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SURFACES WITH POINTWISE 1-TYPE GAUSS MAP OF THE SECOND KIND

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Abstract. In this article, we study generalized slant cylindrical surfaces (GSCS's) with pointwise 1-type Gauss map of the first and second kinds. Our main results state that the right circular cones are the only rational kind GSCS's with pointwise 1-type Gauss map of the second kind.

1. Introduction and Preliminaries

During the late 1970's, B.-Y. Chen introduced the notion of finite type submanifolds in Euclidean or pseudo-Euclidean space, and then the notion has become a useful tool for investigating and characterizing a lot of important submanifolds ([3, 4]). The notion of finite type submanifolds in Euclidean or pseudo-Euclidean space was extended to Gauss maps of submanifolds ([1, 2, 6]).

Suppose that a submanifold M of Euclidean or pseudo-Euclidean space has 1-type Gauss map G. Then the Gauss map G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C, where Δ is the Laplace operator corresponding to the induced metric on M (cf [1, 2, 10]). But, on the important surfaces such as helicoids, catenoids and right circular cones, the Laplacian of the Gauss map take a somewhat different form; namely,

$$\Delta G = f(G + C), \tag{1.1}$$

where f is a non-constant function and C is a constant vector. For this reason, a submanifold is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1.1) for some smooth function f on M and vector C. A submanifold with pointwise 1-type Gauss map is said to be *of the first kind* if the vector C in (1.1) is the zero

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vector. Otherwise, the pointwise 1-type Gauss map is said to be *of the second kind* ([5, 8]).

For the induced metric on the submanifold M, we consider the matrix $g = (g_{ij})$ consisting of the components of the induced metric on M and we denote by $g^{-1} = (g^{ij})$ (resp., $\mathcal{G}$) the inverse matrix (resp., the determinant) of the matrix (g_{ij}). The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{\mathcal{G}}\; g^{ij} \frac{\partial}{\partial x^j}\Big). \tag{1.2}$$

Now, we show that the right circular cone has pointwise 1-type Gauss map of the second kind ([5]).

Example 1.1. Let's consider the right circular cone C_a which is parameterized by

$$x(u,v) = (v\cos u, v\sin u, av), \quad a \geq 0.$$

Then the Gauss map G and its Laplacian ΔG are given by

$$G = \frac{1}{\sqrt{1+a^2}}(a\cos u, a\sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2}\Big(G + \Big(0, 0, \frac{1}{\sqrt{1+a^2}}\Big)\Big),$$

respectively. This implies that the right circular cone has pointwise 1-type Gauss map of the second kind.

In [5], B.-Y. Chen, M. Choi and Y. H. Kim studied surfaces of revolution with pointwise 1-type Gauss map. In [7], U. Dursun studied flat surfaces in Euclidean 3-space with pointwise 1-type Gauss map.

In [9], the author and Y. H. Kim introduced the class of generalized slant cylindrical surfaces (GSCS's). This class includes surfaces of revolution and cylindrical surfaces as special cases. In [8], the author studied GSCS's with pointwise 1-type Gauss map. As a result, he showed that GSCS's with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right circular cones are the only polynomial kind GSCS's with pointwise 1-type Gauss map of the second kind.

In this article, we study the GSCS's with pointwise 1-type Gauss map of the second kind. As a result, we show that the right circular cones are the only rational kind GSCS's with pointwise 1-type Gauss map of the second kind.

From now on, all objects are assumed to be connected and smooth, unless mentioned otherwise.

2. GSCS's with Pointwise 1-type Gauss Map of the First Kind

Consider a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$. We let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where V denotes the unit vector $(0, 0, 1)$. For a constant θ, we let $Y_\theta(s) = \cos\theta N(s) + \sin\theta V$. Then the ruled surface M defined by

$$F(s,t) = X(s) + tY_\theta(s) \tag{2.1}$$

is regular at (s,t) where $1 - \cos\theta\kappa(s)t$ does not vanish. This ruled surface M is called a *slant cylindrical surface (SCS)* over $X(s)$ ([9]).

