

LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT. We study lightlike submanifolds M of a semi-Riemannian manifold \bar{M} with a semi-symmetric non-metric connection subject to the conditions; (a) the characteristic vector field of \bar{M} is tangent to M , (b) the screen distribution on M is totally umbilical in M and (c) the co-screen distribution on M is conformal Killing.

1. INTRODUCTION

In the classical theory of spacetime, while the rest spaces of timelike curves are spacelike subspaces of the tangent spaces, the rest spaces of null curves are lightlike subspaces of the tangent spaces [9]. To investigate this, Hawking and Ellis introduced the notion of so-called screen spaces in section 4.2 of their book [5]. As for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, in a 1996 book [3] Duggal-Bejancu published their work on the general theory of degenerate (lightlike) submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds. The geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). Lightlike hypersurfaces are also studied in the theory of electromagnetism [3].

Ageshe and Chafle [1] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold. Although now we have lightlike version of a large variety of Riemannian submanifolds, a general notion of lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric connection has not

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been introduced as yet. Only there are some limited papers on particular subcases recently studied by Yaşar, Çöken and Yücesan [10], Jin [6] and Jin and Lee [7].

Motivated by the notion of a semi-symmetric non-metric connection on a Riemannian manifold, the objective of this article is to study the geometry of lightlike submanifolds M of a semi-Riemannian manifold \bar{M} with a semi-symmetric non-metric connection. In section 2, we recall some of fundamental formulas in the theory of such a lightlike submanifold and prove some results which will be used in the rest of this article. In section 3, we find the condition that M has an induced symmetric Ricci tensor. In section 4, we study irrotational lightlike submanifolds of a semi-Riemannian space form $\bar{M}(c)$ with a semi-symmetric non-metric connection subject to the conditions; (a) the characteristic vector field of \bar{M} is tangent to M , (b) the screen distribution on M is totally umbilical in M and (c) the co-screen distribution on M is conformal Killing.

2. SEMI-SYMMETRIC NON-METRIC CONNECTIONS

Let (\bar{M}, \bar{g}) be an $(m+n)$ -dimensional semi-Riemannian manifold. A connection $\bar{\nabla}$ on \bar{M} is called a *semi-symmetric non-metric connection* [1] if, for any vector fields X, Y and Z on \bar{M} , $\bar{\nabla}$ and its torsion tensor \bar{T} satisfy

$$(2.1) \quad (\bar{\nabla}_X \bar{g})(Y, Z) = -\pi(Y)\bar{g}(X, Z) - \pi(Z)\bar{g}(X, Y),$$

$$(2.2) \quad \bar{T}(X, Y) = \pi(Y)X - \pi(X)Y,$$

where π is a 1-form associated with a non-zero vector field ζ by $\pi(X) = \bar{g}(X, \zeta)$. We say that ζ is the *characteristic vector field* of \bar{M} . In the entire discussion of this article, we shall assume ζ to be unit spacelike vector field.

Let (M, g) be an m -dimensional lightlike submanifold of \bar{M} . The radical distribution $Rad(TM) = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle TM and the normal bundle TM^\perp , of rank r ($1 \leq r \leq \min\{m, n\}$). Then, in general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and TM^\perp respectively, which called the *screen distribution* and *co-screen distribution* on M , such that

$$(2.3) \quad TM = Rad(TM) \oplus_{orth} S(TM); \quad TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp),$$

where \oplus_{orth} denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over

M . Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and TM^\perp in $S(TM)^\perp$ respectively and let $\{N_1, \dots, N_r\}$ be a lightlike basis of $ltr(TM)$ [3] such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = \bar{g}(X, N_i) = \bar{g}(W, N_i) = 0,$$

for all $X \in \Gamma(S(TM))$ and $W \in \Gamma(S(TM^\perp))$, where the set $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $Rad(TM)$. Then the tangent bundle $T\bar{M}$ is decomposed as follow:

$$(2.4) \quad T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ = \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp).$$

We say that a lightlike submanifold $M \equiv (M, g, S(TM), S(TM^\perp))$ of \bar{M} is

- (1) *r-lightlike* if $1 \leq r < \min\{m, n\}$;
- (2) *co-isotropic* if $1 \leq r = n < m$;
- (3) *isotropic* if $1 \leq r = m < n$;
- (4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)~(4) are particular cases of the class (1) as follows: $S(TM^\perp) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^\perp) = \{0\}$ respectively. The geometry of r -lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, we consider only r -lightlike submanifolds M , with the following local quasi-orthonormal field of frames of \bar{M} :

$$(2.5) \quad \{\xi_1, \dots, \xi_r, N_1, \dots, N_r, F_{r+1}, \dots, F_m, W_{r+1}, \dots, W_n\},$$

where $\{F_{r+1}, \dots, F_m\}$ and $\{W_{r+1}, \dots, W_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^\perp)$ respectively. We use the following range of indices:

$$i, j, k, \dots \in \{1, \dots, r\}, \quad a, b, c, \dots \in \{r + 1, \dots, m\}, \\ \alpha, \beta, \gamma, \dots \in \{r + 1, \dots, n\}, \quad A, B, C, \dots \in \{1, \dots, m\}.$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.3). Then the local Gauss-Weingartan formulas are given by

$$(2.6) \quad \bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^r h_i^\ell(X, Y)N_i + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)W_\alpha,$$

$$(2.7) \quad \bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^r \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^n \rho_{i\alpha}(X)W_\alpha,$$

$$(2.8) \quad \bar{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^r \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^n \theta_{\alpha\beta}(X) W_\beta,$$

$$(2.9) \quad \nabla_X P Y = \nabla_X^* P Y + \sum_{i=1}^r h_i^*(X, P Y) \xi_i,$$

$$(2.10) \quad \nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^r \sigma_{ij}(X) \xi_j,$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms h_i^ℓ and h_α^s on M are called the *local lightlike* and *local screen second fundamental forms* on TM respectively, h_i^* is called the *local second fundamental forms* on $S(TM)$. A_{N_i} , $A_{\xi_i}^*$ and A_{W_α} are linear operators on TM and τ_{ij} , $\rho_{i\alpha}$, $\phi_{\alpha i}$, $\theta_{\alpha\beta}$ and σ_{ij} are 1-forms on TM . We say that

$$h(X, Y) = \sum_{i=1}^r h_i^\ell(X, Y) N_i + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y) W_\alpha$$

is the *second fundamental tensor* of M . Using (2.1), (2.2) and (2.6), we show that

$$(2.11) \quad (\nabla_X g)(Y, Z) = -\pi(Y)g(X, Z) - \pi(Z)g(X, Y) \\ + \sum_{i=1}^r \{h_i^\ell(X, Y)\eta_i(Z) + h_i^\ell(X, Z)\eta_i(Y)\},$$

$$(2.12) \quad T(X, Y) = \pi(Y)X - \pi(X)Y, \quad \forall X, Y, Z \in \Gamma(TM),$$

and each h_i^ℓ and h_α^s are symmetric on TM , where T is the torsion tensor with respect to the induced connection ∇ and η_i is a 1-form on TM such that

$$\eta_i(X) = \bar{g}(X, N_i), \quad \forall X \in \Gamma(TM), \quad i \in \{1, \dots, r\}.$$

