principal curvatures. These austere examples are independently observed in [3] and [10] for the construction of minimal Legendrian submanifolds in S^{2n+1} and for the generalization of austere examples, respectively.

This paper is the first part of the author's study on the special Lagrangian submanifolds in cotangent bundles. Since the list of Stenzel's Ricci-flat metrics[14] contains the cotangent bundles of projective spaces for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions). A natural question is whether the austere submanifolds of each projective space are related the special Lagrangians conormal bundles in the ambient cotangent bundles. This question will be treated in [12]. This paper was prepared in 2006 while the author held a position at Washington University in St. Louis. As the results obtained in [11] overlapped with those in this paper, the author did not publish the paper. But since there is no further development for this study, it is worthy for the author to invite attention to the results, particularly, to the examples and the note on further study at the end of the paper.

2. Preliminaries

Let M be a $2n$-dimensional symplectic manifold with a symplectic form ω, which is a nondegenerate 2-form on M. A *Lagrangian submanifold* L in M is an n-dimensional submanifold where ω vanishes on its tangent vector bundle. Equivalently, the tangent space T_xL of each x in L is a Lagrangian subspace in T_xM. In general, the space of Lagrangian subspaces in a $2n$-dimensional symplectic vector space can be identified with $U(n)/SO(n)$, and each Lagrangian subspace is assigned a phase $\theta \in [0,1]$ defined by

$$U(n)/SO(n) \xrightarrow{\det_{\mathbb{C}}} S^1.$$

Lagrangian submanifolds with constant phase are closely related to calibrated submanifolds in a Calabi-Yau manifold X ; a Kähler manifold with a nonzero holomorphic volume form Ω_X;.

The theory of calibration and calibrated submanifolds was developed by Harvey and Lawson in [6] to produce absolute minimal submanifolds. Recall that a closed differential form ψ of degree k on a Riemannian manifold M is called a *calibration* if it satisfies

$$\psi_x|_V \leq Vol_V,$$

for every oriented k-plane V in T_xM, at each point x in M. Here Vol_V is the volume form on V for the induced metric. A *calibrated submanifold* N is a submanifold where $\psi|_N$ is equal to the induced volume form on N. Note that any other submanifold $\tilde{N}$, homologous to N, satisfies

$$Vol\left(\tilde{N}\right) = \int_{\tilde{N}} Vol_{\tilde{N}} \geq \int_{\tilde{N}} \psi = \int_N \psi = \int_N Vol_N = Vol\left(N\right),$$

and the equality sign holds if and only if $\tilde{N}$ is also a ψ-calibrated submanifold in M.

In [6], Harvey and Lawson showed that for a holomorphic volume form Ω_X of a Calabi-Yau manifold X and each real number θ, $\operatorname{Re} e^{i\theta}\Omega_X$ is a calibration, and $\operatorname{Re} e^{i\theta}\Omega_X$-calibrated submanifolds L equivalently satisfy

$$\left(\operatorname{Im} e^{2\pi i\theta}\Omega_X\right)|_L = 0,\ \omega\,|_L = 0.$$

This calibrated submanifold is called a *special Lagrangian submanifold* (with phase θ). They also related each embbed submanifold X in $\mathbb{R}^n$ to a conormal bundle N^*X in $T^*\mathbb{R}^n$, defined by

$$N^*X := \{\alpha \in \Gamma\left(X,\ T^*\mathbb{R}^n\right) \mid \alpha_x\left(v\right) = 0 \text{ for all } v \in T_xX \text{ and } x \in X\ \},$$

as a Lagrangian submanifold for the standard Kähler form on $T^*\mathbb{R}^n \simeq \mathbb{C}^n$. Furthermore, they proved that each N^*X in $T^*\mathbb{R}^n$ is a special Lagrangian for a phase if and only if X is an *austere* submanifold in $\mathbb{R}^n$. In general, a submanifold N of a manifold M is *austere* if the shape operator of N in M for each normal vector has the set of eigenvalues invariant under the multiplication of -1.