More generally, instead of a line, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_s(t) = z(t)N(s) + w(t)V$, then the parametrized surface defined by

$$H(s,t) = X(s) + Y_s(t) \tag{2.2}$$

is regular at (s,t) where $1 - \kappa(s)z(t)$ does not vanish. This parametrized surface M is called a *generalized slant cylindrical surface (GSCS)* over $X(s)$ ([9]).

In case $W(t)$ is a straight line, the GSCS $H(s,t)$ is nothing but an SCS. If $X(s)$ is a straight line, then the GSCS $H(s,t)$ is nothing but a cylindrical surface. Furthermore, we have the following ([8, 9]).

Proposition 2.1. *If $X(s)$ is a circle, then a GSCS M over $X(s)$ is a surface of revolution.*

Thus, we see that cylindrical surfaces and surfaces of revolution are special families of GSCS's.

Proposition 2.2. *Let M denote a GSCS given by* (2.2)*. Then we have the following.*
(1) *If the mean curvature H is constant, then M is a surface of revolution.*
(2) *If the Gaussian curvature K is constant, then M is either a surface of revolution or an SCS.*

Now, we consider a GSCS M parametrized by (2.2), where $W(t) = (z(t), w(t))$ is a unit speed plane curve, $Y_s(t) = z(t)N(s) + w(t)V$, and $V = (0, 0, 1)$.

Then, we get the following propositions ([8]).

Proposition 2.3. *Let M be a GSCS given by* (2.2). *Suppose that M has pointwise* 1-*type Gauss map G of the first kind. Then M is a surface of revolution.*

Proposition 2.4. *Let M be a GSCS given by* (2.2). *Then the following are equivalent.*

(1) *M has pointwise* 1-*type Gauss map G of the first kind.*
(2) *M has constant mean curvature.*
(3) *M is a surface of revolution with constant mean curvature.*

Remark 2.5. Surfaces of revolution with constant mean curvature are also known as *surfaces of Delaunay* (cf. [11, p.115]).

3. GSCS's with Pointwise 1-type Gauss Map of the Second Kind

Consider a GSCS M parametrized by (2.2). If M is not cylindrical, then $W(t)$ can be parametrized by $W(t) = (t, g(t))$ for some function $g = g(t)$. Hence M is given by

$$H(s,t) = X(s) + tN(s) + g(t)V. \tag{3.1}$$

If $g(t)$ is a polynomial (resp., rational) in t, then M is said to be of polynomial (resp., rational) kind ([5]). $H(s,t)$ is regular at (s,t) where $Q(s,t) = 1 - t\kappa(s) \neq 0$ and we get

$$\begin{aligned} &H_s = Q(s,t)T(s), H_t = N(s) + g'(t)V, \\ &G(s,t) = \frac{1}{P(t)}\{-g'(t)N(s) + V\}, P(t) = \sqrt{1 + g'(t)^2}. \end{aligned} \tag{3.2}$$

The Laplacian Δ on M is given by

$$\begin{aligned} \Delta f = &- P^{-4}Q^{-3}\{\kappa'(s)tP^4 f_s + P^4 Q f_{ss} \\ &- (P^2Q^2\kappa(s) + Q^3 g'g'')f_t + P^2Q^3 f_{tt}\}. \end{aligned} \tag{3.3}$$

Hence, it follows from (3.2) and (3.3) that

$$\begin{aligned} \Delta G = &- \kappa'(s)g'P^{-1}Q^{-3}T(s) \\ &- P^{-7}Q^{-2}\{\kappa(s)^2 g'P^6 + \kappa(s)g''P^2Q \\ &+ g'(g'')^2Q^2 - g'''P^2Q^2 + 3g'(g'')^2Q^2\}N(s) \\ &- P^{-7}Q^{-1}\{(3(g')^2(g'')^2 - (g'')^2 - g'g''' - (g')^3g''')Q + \kappa(s)g'g''P^2\}V. \end{aligned} \tag{3.4}$$