From the facts $h_i^\ell(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $h_\alpha^s(X, Y) = \epsilon_\alpha \bar{g}(\bar{\nabla}_X Y, W_\alpha)$, we know that h_i^ℓ and h_α^s are independent of the choice of a screen distribution. Note that τ_{ij} depend on the section $\xi \in \Gamma(\text{Rad}(TM)|_{\mathcal{U}})$ and $d(\text{tr}(\tau_{ij})) = d(\text{tr}(\bar{\tau}_{ij}))$ [4].

Taking $Y = \xi_j$ to $h_j^\ell(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi_j)$ and $h_\alpha^s(X, Y) = \epsilon_\alpha \bar{g}(\bar{\nabla}_X Y, W_\alpha)$, we get

$$(2.13) \quad h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0, \quad h_\alpha^s(X, \xi_i) = -\epsilon_\alpha \phi_{\alpha i}(X), \quad \forall X \in \Gamma(TM),$$

due to (2.1) and (2.8). From the first equation of (2.13) [denote (2.13)₁], we have

$$(2.14) \quad h_i^\ell(X, \xi_i) = 0, \quad h_i^\ell(\xi_j, \xi_k) = 0, \quad \forall X \in \Gamma(TM).$$

The above local second fundamental forms are related to their shape operators by

$$(2.15) \quad \begin{aligned} h_i^\ell(X, Y) &= g(A_{\xi_i}^* X, Y) + \lambda_i g(X, Y) - \sum_{k=1}^r h_k^\ell(X, \xi_i) \eta_k(Y), \\ \bar{g}(A_{\xi_i}^* X, N_j) &= 0, \quad \tau_{ij}(X) - \sigma_{ji}(X) = \lambda_j \eta_i(X), \end{aligned}$$

$$(2.16) \quad \begin{aligned} \epsilon_\alpha h_\alpha^s(X, Y) &= g(A_{W_\alpha} X, Y) + \epsilon_\alpha \nu_\alpha g(X, Y) - \sum_{i=1}^r \phi_{\alpha i}(X) \eta_i(Y), \\ \bar{g}(A_{W_\alpha} X, N_i) &= \epsilon_\alpha \{ \rho_{i\alpha}(X) - \nu_\alpha \eta_i(X) \}, \end{aligned}$$

$$(2.17) \quad \begin{aligned} h_i^*(X, PY) &= g(A_{N_i} X, PY) + \mu_i g(X, PY) + \pi(PY) \eta_i(X), \\ \eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) + \mu_i \eta_j(X) + \mu_j \eta_i(X) &= 0, \\ \epsilon_\beta \theta_{\alpha\beta} &= -\epsilon_\alpha \theta_{\beta\alpha}, \quad \forall X, Y \in \Gamma(TM), \end{aligned}$$

where $\epsilon_\alpha = \bar{g}(W_\alpha, W_\alpha) (= \pm 1)$ denotes the causal character of W_α , and λ_i, μ_i and ν_α are smooth functions given by $\lambda_i = \pi(\xi_i)$, $\mu_i = \pi(N_i)$ and $\nu_\alpha = \epsilon_\alpha \pi(W_\alpha)$.

Definition 1. A vector field X on a differential manifold (\mathcal{M}, g) with a metric tensor g is said to be *conformal Killing* on \mathcal{M} if $\mathcal{L}_X g = -2fg$ for any smooth function f , where \mathcal{L}_X denotes the Lie derivative with respect to X , that is,

$$(\mathcal{L}_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]), \quad \forall Y, Z \in \Gamma(TM).$$

In particular, if $f = 0$, then X is called a *Killing vector field* on \mathcal{M} . A distribution \mathcal{G} on \mathcal{M} is called a *conformal Killing* (or *Killing*) *distribution* on \mathcal{M} if each vector field belonging to \mathcal{G} is a conformal Killing (or Killing) vector field on \mathcal{M} .

Theorem 2.1. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent:*

- (1) h_i^ℓ vanishes identically on M for all i .
- (2) $A_{\xi_i}^*$ satisfies $A_{\xi_i}^* X = -\lambda_i PX$ for all $X \in \Gamma(TM)$ and i .
- (3) $Rad(TM)$ is a Killing distribution on M .
- (4) ∇ is a semi-symmetric non-metric connection on M .

Proof. (1) \Leftrightarrow (2): If $h_i^\ell = 0$ for all i , then, using (2.15)₁ and the fact that $S(TM)$ is non-degenerate, we have $A_{\xi_i}^* X = -\lambda_i PX$ for all $X \in \Gamma(TM)$. Conversely, if $A_{\xi_i}^* X = -\lambda_i PX$ for all $X \in \Gamma(TM)$, then, from (2.15)₁, we obtain

$$(2.18) \quad h_i^\ell(X, Y) = - \sum_{k=1}^r h_k^\ell(X, \xi_i) \eta_k(Y), \quad \forall X, Y \in \Gamma(TM).$$

Taking the skew-symmetric part of (2.18) and using h_i^ℓ is symmetric, we get

$$\sum_{k=1}^r \{h_k^\ell(X, \xi_i)\eta_k(Y) - h_k^\ell(Y, \xi_i)\eta_k(X)\} = 0.$$

Replacing Y by ξ_j to this and using (2.14)₂, we have $h_j^\ell(X, \xi_i) = 0$ for all j and i . From this and (2.18), we have $h_i^\ell = 0$ for all i .