Recall that the cotangent bundle T^*M of a manifold M has a canonical symplectic structure ω^c, and the conormal bundle N^*X of each submanifold X of M is Lagrangian for ω^c. Furthermore, ω^c is a Kähler form if and only if the corresponding Riemannian metric on M is flat. On the other hand, Stenzel proved that the cotangent bundles of compact symmetric spaces of rank one, including spheres, have complete, Ricci-flat Kähler metrics. Equivalently, each of these cotangent bundles has the holomorphic volume form. In particular, he demonstrated this result on the cotangent bundle of a sphere, which is identified with the affine quadric as in [15]

. In the next section, we consider the cotangent bundles of a sphere and conormal bundles of its submanifolds to get examples of special Lagrangian submanifolds.

3. Special Lagrangian of Cotangent Bundles of Spheres

In this section, we describe Stenzel metric on the cotangent bundles T^*S^n of S^n, and show each conormal bundle N^*X in T^*S^n, for any submanifold X in S^n, is a Lagrangian submanifold for the Kähler form ω_{sz} corresponding to the Stenzel metric. Furthermore, N^*X is special Lagrangian if and only if X is austere in S^n. We also list austere submanifolds in S^n, which come from isoparametric hypersurfaces and related focal submanifolds in S^n.

The cotangent bundle T^*S^n is identified with the tangent bundle TS^n as

$$T^*S^n = \left\{(x,\alpha) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } \langle x,\alpha\rangle \ = 0\right\}$$

where $\langle \cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{n+1}$ and $|x|^2 = \langle x,x\rangle$. As in [15], this can also be identified with the affine quadric

$$Q^n =: \left\{ z \in \mathbb{C}^{n+1} \mid \sum_{k=1}^{n+1} z_k^2 = 1 \right\}$$

along a diffeomorphism Φ defined by

$$\begin{array}{rccl} \Phi: & T^*S^n & \rightarrow & Q^n \\ & (x,\alpha) & \mapsto & x\cosh|\alpha| + \sqrt{-1}\frac{\sinh|\alpha|}{|\alpha|}\alpha. \end{array}$$

This complex affine quadric Q^n has a complex structure induced from the standard complex structure J_0 on $\mathbb{C}^{n+1}$, and that the complex structure of T^*S^n is given by pulling back the complex structure on Q^n. Furthermore, Stenzel showed that Q^n has a Ricci-flat metric corresponding to a Kähler form ω_{sz}, called Stenzel form, defined by

$$\omega_{sz} = \sqrt{-1}\partial\bar{\partial} f(\tau)$$

where $\tau = |z|^2 = \sum_{k=1}^{n+1} z_k \bar{z}_k = \cosh(2|\alpha|)$, and the smooth real valued function $f(\tau)$ is a solution of the ODE

$$\frac{d}{dw}\left(f'(w)\right)^n = cn\left(\sinh w\right)^{n-1}$$

where $w = \cosh^{-1}(\tau)$ and c is a positive constant.

In fact, Stenzel form ω_{sz} is exact on T^*S^n and $\omega_{sz} = d\alpha_{sz}$, where α_{sz} is $-\operatorname{Im}\left(\bar{\partial} f\right)$. Moreover, α_{sz} can be expressed as

$$\alpha_{sz}(v) = \frac{1}{2} f'\left(|z|^2\right)\omega_0(z,v)$$

for $v \in T_zQ$ and $z \in Q$, where ω_0 is the standard symplectic form on $\mathbb{C}^{n+1}$. Therefrom, we can also relate ω_{sz} and ω_0 as follows. For a and b in T_zQ,

$$\begin{aligned}
2\omega_{sz}(a,b) &= 2d\alpha_{sz}(a,b) \\
&= a\left(f'\left(|z|^2\right)\omega_0(z,b)\right) - b\left(f'\left(|z|^2\right)\omega_0(z,a)\right) - 2\alpha_{sz}([a,b]) \\
&= a\left(f'\left(|z|^2\right)\right)\omega_0(z,b) + f'\left(|z|^2\right)a(\omega_0(z,b)) \\
&\quad - \left\{b\left(f'\left(|z|^2\right)\right)\omega_0(z,a) + f'\left(|z|^2\right)b(\omega_0(z,a))\right\} \\
&\quad - f'\left(|z|^2\right)\omega_0(z,\nabla_a b - \nabla_b a) \\
&= f''(|z|^2)\left\{a\left(|z|^2\right)\omega_0(z,b) - b\left(|z|^2\right)\omega_0(z,a)\right\} \\
&\quad + f'\left(|z|^2\right)\{\omega_0(\nabla_a z,b) + \omega_0(z,\nabla_a b) - \omega_0(\nabla_b z,a) - \omega_0(z,\nabla_b a)\} \\
&\quad - f'\left(|z|^2\right)\omega_0(z,\nabla_a b - \nabla_b a) \\
&= 2f''(|z|^2)\{\langle a,z\rangle\omega_0(z,b) - \langle b,z\rangle\omega_0(z,a)\} + 2f'\left(|z|^2\right)\omega_0(a,b).
\end{aligned}$$

