

ON THE FUZZY STABILITY OF CUBIC MAPPINGS USING FIXED POINT METHOD

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ABSTRACT. Let X and Y be vector spaces. We introduce a new type of a cubic functional equation $f : X \rightarrow Y$. Furthermore, we assume X is a vector space and (Y, N) is a fuzzy Banach space and then investigate a fuzzy version of the generalized Hyers-Ulam stability in fuzzy Banach space by using fixed point method for the cubic functional equation.

1. INTRODUCTION

The stability problem of functional equations originated from the question of Ulam [26] concerning the stability of group homomorphisms. It was answered by Hyers [10] on the assumption that the spaces are Banach spaces and generalized by Aoki [1] for the stability of the additive mapping and Rassias [23] for the stability of the linear mapping by considering the unbounded Cauchy difference. The paper [23] has influentially provided in development of what we call the *Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability* of functional equations. Since then the stability problems of several functional equations and various normed spaces have been extensively investigated and generalized by a number of authors [7], [9], [11], [23] and [2].

In [13], Jun et al. considered the following cubic functional equation

$$(1.1) \quad f(ax + y) + f(ax - y) = a(f(x + y) + f(x - y)) + 2a(a^2 - 1)f(x)$$

for all $x, y \in X$ and $a \in \mathbb{Z} (a \neq 0, \pm 1)$. They investigated the Hyers-Ulam-Rassias stability problem for the functional equation (1.1).

In this paper, we will consider a new type of the following cubic functional equation

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$$(1.2) \quad \begin{aligned} & (a+1)f(ax+y) + (a-1)f(ax-y) + 2(a^2-1)f(y) \\ & = 2a^2f(x+y) + 2a^2(a^2-1)f(x) \end{aligned}$$

for all $x, y \in X$ and $a \in \mathbb{Z} (a \neq 0, \pm 1)$.

In 1984, Katsaras [14] and Wu and Fang [27] independently introduced a notion of a fuzzy norm and also Wu and Fang gave the generalization of Kolmogoroff normalized theorem for fuzzy topological linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [3], [8], [16], [28] and [19]. In 1994, Cheng and Mordeson [6] introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [15]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [6]. Bag and Samanta [3] introduced the following definition of fuzzy normed spaces. We will use the definition to prove a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (1.2) in the fuzzy normed vector space setting.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N₁) $N(x, t) = 0$ for $t \leq 0$;
- (N₂) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
- (N₃) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- (N₄) $N(x+y, s+t) \geq \min \{N(x, s), N(y, t)\}$;
- (N₅) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$;
- (N₆) for $x \neq 0$, $N(x, \cdot)$ is continuous on \mathbb{R} .

The pair (X, N) is called a *fuzzy normed vector space*.

Mirmostafae et al. [19] and Mirzavaziri and Moslehian [20] introduced some properties of fuzzy normed vector spaces and examples of fuzzy norms.

Example 1.2. Let $(X, \|\cdot\|)$ be a real normed space. Define

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|} & \text{when } t > 0, t \in \mathbb{R} \\ 0 & \text{when } t \leq 0, \end{cases}$$

where $x \in X$. Then (X, N) is a fuzzy normed space.

The following definitions in fuzzy normed vector space were given in [3].

Definition 1.3. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n -$

$x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n \rightarrow \infty} x_n = x$.

Definition 1.4. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $d > 0$, we have $N(x_{n+d} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy normed space is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 1.5. Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.6 (The alternative of fixed point [17], [25]). *Suppose that we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \rightarrow X$ with Lipschitz constant $0 < L < 1$. Then for each given $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J ;*
- (3) y^* *is the unique fixed point of J in the set*

$$Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\};$$

- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [12] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [4], [5], [21] and [24].

In this paper, we will prove the generalized Hyers-Ulam stability of the cubic functional equation (1.2) in fuzzy Banach spaces by using fixed point method.

2. CUBIC FUNCTIONAL EQUATIONS

Theorem 2.1. *If a mapping $f : X \rightarrow Y$ satisfies the functional equation (1.2) if and only if f satisfies the functional equation (1.1).*

Proof. Suppose the mapping $f : X \rightarrow Y$ satisfies the equation (1.2). It is easy to show that

$$f(0) = 0, \quad f(-x) = -f(x) \quad \text{and} \quad f(ax) = a^3 f(x)$$

for all $x \in X$ and $a \in \mathbb{Z} (a \neq 0, \pm 1)$. Letting $y = ay$ in the equation (1.2), we have

$$(2.1) \quad \begin{aligned} a(a+1)f(x+y) + a(a-1)f(x-y) + 2a(a^2-1)f(y) \\ = 2f(x+ay) + 2(a^2-1)f(x) \end{aligned}$$

for all $x, y \in X$. Now putting $y = -ay$ in (1.2), we get

$$(2.2) \quad \begin{aligned} a(a+1)f(x-y) + a(a-1)f(x+y) - 2a(a^2-1)f(y) \\ = 2f(x-ay) + 2(a^2-1)f(x) \end{aligned}$$

for all $x, y \in X$. Adding two equations (2.1) and (2.2),

$$a^2 f(x+y) + a^2 f(x-y) = f(x+ay) + f(x-ay) + 2(a^2-1)f(x)$$

for all $x, y \in X$. Replacing x by ax in the previous equation,

$$f(ax+y) + f(ax-y) = a \left(f(x+y) + f(x-y) \right) + 2a(a^2-1)f(x),$$

that is, it satisfies the equation (1.1).