(1) \Leftrightarrow (3): From (2.11), (2.12) and the definition of $\mathcal{L}_X g$, we obtain

$$\begin{aligned} (\mathcal{L}_X g)(Y, Z) &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) - 2\pi(X)g(Y, Z) \\ &+ \sum_{k=1}^r \{h_k^\ell(X, Y)\eta_k(Z) + h_k^\ell(X, Z)\eta_k(Y)\}, \quad \forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

Taking $X = \xi_i$ to this and using (2.10), (2.15)₁ and h_i^ℓ is symmetric, we have

$$(\mathcal{L}_{\xi_i} g)(X, Y) = -2h_i^\ell(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

which proves the equivalence of (1) and (3).

(1) \Leftrightarrow (4): If $h_i^\ell = 0$ for all i , then, from (2.11) and (2.12), we have

$$(\nabla_X g)(Y, Z) = -\pi(Y)g(X, Z) - \pi(Z)g(X, Y), \quad T(X, Y) = \pi(Y)X - \pi(X)Y.$$

Thus ∇ is a semi-symmetric non-metric connection on M . Conversely if ∇ is a semi-symmetric non-metric connection on M , from (2.11) we have

$$\sum_{k=1}^r \{h_k^\ell(X, Y)\eta_k(Z) + h_k^\ell(X, Z)\eta_k(Y)\} = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Z by ξ_i to this, we obtain (2.18). By the method of (2) \Rightarrow (1), we have $h_i^\ell = 0$ for all i . \square

Theorem 2.2. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent:*

- (1) $A_{\xi_i}^*$ are self-adjoint on $\Gamma(TM)$ with respect to g , for all i .
- (2) h_i^ℓ satisfy $h_i^\ell(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j .
- (3) $A_{\xi_i}^* \xi_j = 0$ for all i and j , i.e., the image of $\text{Rad}(TM)$ with respect to $A_{\xi_i}^*$ for each i is a trivial vector bundle.

Proof. From (2.15)₁ and the facts g and h_i^ℓ are symmetric, we have

$$g(A_{\xi_i}^* X, Y) - g(X, A_{\xi_i}^* Y) = \sum_{k=1}^r \{h_k^\ell(X, \xi_i)\eta_k(Y) - h_k^\ell(Y, \xi_i)\eta_k(X)\}.$$

(1) \Leftrightarrow (2). If $h_i^\ell(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j , then we have

$$g(A_{\xi_i}^* X, Y) = g(A_{\xi_i}^* Y, X), \quad \forall X, Y \in \Gamma(TM),$$

for all $X, Y \in \Gamma(TM)$, that is, $A_{\xi_i}^*$ are self-adjoint with respect to g . Conversely, if $A_{\xi_i}^*$ are self-adjoint with respect to g , then we have

$$\sum_{k=1}^r \{h_k^\ell(X, \xi_i) \eta_k(Y) - h_k^\ell(Y, \xi_i) \eta_k(X)\} = 0,$$

for all $X, Y \in \Gamma(TM)$. Replacing Y by ξ_j in this equation and using the second equation of (2.14), we have $h_j^\ell(X, \xi_i) = 0$ for all $X \in \Gamma(TM)$, i and j .

(2) \Leftrightarrow (3). Replacing X by ξ_j to (2.15)₁ and using (2.14)₂, we have

$$h_i^\ell(X, \xi_j) = g(A_{\xi_i}^* \xi_j, X), \quad \forall X \in \Gamma(TM).$$

As $S(TM)$ is non-degenerate, we have the equivalence of (2) and (3). □

In the sequel, we call self-adjoint $A_{\xi_i}^*$ the *lightlike shape operators* of M . In this case, we say that M is ℓ -irrotational. It follows from the equivalence of (1) and (3) in Theorem 2.2 that the radical distribution $Rad(TM)$ of an ℓ -irrotational lightlike submanifold M is always integrable and auto-parallel due to (2.10).

Theorem 2.3. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent:*

- (1) A_{W_α} are self-adjoint on $\Gamma(TM)$ with respect to g for all α .
- (2) h_α^s satisfy $h_\alpha^s(X, \xi_i) = 0$ for all $X \in \Gamma(S(TM))$, α and i .
- (3) $\phi_{\alpha i}(X) = 0$ for all $X \in \Gamma(S(TM))$, α and i .
- (4) $A_{W_\alpha} \xi_i = \sum_{j=1}^r \epsilon_\alpha \rho_{j\alpha}(\xi_i) \xi_j$ i.e., the radical distribution $Rad(TM)$ is invariant vector bundle under the operators A_{W_α} .

Proof. From (2.16)₁ and the facts g and h_α^s are symmetric, we have

$$(2.19) \quad g(A_{W_\alpha} X, Y) - g(X, A_{W_\alpha} Y) = \sum_{j=1}^r \{\phi_{\alpha j}(X) \eta_j(Y) - \phi_{\alpha j}(Y) \eta_j(X)\}.$$

Replace Y by ξ_i in this and using the fact $\phi_{\alpha j}(\xi_i) = \phi_{\alpha i}(\xi_j)$ due to (2.13)₂, we have $g(A_{W_\alpha} \xi_i, X) = -\phi_{\alpha i}(PX)$ for all $X \in \Gamma(TM)$. Also, from (2.19), we have

$$g(A_{W_\alpha} PX, PY) - g(PX, A_{W_\alpha} PY) = g(A_{W_\alpha} \xi_j, \xi_k) - g(\xi_j, A_{W_\alpha} \xi_k) = 0.$$

(1) \Leftrightarrow (3). If $\phi_{\alpha i}(PX) = 0$ for all $X \in \Gamma(TM)$, i and α , then $g(\xi_i, A_{W_\alpha} X) - g(A_{W_\alpha} \xi_i, X) = \phi_{\alpha i}(PX) = 0$. Thus $g(A_{W_\alpha} X, Y) = g(X, A_{W_\alpha} Y)$ for all $X, Y \in$

$\Gamma(TM)$, i.e., A_{W_α} are self-adjoint with respect to g . Conversely, if A_{W_α} are self-adjoint with respect to g , then we have $\sum_{k=1}^r \{\phi_{\alpha k}(X)\eta_k(Y) - \phi_{\alpha k}(Y)\eta_k(X)\} = 0$ for all $X, Y \in \Gamma(TM)$. Replace X by PX and Y by ξ_i in this equation, we have $\phi_{\alpha i}(PX) = 0$ for all $X \in \Gamma(TM)$, i and α .