Here, we used that $a(\omega_0(z,b)) = \omega_0(\nabla_a z,b) + \omega_0(z,\nabla_a b)$ since ω_0 is a Kähler form, and $a|z|^2 = 2\langle a,z\rangle$. On the quadric Q, we can define the holomorphic volume form Ω_Q as a $(n,0)$-form satisfying

$$\Omega_Q(a_1,a_2,...,a_n) = dz_1 \wedge ... \wedge dz_{n+1}(z,a_1,...,a_n)$$

for $a_1,a_2,...,a_n \in T_zQ$ and $z \in Q$.

In the following theorem, we consider the conormal bundles in T^*S^n and their diffeomorphic images in Q^n to relate to Lagrangian submanifolds in T^*S^n for ω_{sz}. Furthermore, we show the correspondence between the austere condition in S^n and the special Lagrangian condition in T^*S^n.

Theorem 1. *For each k-dimensional submanifold X in S^n, its conormal bundle N^*X is a Lagrangian submanifold of T^*S^n for ω_{sz}. Furthermore, N^*X is a special Lagrangian if and only if X is an austere submanifold in S^n.*

Proof. First, we identify $N^*X \subset T^*S^n$ with the normal bundle NX in TS^n defined by

$$NX := \left\{(x,v(x)) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \mid \begin{array}{l} |x|=1,\ \langle x,v(x)\rangle = 0 \text{ and} \\ \langle v,v(x)\rangle = 0 \text{ all } v \in T_xX \end{array}\right\}.$$

For a fixed $(a,v(a))$ in NX, we consider a local parametrization near $(a,v(a))$ defined by

$$(x, v(x)) \longmapsto (x_1, ..., x_k, t_1, ..., t_{n-k}) \in \mathbb{R}^n$$

where x is a local coordinate near a in X and $v(x) = \sum_{i=1}^{n-k} t_i n_i(x)$ such that $n_1, ..., n_{n-k}$ are orthonormal unit vector fields on X in S^n near a. Furthermore, an adapted orthonormal moving frame of $\mathbb{R}^{n+1}$ along the submanifold X consists of $(x, e_1(x), ..., e_k(x), n_1(x), ..., n_{n-k}(x))$ where $e_1, ..., e_k$ is an orthonormal basis of tangent vector fields of X near a.

We consider

$$\Phi(a, v(a)) = a \cosh |c| + \sqrt{-1} \frac{\sinh |c|}{|c|} v(a)$$

where $|c|^2 := |v(a)|^2 = \sum_{i=1}^{n-k} c_i^2$, and compute a basis of tangent space of $\Phi(NX)$ at $\Phi(a, v(a))$ defined by

$$E_i := \Phi_* \left(\frac{\partial}{\partial x_i} \right) \text{ for } i = 1, ..., k$$
$$N_j := \Phi_* \left(\frac{\partial}{\partial t_j} \right) \text{ for } i = 1, ..., n-k.$$

Therefore, we have

$$E_i = e_i \cosh |c| + \sqrt{-1} A^{v(a)}(e_i) \frac{\sinh |c|}{|c|} \text{ for } i = 1, ..., k$$

where $A^v(e_i)$ is the second fundamental form in the direction of the normal vector $v(a)$ to NX, namely $A^v(e_i) = \nabla_{e_i} v$ for the Levi-Civita connection ∇ on S^n. Similarly,

$$N_j = a(\cosh |c|)_j + \sqrt{-1} \sum_{i=1}^{n-k} n_i(a) \left(c_i \frac{\sinh |c|}{|c|} \right)_j \text{ for } j = 1, ..., n-k$$

where $(\ )_j := \frac{\partial(\ )}{\partial t_j}$. In fact, we can choose $e_1, ..., e_k$ as to be the principal vectors of $A^{v(a)}$ with principal curvatures $\lambda_1, ..., \lambda_k$ respectively, and $E_1, ..., E_k$ can be written as

$$E_i = e_i \cosh |c| + \sqrt{-1} \lambda_i e_i \frac{\sinh |c|}{|c|} \text{ for } i = 1, ..., k.$$