Suppose that the Gauss map G satisfies (1.1) with nonzero constant vector C. Hereafter, we may assume that $f \neq 0$, because otherwise, M is a plane. Letting $C = C_1(s)T(s) + C_2(s)N(s) + C_3V$, we have the following:

$$PQ^3C_1(s)f(s,t) + \kappa'(s)g'(t) = 0, \tag{3.5}$$

$$\begin{aligned} &P^6Q^2f(s,t)\{-g'(t) + PC_2(s)\} + \kappa(s)^2g'P^6 \\ &+ \kappa(s)g''P^2Q + 4g'(g'')^2Q^2 - g'''P^2Q^2 = 0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &P^6Qf(s,t)\{1 + C_3P\} + \{3(g')^2(g'')^2 \\ &- (g'')^2 - g'g''' - (g')^3g'''\}Q + \kappa(s)g'g''P^2 = 0. \end{aligned} \tag{3.7}$$

Suppose that M is a GSCS of rational kind, that is, $g(t)$ is a rational function in t. Then both of $g(t)$ and $g'(t)$ are rational functions in t. Denote by $g'(t) = r(t)/q(t)$, where $r(t)$ and $q(t)$ are relatively prime polynomials.

Lemma 3.1. *Suppose that $C_1(s)$ vanishes identically. Then M is a right circular cone.*

Proof. If $C_1(s)$ vanishes identically, then (3.5) shows that $\kappa'(s)g'(t)$ vanishes identically. If $g'(t) = 0$, then M is a plane. Otherwise, $\kappa(s)$ is a constant. First, suppose that $\kappa(s)$ is a nonzero constant. Then Proposition 2.1 shows that M is a surface of rotation. Thus, it follows from [5] that M is a right circular cone.

Now, suppose that $\kappa(s)$ vanishes identically. Then $X(s)$ is a straight line with constant vector fields T and N. Hence M is a cylindrical surface over a plane curve $W(t) = (t, g(t))$ with constants C_2 and C_3. Furthermore, (3.6) and (3.7) reduce to, respectively,

$$P^6f(s,t)\{-g'(t) + PC_2\} + 4g'(g'')^2 - g'''P^2 = 0, \tag{3.8}$$

and

$$P^6f(s,t)\{1 + C_3P\} + \{3(g')^2(g'')^2 - (g'')^2 - g'g''' - (g')^3g'''\} = 0. \tag{3.9}$$

By eliminating $f(s,t)$, we get

$$\sqrt{1 + (g')^2}\{C_2A - C_3D - C_2B\} = \{g'A - g'B + D\}, \tag{3.10}$$

where

$$\begin{aligned} &A = 3(g')^2(g'')^2 - (g')^3g''', \quad B = (g'')^2 + g'g''', \\ &D = 4g'(g'')^2 - (g')^2g''' - g'''. \end{aligned} \tag{3.11}$$

From (3.10), we also obtain

$$\{1 + (g')^2\}\{C_2A - C_3D - C_2B\}^2 = \{g'A - g'B + D\}^2. \tag{3.12}$$

Case 1. First, suppose that $g'(t) = r(t)/q(t)$ satisfies $\deg r(t) > \deg q(t)$. Then we put $g'(t) = r(t)/q(t) = s(t) + u(t)/q(t)$, where $s(t)$, $q(t)$ and $u(t)$ are polynomials given by

$$\begin{aligned} q(t) =& t^m + \cdots + q_m, \quad s(t) = s_0 t^l + \cdots + s_l, \quad l \geq 1, \\ u(t) =& u_0 t^n + \cdots + u_n, \quad n < m. \end{aligned} \tag{3.13}$$

By comparing the leading coefficients of both sides of $q(t)^{14}$ times of (3.12), we get $C_2^2 = 1$, and hence again we get $C_3 = 0$. This shows that the leading coefficient of $q(t)^{14}\{A^2 - 2g'AD\}$ becomes zero, which is a contradiction.

Case 2. Second, suppose that $g'(t) = u(t)/q(t)$ satisfies $\deg u(t) < \deg q(t)$, where $q(t)$ and $u(t)$ are relatively prime polynomials given in (3.13).