(2) \Leftrightarrow (3). By (2.13)₂, we have the equivalence of (2) and (3).

(3) \Leftrightarrow (4). If $\phi_{\alpha i}(PX) = 0$ for all $X \in \Gamma(TM)$, i and α , then $g(A_{W_\alpha} \xi_i, X) = 0$. From this, we have $P(A_{W_\alpha} \xi_i) = 0$. In general, since

$$A_{W_\alpha} X = \sum_{i=1}^r \epsilon_\alpha \rho_{i\alpha}(X) \xi_i + P(A_{W_\alpha} X),$$

we get $A_{W_\alpha} \xi_i = \sum_{j=1}^r \epsilon_\alpha \rho_{j\alpha}(\xi_i) \xi_j$. Conversely if $A_{W_\alpha} \xi_i = \sum_{j=1}^r \epsilon_\alpha \rho_{j\alpha}(\xi_i) \xi_j$, from the second equation in this proof, we have $\phi_{\alpha i}(PX) = 0$ for all i and α . \square

In the sequel, we call self-adjoint A_{W_α} the *screen shape operators* of M . In this case, we say that M is *s-irrotational*.

Definition 2. An r -lightlike submanifold M is called *irrotational* [8] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(Rad(TM))$.

Note that M is irrotational if and only if

$$(2.20) \quad h_j^\ell(X, \xi_i) = 0, \quad h_\alpha^s(X, \xi_i) = \phi_{\alpha i} = 0, \quad \forall X \in \Gamma(TM).$$

If M is irrotational, then $A_{\xi_i}^*$ and A_{W_α} are the lightlike and screen shape operators of M respectively and M is both ℓ -irrotational and s -irrotational.

Theorem 2.4. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. If $S(TM^\perp)$ is a conformal Killing distribution on \bar{M} , then there exist smooth functions B_α s such that*

$$(2.21) \quad h_\alpha^s(X, Y) = \epsilon_\alpha B_\alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

In particular, if $S(TM^\perp)$ is a Killing distribution on \bar{M} , then $h_\alpha^s = 0$ for all α .

Proof. Using (2.1), (2.2) and the definition of $\bar{\mathcal{L}}_X \bar{g}$, we have

$$(\bar{\mathcal{L}}_X \bar{g})(Y, Z) = \bar{g}(\bar{\nabla}_Y X, Z) + \bar{g}(Y, \bar{\nabla}_Z X) - 2\pi(X) \bar{g}(Y, Z).$$

Using (2.8) and (2.16), we have $\bar{g}(\bar{\nabla}_X W_\alpha, Y) = \epsilon_\alpha \{\nu_\alpha g(X, Y) - h_\alpha^s(X, Y)\}$. Thus

$$(\bar{\mathcal{L}}_{W_\alpha} \bar{g})(X, Y) = -2\epsilon_\alpha h_\alpha^s(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

From this equation we deduce our assertions. \square

3. INDUCED RICCI CURVATURE TENSORS

Denote by \bar{R} , R and R^* the curvature tensors of the semi-symmetric non-metric connection $\bar{\nabla}$ on \bar{M} , the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$(3.1) \quad \bar{g}(\bar{R}(X, Y)Z, PU) = g(R(X, Y)Z, PU) + \sum_{i=1}^r \{h_i^\ell(X, Z)g(A_{N_i}Y, PU) - h_i^\ell(Y, Z)g(A_{N_i}X, PU)\} + \sum_{\alpha=r+1}^n \{h_\alpha^s(X, Z)g(A_{W_\alpha}Y, PU) - h_\alpha^s(Y, Z)g(A_{W_\alpha}X, PU)\},$$

$$(3.2) \quad \bar{g}(\bar{R}(X, Y)Z, \xi_i) = (\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z) - \pi(X)h_i^\ell(Y, Z) + \pi(Y)h_i^\ell(X, Z) + \sum_{k=1}^r \{\tau_{ki}(X)h_k^\ell(Y, Z) - \tau_{ki}(Y)h_k^\ell(X, Z)\} + \sum_{\alpha=r+1}^n \{\phi_{\alpha i}(X)h_\alpha^s(Y, Z) - \phi_{\alpha i}(Y)h_\alpha^s(X, Z)\},$$

$$(3.3) \quad \bar{g}(\bar{R}(X, Y)Z, N_i) = \bar{g}(R(X, Y)Z, N_i) + \sum_{j=1}^r \{h_j^\ell(X, Z)\eta_i(A_{N_j}Y) - h_j^\ell(Y, Z)\eta_i(A_{N_j}X)\} + \sum_{\alpha=r+1}^n \{h_\alpha^s(X, Z)\eta_i(A_{W_\alpha}Y) - h_\alpha^s(Y, Z)\eta_i(A_{W_\alpha}X)\},$$

$$(3.4) \quad \epsilon_\alpha \bar{g}(\bar{R}(X, Y)Z, W_\alpha) = (\nabla_X h_\alpha^s)(Y, Z) - (\nabla_Y h_\alpha^s)(X, Z) - \pi(X)h_\alpha^s(Y, Z) + \pi(Y)h_\alpha^s(X, Z) + \sum_{i=1}^r \{\rho_{i\alpha}(X)h_i^\ell(Y, Z) - \rho_{i\alpha}(Y)h_i^\ell(X, Z)\} + \sum_{\beta=r+1}^n \{\theta_{\beta\alpha}(X)h_\beta^s(Y, Z) - \theta_{\beta\alpha}(Y)h_\beta^s(X, Z)\},$$

$$(3.5) \quad \bar{g}(\bar{R}(X, Y)N_i, \xi_j) = h_j^\ell(Y, A_{N_i}X) - h_j^\ell(X, A_{N_i}Y) + 2d\tau_{ij}(X, Y) + \sum_{k=1}^r \{\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)\}$$