Now we show $\Phi(NX)$ is a Lagrangian submanifold for ω_{sz}. Recall that

$$\omega_{sz}(a, b) = f''(|z|^2) \{ \langle a, z \rangle \omega_0(z, b) - \langle b, z \rangle \omega_0(z, a) \} + f'\left(|z|^2\right) \omega_0(a, b).$$

for a and b in T_zQ. Therefore, we only need to check the followings.

$$\begin{aligned}
&\omega_0\left(\Phi\left(a,v\left(a\right)\right),E_i\right)\\
&=\left\langle J_0\left(a\cosh\left|c\right|,\frac{\sinh\left|c\right|}{\left|c\right|}v\left(a\right)\right),\left(e_i\cosh\left|c\right|,\lambda_ie_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)\right\rangle\\
&=\left\langle\left(-\frac{\sinh\left|c\right|}{\left|c\right|}v\left(a\right),a\cosh\left|c\right|,\right),\left(e_i\cosh\left|c\right|,\lambda_ie_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)\right\rangle\\
&=0,
\end{aligned}$$

since $\langle v\left(a\right),e_i\left(a\right)\rangle=0=\langle a,e_i\left(a\right)\rangle$. Similarly, $\omega_0\left(\Phi\left(a,v\left(a\right)\right),N_j\right)=0$.

$$\begin{aligned}
&\omega_0\left(E_i,E_j\right)\\
&=\left\langle J_0\left(e_i\cosh\left|c\right|,\lambda_ie_i\frac{\sinh\left|c\right|}{\left|c\right|}\right),\left(e_j\cosh\left|c\right|,\lambda_je_j\frac{\sinh\left|c\right|}{\left|c\right|}\right)\right\rangle\\
&=\left\langle\left(-\lambda_ie_i\frac{\sinh\left|c\right|}{\left|c\right|},e_i\cosh\left|c\right|\right),\left(e_j\cosh\left|c\right|,\lambda_je_j\frac{\sinh\left|c\right|}{\left|c\right|}\right)\right\rangle\\
&=\frac{\sinh\left|c\right|\cosh\left|c\right|}{\left|c\right|}\left(\lambda_j-\lambda_i\right)\delta_{ij}=0,
\end{aligned}$$

$$\begin{aligned}
&\omega_0\left(F_i,F_j\right)\\
&=\left\langle J_0\left(a\left(\cosh\left|c\right|\right)_i,\sum_{i=1}^{n-k}n_i\left(a\right)\left(c_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)_i\right),\right.\\
&\qquad\qquad\left.\left(a\left(\cosh\left|c\right|\right)_j,\sum_{i=1}^{n-k}n_i\left(a\right)\left(c_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)_j\right)\right\rangle\\
&=\left\langle\left(-\sum_{i=1}^{n-k}n_i\left(a\right)\left(c_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)_i,a\left(\cosh\left|c\right|\right)_i\right),\right.\\
&\qquad\qquad\left.\left(a\left(\cosh\left|c\right|\right)_j,\sum_{i=1}^{n-k}n_i\left(a\right)\left(c_i\frac{\sinh\left|c\right|}{\left|c\right|}\right)_j\right)\right\rangle\\
&=0
\end{aligned}$$

since $\langle a,n_k\left(a\right)\rangle=0$, and similarly we have $\omega_0\left(E_i,F_j\right)=0$.

From the above calculation, we conclude that for any submanifold X in S^n, $\Phi(NX)$ is a Lagrangian submanifold in Q^n for ω_{sz}.

In the following, we show that $\Phi\left(NX\right)$ is a special Lagrangian submanifold if and only if X is austere in S^n.