By comparing the leading coefficients of both sides of $q(t)^{14}$ times of (3.12), we get $C_3^2 = 1$, and hence again we get $C_2 = 0$. This shows that the leading coefficient of $q(t)^{14}g'D(g'D + 2B)$ becomes zero, which is a contradiction.

Case 3. Finally, suppose that $g'(t) = r(t)/q(t)$ satisfies $\deg r(t) = \deg q(t)$, where $q(t)$ and $r(t)$ are relatively prime polynomials given in (3.12). Hence we have $g'(t) = r(t)/q(t) = a + u(t)/q(t)$ for some nonzero constant a and a polynomial $u(t)$ with $\deg u < \deg q$.

In this case, first suppose that $C_2A - C_3D - C_2B = 0$ in (3.10), then we have $g'A - g'B + D = 0$. Hence we get $A - B = D = 0$. Thus, from $D = 0$ we get

$$g''' = \frac{4g'(g'')^2}{1 + (g')^2}, \tag{3.14}$$

and hence we obtain

$$A - B = -(g'')^2\{1 + (g')^2\} = 0. \tag{3.15}$$

This shows that $g(t)$ is a linear function, and hence M is nothing but a plane.

Now, suppose that $g(t)$ is not a linear function. Then, the above discussion shows that $P = \sqrt{1 + g'(t)^2}$ is a rational function in t. Hence, there exists a polynomial $p(t)$ satisfying

$$(1 + a^2)q^2 + 2auq + u^2 = p^2. \tag{3.16}$$

Thus, we see that $q(t), u(t)$ and $p(t)$ satisfy

$$(p - \sqrt{1 + a^2}q)(p + \sqrt{1 + a^2}q) = u(2aq + u). \tag{3.17}$$

Since the leading coefficient of $p(t)$ is $\pm\sqrt{1 + a^2}$, without loss of generality, we may assume that the leading term of $p(t)$ is given by $\sqrt{1 + a^2}t^m$. Then, by considering the leading terms of polynomials in (3.17), we get

$$p + \sqrt{1+a^2}q = \frac{a}{\sqrt{1+a^2}}(2aq + u), \tag{3.18}$$

and hence

$$p - \sqrt{1+a^2}q = \frac{\sqrt{1+a^2}}{a}u. \tag{3.19}$$

From (3.19), we get

$$p = \sqrt{1+a^2}q + \frac{\sqrt{1+a^2}}{a}u. \tag{3.20}$$

By substituting p in (3.20) into (3.18), we obtain

$$2aq + u = 0, \tag{3.21}$$

which is a contradiction. □

Lemma 3.2. *Suppose that M is a GSCS of rational kind with pointwise 1-type Gauss map. Then $C_1(s)$ vanishes identically.*

Proof. Suppose that $C_1(s) \neq 0$ on an interval I. Then we have $\kappa'(s)g'(t) \neq 0$ on I. It follows from (3.5) and (3.7) that

$$\begin{aligned} &C_3\kappa'(s)g'P^6 + \kappa'(s)g'P^5 \\ &= C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3g'''\} + C_1(s)\kappa(s)g'g''P^2Q^2 \\ &\quad - C_1(s)Q^3\{(g'')^2 + g'g'''\}. \end{aligned} \tag{3.22}$$

From now on, we proceed on the interval I. Since $g'(t)$ and P^2 are rational functions, (3.22) shows that $P = \sqrt{1+g'(t)^2}$ is also a rational function in t. Hence, there exists a polynomial $p(t)$ satisfying $q^2(t) + r^2(t) = p^2(t)$, where $q(t), r(t)$ and $p(t)$ are relatively prime. We put

$$\begin{aligned} &R(t) = C_3\kappa'(s)g'P^6, \quad R_1(t) = -\kappa'(s)g'P^5, \\ &R_2(t) = C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3g'''\}, \\ &R_3(t) = C_1(s)\kappa(s)g'g''P^2Q^2, \quad R_4(t) = -C_1(s)Q^3\{(g'')^2 + g'g'''\}. \end{aligned} \tag{3.23}$$