$$\begin{aligned}
& + \sum_{\alpha=r+1}^n \{\rho_{i\alpha}(Y)\phi_{\alpha j}(X) - \rho_{i\alpha}(X)\phi_{\alpha j}(Y)\} \\
= & h_i^*(X, A_{\xi_j}^* Y) - h_i^*(Y, A_{\xi_j}^* X) + 2d\sigma_{ji}(X, Y) \\
& + \sum_{k=1}^r \{\sigma_{jk}(X)\sigma_{ki}(Y) - \sigma_{jk}(Y)\sigma_{ki}(X)\} \\
& + \sum_{\alpha=r+1}^n \{\rho_{i\alpha}(Y)\phi_{\alpha j}(X) - \rho_{i\alpha}(X)\phi_{\alpha j}(Y)\} \\
& + \sum_{\alpha=r+1}^n \nu_\alpha \{\phi_{\alpha j}(Y)\eta_i(X) - \phi_{\alpha j}(X)\eta_i(Y)\} \\
& + \sum_{k=1}^r \{h_k^\ell(Y, \xi_j)\eta_i(A_{N_k} X) - h_k^\ell(X, \xi_j)\eta_i(A_{N_k} Y)\},
\end{aligned}$$

$$\begin{aligned}
(3.6) \quad \bar{g}(R(X, Y)PZ, PU) & = g(R^*(X, Y)PZ, PU) \\
& + \sum_{i=1}^r \{h_i^*(X, PZ)g(A_{\xi_i}^* Y, PU) - h_i^*(Y, PZ)g(A_{\xi_i}^* X, PU)\},
\end{aligned}$$

$$\begin{aligned}
(3.7) \quad \bar{g}(R(X, Y)PZ, N_i) & = (\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ) \\
& - \pi(X)h_i^*(Y, PZ) + \pi(Y)h_i^*(X, PZ) \\
& + \sum_{j=1}^r \{\sigma_{ji}(Y)h_j^*(X, PZ) - \sigma_{ji}(X)h_j^*(Y, PZ)\}.
\end{aligned}$$

Let \bar{Ric} be the Ricci tensor of \bar{M} and let $R^{(0,2)}$ denote the induced Ricci type tensor on M given respectively by

$$\begin{aligned}
\bar{Ric}(X, Y) & = \text{trace}\{Z \rightarrow \bar{R}(Z, X)Y\}, \quad \forall X, Y \in \Gamma(T\bar{M}), \\
R^{(0,2)}(X, Y) & = \text{trace}\{Z \rightarrow R(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM).
\end{aligned}$$

Using the quasi-orthonormal frame field (2.5), the above equations reduce to

$$\begin{aligned}
(3.8) \quad \bar{Ric}(X, Y) & = \sum_{i=1}^r \bar{g}(\bar{R}(\xi_i, X)Y, N_i) + \sum_{a=r+1}^m \epsilon_a \bar{g}(\bar{R}(F_a, X)Y, F_a) \\
& + \sum_{i=1}^r \bar{g}(\bar{R}(N_i, X)Y, \xi_i) + \sum_{\alpha=r+1}^n \epsilon_\alpha \bar{g}(\bar{R}(W_\alpha, X)Y, W_\alpha),
\end{aligned}$$

$$(3.9) \quad R^{(0,2)}(X, Y) = \sum_{i=1}^r \bar{g}(R(\xi_i, X)Y, N_i) + \sum_{a=r+1}^m \epsilon_a g(R(F_a, X)Y, F_a),$$

respectively, where ϵ_a denotes the causal character of F_a . Substituting (3.1) and (3.3) into (3.8) and using (2.13), (2.15) and (2.16), we obtain

$$\begin{aligned}
 (3.10) \quad R^{(0,2)}(X, Y) &= \bar{Ric}(X, Y) + \sum_{i=1}^r h_i^\ell(X, Y)trA_{N_i} + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)trA_{W_\alpha} \\
 &\quad - \sum_{i=1}^r h_i^\ell(A_{N_i}X, Y) - \sum_{\alpha=r+1}^n h_\alpha^s(A_{W_\alpha}X, Y) \\
 &\quad - \sum_{i=1}^r \bar{g}(\bar{R}(N_i, X)Y, \xi_i) - \sum_{\alpha=r+1}^n \epsilon_\alpha \bar{g}(\bar{R}(W_\alpha, X)Y, W_\alpha),
 \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called the *induced Ricci tensor* [3, 4] of M , denoted by Ric , if it is symmetric. M is *Ricci flat* if its induced Ricci tensor vanishes identically on M .

Theorem 3.1. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection $\bar{\nabla}$. Then the Ricci type tensor $R^{(0,2)}$ of the induced connection ∇ on M of $\bar{\nabla}$ is symmetric if and only if the 1-form $d(tr(\tau_{ij}))$ is closed, i.e., $d(tr(\tau_{ij})) = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

Proof. Using (2.16), (2.19), (3.5)₁, (3.10) and the first Bianchi's identity, we get

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d(tr(\tau_{ij}))(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

From which, we show that $R^{(0,2)}$ is symmetric if and only if $d(tr(\tau_{ij})) = 0$. □

Note 1. For any r -lightlike submanifold M of a semi-Riemannian manifold \bar{M} admits a Levi-Civita connection, the similar result proved by Duggal and Jin [4].

A semi-Riemannian manifold \bar{M} of constant curvature c is called a *space form* and denote it by $\bar{M}(c)$. In this case, the curvature tensor \bar{R} of \bar{M} is given by

$$(3.11) \quad \bar{R}(X, Y)Z = c\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\}, \quad \forall X, Y, Z \in \Gamma(T\bar{M}).$$

In case the ambient manifold \bar{M} is a semi-Riemannian space form $\bar{M}(c)$, we have

$$\begin{aligned}
 (3.12) \quad R^{(0,2)}(X, Y) &= (m - 1)cg(X, Y) \\
 &\quad + \sum_{i=1}^r h_i^\ell(X, Y)trA_{N_i} + \sum_{\alpha=r+1}^n h_\alpha^s(X, Y)trA_{W_\alpha} \\
 &\quad - \sum_{i=1}^r h_i^\ell(A_{N_i}X, Y) - \sum_{\alpha=r+1}^n h_\alpha^s(A_{W_\alpha}X, Y),
 \end{aligned}$$

for all $X, Y \in \Gamma(TM)$.

4. TANGENTIAL CHARACTERISTIC VECTOR FIELD

In this section, we may assume that ζ is tangent to M . Then we show that $\lambda_i = \pi(\xi_i) = 0$, $\nu_\alpha = \epsilon_\alpha \pi(W_\alpha) = 0$ and $\tau_{ij} = \sigma_{ji}$ due to (2.15)₃, for all i, j and α .