Consider

$$\begin{aligned}
&\Omega_Q\left(E_1,...,E_k,N_1,...,N_{n-k}\right)\\
&=dz_1\wedge...\wedge dz_{n+1}\left(z,E_1,...,E_k,N_1,...,N_{n-k}\right)\\
&=Det_{\mathbb{C}}\left(z,E_1,...,E_k,N_1,...,N_{n-k}\right)\\
&=\begin{vmatrix}
\cosh|c| & a\left(\cosh|c|\right)_1 & .. & a\left(\cosh|c|\right)_j\\
\sqrt{-1}\frac{\sinh|c|}{|c|}c_1 & \sqrt{-1}\left(c_1\frac{\sinh|c|}{|c|}\right)_1 & .. & \sqrt{-1}\left(c_1\frac{\sinh|c|}{|c|}\right)_{n-k}\\
.. & .. & .. & ..\\
\sqrt{-1}\frac{\sinh|c|}{|c|}c_{n-k} & \sqrt{-1}\left(c_{n-k}\frac{\sinh|c|}{|c|}\right)_1 & .. & \sqrt{-1}\left(c_{n-k}\frac{\sinh|c|}{|c|}\right)_{n-k}
\end{vmatrix}\\
&\quad\times\prod_{i=1}^{k}\left(\cosh|c|+\sqrt{-1}\lambda_i\frac{\sinh|c|}{|c|}\right)\times B_1\\
&=\left(\sqrt{-1}\right)^{n-k}B_2\prod_{i=1}^{k}\left(\cosh|c|+\sqrt{-1}\lambda_i\frac{\sinh|c|}{|c|}\right)\times B_1
\end{aligned}$$

where B_1 and B_2 are real constants. Since $\Phi(NX)$ is already Lagrangian for ω_{sz}, it is a special Lagrangian submanifold for $\mathrm{Re}\left(\left(\sqrt{-1}\right)^{k-n}\Omega_Q\right)$ if it satisfies

$$\mathrm{Im}\left\{\prod_{i=1}^{k}\left(\cosh|c|+\sqrt{-1}\lambda_i\frac{\sinh|c|}{|c|}\right)\right\}=0$$

for all $|c|$. And an equivalent statement of this is that each symmetric polynomials of λ_1,...,λ_k with odd degrees vanishes. Since this condition is also equivalent to the austere condition for X in S^n, we conclude that $\Phi(NX)$ is a special Lagrangian with phase $\pi(k-n)/2$ if and only if the k-dimensional submanifold X in S^n is austere. □

By applying this theorem to the following list of austere examples, we have many special Lagrangian conormal bundles in T^*S^n. The comparison between these examples and old examples of special Lagrangians submanifolds will be described in [12].

Examples of austere submanifolds in S^n. In a sphere S^n, there is an isoparametric hypersurface M which is a hypersurface with constant principal curvatures. Münzner showed that these principal curvatures $\lambda_1>...>\lambda_g$ of M satisfy

$$\lambda_k=\cot\left(t+\frac{\pi(k-1)}{g}\right)$$

for $k = 1, ..., g$ with $0 < t < \frac{\pi}{g}$, and the corresponding multiplicities m_1, ... , m_g satisfy

$$\begin{cases} m_1 = ... = m_g & \text{if } g \text{ odd} \\ m_1 = ... = m_{g-1},\ m_2 = ... = m_g & \text{if } g \text{ even.} \end{cases}$$

In fact, this gives an isoparametric family of parallel hypersurfaces M_t for $0 < t < \frac{\pi}{g}$. Furthermore, Münzner also showed that this family has exactly two focal submanifolds M_0 and $M_{\pi/g}$, which sat apart with a distance π/g, and the ambient sphere S^n is divided into two disk bundles over M_0 and $M_{\pi/g}$. Therefrom, Münzner obtained that the number of distinct principal curvatures g is 1, 2, 3, 4 or 6. Note the principal curvatures of the focal submanifold M_0 (resp. $M_{\pi/g}$) for any unit normal are λ_k for $k = 2, ..., g$ where $t = 0$ (resp. λ_k for $k = 1, ..., g-1$ where $t = \pi/g$) with the above multiplicities. Münzner reproved Cartan's result that M_0 and $M_{\pi/g}$ are minimal.

From the above description of principal curvatures on isoparametric hypersurfaces and focal submanifolds, one can easily obtain the following list of austere submanifolds in S^n. Note that there is only one minimal isoparametric hypersurface among M_t for $0 < t < \frac{\pi}{g}$.

(1) For each g, the focal submanifolds M_0 and $M_{\pi/g}$ are austere.

(2) For odd g, the minimal isoparametric hypersurface $M_{\pi/2g}$ is austere.

(3) For even g, the minimal isoparametric hypersurface is austere if $m_1 = m_2$. Note, when $g = 4$ or 6, $m_1 = m_2$ implies $m_1 = m_2 = 1$ or 2 ([1]), and all of these are homogeneous isoparametric hypersurfaces by old result of Cartan for $g = 4$ and results in [5] and [13] for $g = 6$.