Conversely, suppose the mapping $f : X \rightarrow Y$ satisfies the equation (1.1). It also satisfies the following properties :

$$f(0) = 0, \quad f(-x) = -f(x) \quad \text{and} \quad f(ax) = a^3 f(x)$$

for all $x \in X$ and $a \in \mathbb{Z} (a \neq 0, \pm 1)$. Letting $y = ay$ in the equation (1.1), we get

$$(2.3) \quad a^3 f(x+y) + a^3 f(x-y) = a \left(f(x+ay) + f(x-ay) \right) + 2a(a^2-1)f(x)$$

for all $x, y \in X$. Exchanging x and y in the previous equation,

$$(2.4) \quad f(ax+y) - f(ax-y) + 2(a^2-1)f(y) = a^2 \left(f(x+y) - f(x-y) \right)$$

for all $x, y \in X$. Replacing x by ax in the equation (2.3),

$$(2.5) \quad af(ax+y) + af(ax-y) = a^2 \left(f(x+y) + f(x-y) \right) + 2a^2(a^2-1)f(x)$$

for all $x, y \in X$. Now adding two equations (2.4) and (2.5), we have the equation (1.2), as desired. \square

3. FUZZY STABILITY OF CUBIC MAPPINGS

Let us fix some notations which will be used throughout this paper. We assume X is a vector space and (Y, N) is a fuzzy Banach space. Using fixed point method, we will prove the generalized Hyers-Ulam stability of the functional equation satisfying equation (1.2) in fuzzy Banach space. For a given mapping $f : X \rightarrow Y$, let

$$(3.1) \quad D_a f(x, y) := (a + 1)f(ax + y) + (a - 1)f(ax - y) \\ - 2a^2 f(x + y) - 2a^2(a^2 - 1)f(x) + 2(a^2 - 1)f(y)$$

for all $x, y \in X$ and $a \in \mathbb{Z}(a \neq 0, \pm 1)$.

Theorem 3.1. *Let $a \in \mathbb{Z}(a \neq 0, \pm 1)$ and $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\phi(x, y) \leq \frac{L}{|a|^3} \phi(ax, ay)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$(3.2) \quad N(D_a f(x, y), t) \geq \frac{t}{t + \phi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n \rightarrow \infty} a^{3n} f\left(\frac{x}{a^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$(3.3) \quad N(f(x) - C(x), t) \geq \frac{2a^4(1 - L)t}{2a^4(1 - L)t + L\phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. By letting $y = 0$ in the inequality (3.2), we have

$$(3.4) \quad N\left(2af(ax) - 2a^4f(x), t\right) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$. We consider the set

$$S := \{g : X \rightarrow X\}$$

and the mapping d defined on $S \times S$ by

$$d(g, h) = \inf\{\mu \in \mathbb{R}^+ \mid N(g(x) - h(x), \mu t) \geq \frac{t}{t + \phi(x, 0)}, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then (S, d) is a complete generalized metric space; see [18, Lemma 2.1]. Now let's consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := a^3 g\left(\frac{x}{a}\right)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N\left(g(x) - h(x), \varepsilon t\right) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(a^3g\left(\frac{x}{a}\right) - a^3h\left(\frac{x}{a}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{a}\right) - h\left(\frac{x}{a}\right), \frac{L}{|a|^3}\varepsilon t\right) \geq \frac{\frac{L}{|a|^3}t}{\frac{L}{|a|^3}t + \phi\left(\frac{x}{a}, 0\right)} \\ &\geq \frac{\frac{L}{|a|^3}t}{\frac{L}{|a|^3}t + \frac{L}{|a|^3}\phi(x, 0)} = \frac{t}{t + \phi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. Hence we get

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. The inequality (3.4) implies that

$$N\left(f(x) - a^3f\left(\frac{x}{a}\right), \frac{L}{2a^4}t\right) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(f, Jf) \leq \frac{L}{2a^4}$. By Theorem 1.6, there exists a mapping $C : X \rightarrow Y$ such that