Proposition 4.1. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. If ζ is tangent to M , then there exists a screen distribution $S(TM)$ such that it contains ζ .*

Proof. Assume that ζ belongs to $Rad(TM)$. Then we have

$$\zeta = \sum_{i=1}^r \mu_i \xi_i, \quad 1 = \bar{g}(\zeta, \zeta) = \sum_{i,j=1}^r \mu_i \mu_j g(\xi_i, \xi_j) = 0.$$

It is a contradiction. Thus ζ does not belong to $Rad(TM)$. Due to the decomposition (2.3)₁, this result enables one to choose a screen distribution $S(TM)$ which contains ζ . We call such a $S(TM)$ the *natural screen distribution* of M . \square

Note 2. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ considered by Kupeli [8]. Thus all $S(TM)$ are mutually isomorphic. For this reason, in this section we consider only r -lightlike submanifolds M of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection equipped with a natural screen distribution $S(TM)$.

Theorem 4.1. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. If the characteristic vector field ζ is tangent to M , then we have*

$$(4.1) \quad h_i^\ell(X, \zeta) = 0, \quad \forall X \in \Gamma(TM).$$

Moreover, if ζ is tangent to M and M is s -irrotational, then we have

$$(4.2) \quad h_\alpha^s(X, \zeta) = 0, \quad \forall X \in \Gamma(TM).$$

Proof. Using the equations of (3.5) and the facts $\nu_\alpha = 0$ and $\tau_{ij} = \sigma_{ji}$, we get

$$\begin{aligned} h_j^\ell(Y, A_{N_i} X) - h_j^\ell(X, A_{N_i} Y) &= h_i^*(X, A_{\xi_j}^* Y) - h_i^*(Y, A_{\xi_j}^* X) \\ &+ \sum_{k=1}^r \{h_k^\ell(Y, \xi_j) \eta_i(A_{N_k} X) - h_k^\ell(X, \xi_j) \eta_i(A_{N_k} Y)\}, \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Using (2.15)₁, (2.17)_{1,2} and the fact $\lambda_i = \mu_i = 0$, we have

$$\pi(A_{\xi_j}^* X) \eta_i(Y) = \pi(A_{\xi_j}^* Y) \eta_i(X), \quad \forall X, Y \in \Gamma(TM).$$

Taking $X = \xi_i$ and $Y = \zeta$ to (2.15)₁ and using (2.14)_{1,2}, we show that $\pi(A_{\xi_i}^* \xi_i) = 0$ for all i . Replacing Y by ξ_j to the last equation and using (2.15)₁, we have

$$h_i^\ell(X, \zeta) = \pi(A_{\xi_i}^* X) = 0, \quad \forall X \in \Gamma(TM).$$

Taking the scalar product with W_α and N_i to the Weingarten equations $\bar{R}(X, Y)N_i$ and $\bar{R}(X, Y)W_\alpha$ for M respectively, and then, comparing this two results, we get

$$\begin{aligned} & \epsilon_\alpha \{h_\alpha^s(Y, A_{N_i} X) - h_\alpha^s(X, A_{N_i} Y) + 2d\rho_{i\alpha}(X, Y) \\ & \quad + \sum_{j=1}^r [\tau_{ij}(Y)\rho_{j\alpha}(X) - \tau_{ij}(X)\rho_{j\alpha}(Y)] \\ & \quad + \sum_{\beta=r+1}^n [\rho_{i\beta}(Y)\theta_{\beta\alpha}(X) - \rho_{i\beta}(X)\theta_{\beta\alpha}(Y)]\} \\ & = \bar{g}(\nabla_X(A_{W_\alpha} Y) - \nabla_Y(A_{W_\alpha} X) - A_{W_\alpha}[X, Y], N_i) \\ & \quad + \sum_{j=1}^r \{\phi_{\alpha j}(Y)\eta_i(A_{N_j} X) - \phi_{\alpha j}(X)\eta_i(A_{N_j} Y)\} \\ & \quad + \sum_{\beta=r+1}^n \{\theta_{\alpha\beta}(Y)\eta_i(A_{W_\beta} X) - \theta_{\alpha\beta}(X)\eta_i(A_{W_\beta} X)\}, \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Using this, (2.16)₂ and (2.17)₃, we have

$$\begin{aligned} (4.3) \quad & \epsilon_\alpha \{h_\alpha^s(Y, A_{N_i} X) - h_\alpha^s(X, A_{N_i} Y) + 2d\rho_{i\alpha}(X, Y) \\ & \quad + \sum_{j=1}^r [\tau_{ij}(Y)\rho_{j\alpha}(X) - \tau_{ij}(X)\rho_{j\alpha}(Y)]\} \\ & = \bar{g}(\nabla_X(A_{W_\alpha} Y) - \nabla_Y(A_{W_\alpha} X) - A_{W_\alpha}[X, Y], N_i) \\ & \quad + \sum_{j=1}^r \{\phi_{\alpha j}(Y)\eta_i(A_{N_j} X) - \phi_{\alpha j}(X)\eta_i(A_{N_j} Y)\}. \end{aligned}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(A_{W_\alpha} Y, N_i) = \epsilon_\alpha \rho_{i\alpha}(Y)$ and using (2.1), (2.7) and (2.8), we get

$$\begin{aligned} \bar{g}(\nabla_X(A_{W_\alpha} Y), N_i) & = \epsilon_\alpha X(\rho_{i\alpha}(Y)) + \pi(A_{W_\alpha} Y)\eta_i(X) + g(A_{W_\alpha} Y, A_{N_i} X) \\ & \quad - \sum_{j=1}^r \epsilon_\alpha \tau_{ij}(X)\rho_{j\alpha}(Y), \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

Substituting this equation into (4.3) and using (2.16)₁, we have

$$\pi(A_{W_\alpha} X)\eta_i(Y) = \pi(A_{W_\alpha} Y)\eta_i(X), \quad \forall X, Y \in \Gamma(TM).$$

Replacing Y by ξ_i to this equation and using $\eta_i(\xi_i) = 1$, we have

$$\pi(A_{W_\alpha} X) = \pi(A_{W_\alpha} \xi_i) \eta_i(X), \quad \forall X \in \Gamma(TM).$$

As M is s -irrotational, taking $X = \xi_i$ and $Y = \zeta$ to (2.16)₁, we get $\pi(A_{W_\alpha} \xi_i) = -\phi_{\alpha i}(\zeta) = 0$ due to (3) in Theorem 2.3. Replacing Y by ζ to (2.16)₁ and using the above result, we have $h_\alpha^s(X, \zeta) = \pi(A_{W_\alpha} X) = 0$ for all $X \in \Gamma(TM)$. \square

Definition 3. An r -lightlike submanifold M of \bar{M} is said to be *totally umbilical* [4] if there is a smooth vector field $\mathcal{H} \in \Gamma(\text{tr}(TM))$ such that

$$h(X, Y) = \mathcal{H} g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$, we say that M is *totally geodesic*.