4. Further Study

From the isoparametric geometry, we have many examples of special Lagrangian conormal bundles in T^*S^n corresponding to austere submanifolds in S^n. Since this approach is new, it is interesting to characterize these examples along the isoparametric geometry. For example, a large class of isoparametric hypersurfaces in spheres are given by orthogonal representations of compact Lie groups, and a classification of homogeneous isoparametric hypersurfaces are related to the classification of orthogonal representations of cohomogeneity two (see[7]). Furthermore, there are also nonhomogeneous isoparametric hypersurfaces. Author expects these two directions may provide many interesting approaches to special Lagrangian submanifolds.

Note that Stenzel's result[14] contains cotangent bundles of projective spaces for each normed algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. It is natural to ask whether the austere submanifolds of each projective space are related to any special Lagrangian conormal bundles in the ambient cotangent bundles. In particular, the cotangent bundle $T^*\mathbb{CP}^n$ is especially interesting in that it is an example of noncompact manifolds with the hyperkähler structures (Calabi[4] and Hitchin[8]), and Lagrangian submanifolds in $T^*\mathbb{CP}^n$ are related to isotropic submanifolds in $\mathbb{CP}^n$. On the other hand, we can fine austere submanifolds in $\mathbb{CP}^n$ from isoparametric geometry on $\mathbb{CP}^n$ and presumably produce special Lagrangian conormal bundles in $T^*\mathbb{CP}^n$. The relationship between these two types of Lagrangian submanifolds in $T^*\mathbb{CP}^n$ must be explained along the hyperkähler structure on $T^*\mathbb{CP}^n$.

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References

1. U. Abresch: Isoparametric hypersurfaces with four or six distinct principal curvatures. Necessary conditions on the multiplicities. *Math. Ann.* **264** (1983), no. 3, 283-302.
2. H. Anciaux: Special Lagrangian submanifolds in the complex sphere. *Ann. Fac. Sci. Toulouse Math.* (6) **16** (2007), no. 2, 215-227.
3. V. Borrelli & C. Gorodski: Minimal Legendrian submanifolds of S^{2n+1}*and absolutely area-minimizing cones. Differential Geom. Appl.* **21** (2004), no. 3, 337-347.
4. E. Calabi: Mériques Kähleriennes et fibrés holomorphes. *Ann. Ec. Norm. Sup.* **12** (1979), 269-294.
5. J. Dorfmeister & E. Neher: Isoparametric hypersurfaces, case g=6,m=1. *Comm. Algebra* **13** (1985), no. 11, 2299-2368.
6. R. Harvey & B. Lawson: Calibrated geometries. *Acta Math.* **148** (1982), 47-157.
7. W. Hsiang & B. Lawson: Minimal submanifolds of low cohomogeneity. *J. Differential Geometry* **5** (1971), 1-38.
8. N. Hitchin: Hyperkahler manifolds. *Seminaire Bourbaki, Asterisque* **206**(1992) 137-166.
9. M. Ionel & M. Min-Oo: Cohomogeneity One Special Lagrangian Submanifolds in the Deformed Conifold. [math/0504557]

10. M. Kimura: *A generalization of Cartan hypersurfaces.* Proceedings of the Ninth International Workshop on Differential Geometry, 51-59, Kyungpook Nat'l. Univ., Taegu, 2005.
11. S. Karigiannis & M. Min-Oo: Calibrated subbundles in noncompact manifolds of special holonomy. *Ann. Global Anal. Geom.* **28** (2005), no. 4, 371-394.
12. J.H. Lee: Special Lagragnian Submanifolds in Cotangent Bundles. *On progresss.*
13. R. Miyaoka: *A new proof of the homogeneity of isoparametric hypersurfaces with* $(g,m) = (6,1)$. Geometry and topology of submanifolds, X (Beijing/Berlin, 1999), 178–199, World Sci. Publ., River Edge, NJ, 2000.
14. M.B. Stenzel: Ricci-flat metrics on the complexification of a compact rank one symmetric space. *Manuscripta Math.* **80** (1993), no. 2, 151-163.
15. R. Szöke: Complex structures on tangent bundles of Riemannian manifolds. *Math. Ann.* **291** (1991), 409-428.

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