Then, for each $i = 1, 2, 3, 4$, R_i is a rational function, which satisfies

$$R(t) = R_1(t) + R_2(t) + R_3(t) + R_4(t). \tag{3.24}$$

Since, for each $i = 1, 2, 3, 4$, $q^6R_i(t)$ is a polynomial, it follows from (3.23) and (3.24) that

$$q^6R(t) = \frac{C_3\kappa'(s)r(t)p(t)^6}{q(t)} \tag{3.25}$$

is a polynomial in t. Since $p(t), q(t), r(t)$ are relatively prime, it follows from $\kappa'(s) \neq 0$ that $C_3 = 0$. Thus (3.22) becomes

$$\left\{\frac{\kappa'(s)}{C_1(s)}\right\}^2 (g')^2\{1+(g')^2\}^5 = [Q^3\{3(g')^2(g'')^2 - (g')^3g''' - (g'')^2 - g'g'''\} + \kappa(s)g'g''Q^2\{1+(g')^2\}]^2. \tag{3.26}$$

We also get from (3.5) and (3.6) that

$$S_1(t) + S_2(t) = 0, \tag{3.27}$$

where we denote

$$\begin{aligned} S_1(t) &= \kappa'(s)g'(t)^2P^5 - \kappa'(s)C_2(s)g'(t)P^6 + C_1(s)\kappa(s)^2g'(t)P^6Q, \\ S_2(t) &= \kappa(s)C_1(s)g''(t)P^2Q^2 + 4C_1(s)g'(g'')^2Q^3 - C_1(s)g'''(t)P^2Q^3. \end{aligned} \tag{3.28}$$

Since $q(t)^5S_2(t)$ is a polynomial in t, (3.27) shows that

$$q(t)^5S_1(t) = \frac{q(t)^7S_1(t)}{q(t)^2} = \frac{rp^5(Ar+Bp)}{q(t)^2} \tag{3.29}$$

is a polynomial in t, where we denote

$$A = \kappa'(s), B = C_1(s)\kappa(s)^2Q - \kappa'(s)C_2(s). \tag{3.30}$$

Since $p(t), q(t), r(t)$ are relatively prime, we see that

$$Ar + Bp = u(t)q(t)^2, \tag{3.31}$$

where $u(t)$ is a polynomial in t.

Case 1. Suppose that $\deg q(t) \geq \deg r(t)$. Then we have $\deg p(t) = \deg q(t)$. Since $\kappa(s) \neq 0$, we have $\deg B(t) = 1$. Hence (3.31) shows that $\deg q(t) = 1$, and hence $g'(t) = r(t)/q(t)$ is a linear fractional function in t. But, in this case, $q^2(t) + r^2(t)$ can not be a square of a linear function. This is a contradiction.

Case 2. Suppose that $\deg q(t) < \deg r(t)$. Then we put $g'(t) = r(t)/q(t) = s(t) + u(t)/q(t)$, where $s(t)$, $q(t)$ and $u(t)$ are polynomials given in (3.13).

Note that Q is a polynomial in t given by $Q = 1-\kappa(s)t$. Then, it is straightforward to show that the highest degree of left side of $q(t)^{12}$ times of (3.26) is $12(m+l)$ with leading coefficient $\{\kappa'(s)/C_1(s)\}^2s_0^{12}$, and the highest degree of right side of $q(t)^{12}$ times of (3.26) is $12m + 8l + 2$. This shows that $\kappa'(s) = 0$ on I, which is a contradiction. □

Summarizing above, we obtain

Theorem 3.3. *Suppose that a GSCS M of rational kind has pointwise 1-type Gauss map G of the second kind. Then M is a right circular cone.*

Hence, combining the results in [5] and [10], we get

Corollary 3.4. *Suppose that a GSCS M has pointwise* 1*-type Gauss map G of the second kind. Then the following are equivalent.*

(1) *M is of rational kind.*

(2) *M is of polynomial kind.*

(3) *M is a right circular cone.*

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