(1) C is a fixed point of J , that is,

$$(3.5) \quad C\left(\frac{x}{a}\right) = \frac{1}{a^3}C(x)$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set $M = \{g \in S \mid d(f, g) < \infty\}$. This means that C is a unique mapping satisfying the equation (3.5) such that

$$\inf\{\mu \in \mathbb{R}^+ \mid N(f(x) - C(x), \mu t) \geq \frac{t}{t + \phi(x, 0)}, \forall x \in X \text{ and } t > 0\}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, C) \rightarrow 0$ as $n \rightarrow \infty$. This implies the following equality

$$\text{N-}\lim_{n \rightarrow \infty} a^{3n}f\left(\frac{x}{a^n}\right) = C(x)$$

for all $x \in X$ and all $t > 0$;

(3) $d(f, C) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{1}{1-L} \cdot \frac{L}{2a^4} = \frac{L}{2a^4(1-L)}.$$

It implies that

$$N\left(f(x) - C(x), \frac{L}{2a^4(1-L)}t\right) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$. By replacing t by $\frac{2a^4(1-L)}{L}t$, we have

$$N\left(f(x) - C(x), t\right) \geq \frac{2a^4(1-L)t}{2a^4(1-L)t + L\phi(x, 0)}$$

for all $x \in X$ and all $t > 0$. That is, the inequality (3.3) holds. By letting $x = \frac{x}{a^n}$ and $y = \frac{y}{a^n}$ in the inequality (3.2), we have

$$N\left(a^{3n}D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), |a|^{3n}t\right) \geq \frac{t}{t + \phi\left(\frac{x}{a^n}, \frac{y}{a^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Replacing t by $\frac{t}{|a|^{3n}}$,

$$N\left(a^{3n}D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), t\right) \geq \frac{\frac{t}{|a|^{3n}}}{\frac{t}{|a|^{3n}} + \phi\left(\frac{x}{a^n}, \frac{y}{a^n}\right)} \geq \frac{t}{t + L^n \phi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \frac{t}{t + L^n \phi(x, y)} = 1$ for all $x, y \in X$ and all $t > 0$, we may conclude that

$$N\left(D_a C(x, y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \rightarrow Y$ is the cubic mapping. □

Corollary 3.2. *Let $\theta \geq 0, p > 3$ be a real number and X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is a mapping satisfying*

$$(3.6) \quad N(D_a f(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n \rightarrow \infty} a^{3n}f\left(\frac{x}{a^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2a(a^p - a^3)t}{2a(a^p - a^3)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and $L = a^{3-p}$. □

Theorem 3.3. *Let $a \in \mathbb{Z} (a \neq 0, \pm 1)$ and $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\phi(x, y) \leq |a|^3 L \phi\left(\frac{x}{a}, \frac{y}{a}\right)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$(3.7) \quad N(D_a f(x, y), t) \geq \frac{t}{t + \phi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n \rightarrow \infty} a^{-3n} f(a^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$(3.8) \quad N(f(x) - C(x), t) \geq \frac{2a^4(1-L)t}{2a^4(1-L)t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1. Consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := \frac{1}{a^3}g(ax)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{a^3}g(ax) - \frac{1}{a^3}h(ax), L\varepsilon t\right) \\ &= N(g(ax) - h(ax), |a|^3 L\varepsilon t) \geq \frac{|a|^3 L\varepsilon t}{|a|^3 L\varepsilon t + \phi(ax, 0)} \\ &\geq \frac{|a|^3 L\varepsilon t}{|a|^3 L\varepsilon t + |a|^3 L\phi(x, 0)} = \frac{t}{t + \phi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. Hence we get

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. The inequality (3.4) implies that

$$N\left(f(x) - \frac{1}{a^3}f(ax), \frac{1}{2a^4}t\right) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(f, Jf) \leq \frac{1}{2a^4}$. By Theorem 1.6, there exists a mapping $C : X \rightarrow Y$ such that

(1) C is a fixed point of J , that is,

$$(3.9) \quad C(ax) = a^3 C(x)$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set $M = \{g \in S \mid d(f, g) < \infty\}$. This means that C is a unique mapping

satisfying the equation (3.9) such that

$$\inf \{ \mu \in \mathbb{R}^+ \mid N(f(x) - C(x), \mu t) \geq \frac{t}{t + \phi(x, 0)}, \forall x \in X \text{ and } t > 0 \}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, C) \rightarrow 0$ as $n \rightarrow \infty$. This implies the following equality

$$N\text{-}\lim_{n \rightarrow \infty} \frac{1}{a^{3n}} f(a^n x) = C(x)$$

for all $x \in X$ and all $t > 0$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{1}{1-L} \cdot \frac{1}{2a^4} = \frac{1}{2a^4(1-L)}.$$

This implies the inequality (3.8) holds. The remains of the proof is similar to the proof of Theorem 3.1. \square

Corollary 3.4. *Let $\theta \geq 0$, $p < 3$ be a real number and X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is a mapping satisfying*

$$(3.10) \quad N(D_a f(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{a^{3n}} f(a^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2a(a^3 - a^p)t}{2a(a^3 - a^p)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and $L = a^{p-3}$. \square

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