It is easy to see [4] that M is totally umbilical if and only if, on each coordinate neighborhood \mathcal{U} , there exist smooth functions A_i and B_α such that

$$(4.4) \quad h_i^\ell(X, Y) = A_i g(X, Y), \quad h_\alpha^s(X, Y) = B_\alpha g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Corollary 1. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection such that ζ is tangent to M . If M is totally umbilical, then M is totally geodesic.*

Proof. As M is totally umbilical, M is s -irrotational by Theorem 2.3. Thus, by Theorem 4.1, we get $h_i^\ell(X, \zeta) = 0$ and $h_\alpha^s(X, \zeta) = 0$ for all $X \in \Gamma(TM)$. From this results and (4.4), we have $A_i \pi(X) = 0$ and $B_\alpha \pi(X) = 0$. Replacing X by ζ to this results, we get $A_i = B_\alpha = 0$ for all i and α . Thus M is totally geodesic. \square

Definition 4. We say that $S(TM)$ is *totally umbilical* [3] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ_i such that

$$(4.5) \quad h_i^*(X, PY) = C_i g(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad i.$$

In case $C_i = 0$ on \mathcal{U} , we say that $S(TM)$ is *totally geodesic* in M .

Theorem 4.2. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection such that ζ is tangent to M . If $S(TM)$ is totally umbilical in M , then $S(TM)$ is totally geodesic in M .*

Proof. Taking $i = j$ to (2.17)₂ and using $\mu_i = 0$, we have $\bar{g}(A_{N_i} Y, N_i) = 0$. Applying $\bar{\nabla}_X$ to this equation and using (2.1), (2.6), (2.7) and (2.17)₂, we have

$$\bar{g}(\nabla_X(A_{N_i} Y), N_i) = \pi(A_{N_i} Y) \eta_i(X) + g(A_{N_i} X, A_{N_i} Y) + \sum_{j=1}^r \tau_{ij}(X) \eta_i(A_{N_j} Y),$$

for all $X, Y \in \Gamma(TM)$. Substituting this equation into the following equation

$$\begin{aligned}
 0 &= \bar{g}(\bar{R}(X, Y)N_i, N_i) \\
 &= \bar{g}(-\nabla_X(A_{N_i}Y) + \nabla_Y(A_{N_i}X) + A_{N_i}[X, Y], N_i) \\
 &\quad + \sum_{j=1}^r \{\tau_{ij}(X)\eta_i(A_{N_j}Y - \tau_{ij}(Y)\eta_i(A_{N_j}X)\} \\
 &\quad + \sum_{\alpha=r+1}^n \{\rho_{i\alpha}(X)\eta_i(A_{W_\alpha}Y) - \rho_{i\alpha}(Y)\eta_i(A_{W_\alpha}X)\} \\
 &= -\bar{g}(\nabla_X(A_{N_i}Y), N_i) + \bar{g}(\nabla_Y(A_{N_i}X), N_i) \\
 &\quad + \sum_{j=1}^r \{\tau_{ij}(X)\eta_i(A_{N_j}Y) - \tau_{ij}(Y)\eta_i(A_{N_j}X)\}
 \end{aligned}$$

due to (2.16)₂ and the fact $\nu_\alpha = 0$, we have

$$\pi(A_{N_i}X)\eta_i(Y) = \pi(A_{N_i}Y)\eta_i(X), \quad \forall X, Y \in \Gamma(TM).$$

Replacing Y by ξ_j to this equation and using $\eta_i(\xi_j) = \delta_{ij}$, we have

$$\pi(A_{N_j}X) = \pi(A_{N_i}\xi_j)\eta_i(X), \quad \forall X \in \Gamma(TM).$$

As $S(TM)$ is totally umbilical in M , replacing PY by ζ to (2.17)₁, we have

$$\pi(A_{N_j}X) = g(A_{N_j}X, \zeta) = C_j\pi(X) - \eta_j(X), \quad \pi(A_{N_i}\xi_j) = g(A_{N_i}\xi_j, \zeta) = -\delta_{ij}.$$

From the last two results, we have $C_j\pi(X) = 0$ for any $X \in \Gamma(TM)$. Replacing X by ζ to this and using $\pi(\zeta) = 1$, we get $C_j = 0$ for all j . Thus $S(TM)$ is totally geodesic in M . □

Theorem 4.3. *Let M be an r -lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M and $S(TM)$ is totally umbilical in M .*

- (1) *If M is s -irrotational, then we have $c = 0$.*
- (2) *If M is ℓ -irrotational, then M is locally a product manifold $M_r \times M_{m-r}$ where M_r and M_{m-r} are leaves of the integrable distributions $Rad(TM)$ and $S(TM)$ respectively.*

Proof. (1) As $S(TM)$ is totally umbilical in M , we have $h_i^* = 0$ by Theorem 4.2. Thus (3.7) reduce to $\bar{g}(R(X, Y)PZ, N_i) = 0$ for all $X, Y, Z \in \Gamma(TM)$. Substituting this result and (3.11) into (3.3) and using (2.16)₂ with $\nu_\alpha = 0$, we get

$$\begin{aligned}
 (4.6) \quad & c \{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
 & = \sum_{j=1}^r \{h_j^\ell(X, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y, PZ)\eta_i(A_{N_j}X)\} \\
 & \quad + \sum_{\alpha=r+1}^n \epsilon_\alpha \{h_\alpha^s(X, PZ)\rho_{i\alpha}(Y) - h_\alpha^s(Y, PZ)\rho_{i\alpha}(X)\}.
 \end{aligned}$$

Taking $X = \xi_i$ and $Y = PZ = \zeta$ to this and using (4.1) and (4.2), we have $c = 0$.

(2) As $A_{\xi_i}^*$ is self-adjoint, we have $A_{\xi_i}^* \xi_j = 0$ due to (3) in Theorem 2.2. From this and (2.10), we show that $Rad(TM)$ is an auto-parallel distribution on M . As $S(TM)$ is totally umbilical in M , we have $h_i^* = 0$ and $\nabla_X Y = \nabla_X^* Y$ for all $X, Y \in \Gamma(S(TM))$ due to (2.9). Thus $S(TM)$ is also an auto-parallel distribution on M . By the decomposition theorem of de Rham [2], we have $M = M_r \times M_{m-r}$ where M_r and M_{m-r} are leaves of $Rad(TM)$ and $S(TM)$ of M respectively. \square

Theorem 4.4. *Let M be an r -lightlike submanifold of a semi-Riemannian manifold \bar{M} admitting a semi-symmetric non-metric connection. If $S(TM^\perp)$ is a conformal Killing distribution on \bar{M} , then $S(TM^\perp)$ is a Killing distribution on \bar{M} .*

Proof. As $S(TM^\perp)$ is a conformal Killing distribution on \bar{M} , by (2.21) and (2) in Theorem 2.3, we show that M is s -irrotational. Thus we have $h_\alpha^s(X, \zeta) = 0$ for all $X \in \Gamma(TM)$ by Theorem 4.1. Taking $X = Y = \zeta$ to (2.21), we have $B_\alpha = 0$. Thus $h_\alpha^s = 0$ for all α . This implies $S(TM^\perp)$ is a Killing distribution on \bar{M} . \square

Theorem 4.5. *Let M be an ℓ -irrotational r -lightlike submanifold of $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M . If $S(TM)$ is totally umbilical in M and $S(TM^\perp)$ is conformal Killing, then M is Ricci flat.*

Proof. As $S(TM)$ is totally umbilical, we have $h_i^* = 0$. From (2.17)₁, we get

$$(4.7) \quad A_{N_i} X = -\eta_i(X)\zeta + \sum_{j=1}^r \eta_j(A_{N_i} X)\xi_j, \quad \forall X \in \Gamma(TM),$$

$$(4.8) \quad tr A_{N_i} = \sum_{a=r+1}^m \epsilon_a g(A_{N_i} F_a, F_a) + \sum_{k=1}^r \bar{g}(A_{N_i} \xi_k, N_k) = 0 + \sum_{k=1}^r \eta_k(A_{N_i} \xi_k).$$

As $S(TM^\perp)$ is conformal Killing, we get $c = 0$ and $h_\alpha^s = 0$ by Theorem 4.3 and Theorem 4.4. Also as M is ℓ -irrotational, $h_i^\ell(X, \xi_j) = 0$. Thus (4.6) reduce to

$$\sum_{j=1}^r \{h_j^\ell(X, Z)\eta_i(A_{N_j}Y) - h_j^\ell(Y, Z)\eta_i(A_{N_j}X)\} = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing Y by ξ_i to this result, we have

$$(4.9) \quad \sum_{j=1}^r \{h_j^\ell(X, Y)\eta_i(A_{N_j}\xi_i) - h_j^\ell(\xi_i, Y)\eta_i(A_{N_j}X)\} = 0, \quad \forall X, Y \in \Gamma(TM).$$

Substituting (4.7) and (4.8) into (3.12) and using (4.9), we have

$$\begin{aligned} R^{(0,2)}(X, Y) &= \sum_{i=1}^r \{h_i^\ell(X, Y)tr A_{N_i} - h_i^\ell(A_{N_i}X, Y)\} \\ &= \sum_{i,k=1}^r \{h_i^\ell(X, Y)\eta_k(A_{N_i}\xi_k) - h_i^\ell(Y, \xi_k)\eta_k(A_{N_i}X)\} = 0. \end{aligned}$$

Thus M is Ricci flat. □

Theorem 4.6. *Let M be an r -lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M . If $S(TM)$ is totally umbilical in M and $S(TM^\perp)$ is a conformal Killing distribution on \bar{M} , then the following assertions are equivalent:*

- (1) M is flat, i.e., the curvature tensor R of M satisfies $R = 0$.
- (2) The local lightlike second fundamental form h_i^ℓ of M satisfies $h_i^\ell = 0$.
- (3) The induced connection ∇ of M is a semi-symmetric non-metric connection.

Proof. First, using (3.1)~(3.4), (4.7) and the fact $c = h_\alpha^s = 0$, we show that

$$\begin{aligned} R(X, Y)Z &= \sum_{i=1}^r \{h_i^\ell(Y, Z)A_{N_i}X - h_i^\ell(X, Z)A_{N_i}Y\} \\ &= \sum_{i=1}^r \{h_i^\ell(X, Z)\eta_i(Y) - h_i^\ell(Y, Z)\eta_i(X)\}\zeta \\ &\quad + \sum_{i=1}^r \{h_i^\ell(Y, Z)\eta_k(A_{N_i}X) - h_i^\ell(X, Z)\eta_k(A_{N_i}Y)\}\xi_k. \end{aligned}$$

(1) \Leftrightarrow (2). If $h_i^\ell = 0$ for all i , then we have $R = 0$. Conversely, if $R = 0$, then, taking the scalar product with ζ to the last equation and using $\lambda_i = 0$, we have

$$\sum_{i=1}^r \{h_i^\ell(X, Z)\eta_i(Y) - h_i^\ell(Y, Z)\eta_i(X)\} = 0.$$

Replacing Y by ξ_k to this, we obtain

$$(4.10) \quad h_k^\ell(X, Y) = \sum_{i=1}^r h_i^\ell(Y, \xi_k)\eta_i(X).$$

Taking the skew-symmetric part of this equation, we have

$$\sum_{k=1}^r \{h_k^\ell(X, \xi_i) \eta_k(Y) - h_k^\ell(Y, \xi_i) \eta_k(X)\} = 0.$$

Replacing Y by ξ_j to this and using (2.14)₂, we have $h_j^\ell(X, \xi_i) = 0$ for all $X \in \Gamma(TM)$, i and j . Substituting this into (4.10), we have $h_i^\ell = 0$ for all i .

(2) \Leftrightarrow (3). The equivalence of (2) and (3) follows from Theorem 2.1. \square

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