J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2012.19.4.383
Volume 19, Number 4 (November 2012), Pages 383–396

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

ON THE STABILITY OF A MIXED TYPE QUADRATIC AND CUBIC FUNCTIONAL EQUATION

Chang-ju Lee [a] and Yang-Hi Lee [b,*]

Abstract. In this paper, we investigate a fuzzy version of stability for the functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y) = 0$$

in the sense of M. Mirmostafaee and M. S. Moslehian.

1. Introduction

A classical question in the theory of functional equations is "when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?". Such a problem, called *a stability problem of the functional equation,* was formulated by S. M. Ulam [14] in 1940. In the next year, D. H. Hyers [5] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [12] for linear mappings by considering the stability problem with unbounded Cauchy differences (see [4,8,9]).

In 1984, A. K. Katsaras [6] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, T. Bag and S.K. Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [7]. In 2008, M. Mirmostafaee and M. S. Moslehian [10] proved a fuzzy version of stability for *the quadratic functional equation:*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0. \tag{1.1}$$

Received by the editors August 7, 2012. Accepted October 16, 2012.
2010 *Mathematics Subject Classification.* 46S40, 39B52.
Key words and phrases. fuzzy normed space, fuzzy almost cubic-quadratic mapping.
*Corresponding author
This work was supported by Gongju National University of Education Grant.

In 2009, M. Mursaleen and S. A. Mohiuddine [11] proved a fuzzy version of stability for *the cubic functional equation:*

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)=0. \tag{1.2}$$

A solution of (1.1) is called a quadratic mapping and a solution of (1.2) is called *a cubic mapping.* The functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-3f(y)+3f(-y)=0. \tag{1.3}$$

is called *the mixed type quadratic and cubic functional equation,* since the function $f(x)=ax^3+bx^2+c$ is its solution. Every solution of the quadratic and cubic functional equation is said to be *a quadratic and cubic mapping.* In 2010, W. Towanlong and P. Nakmahachalasint [13] obtained a stability of the functional equation (1.3). In their processing, they took a cubic mapping C and a quadratic mapping Q such that C is approximate to the odd part $\frac{f(x)-f(-x)}{2}$ of f and Q is close to the even part $\frac{f(x)+f(-x)}{2}-f(0)$ of f, respectively.

In this paper, we get a general stability result of the functional equation (1.3) in the fuzzy normed linear space in the manner of M. Mirmostafaee and M. S. Moslehian [10]. To do it, we introduce a Cauchy sequence $\{J_nf(x)\}$ starting from a given mapping f, which converges to the desired mapping F in the fuzzy sense. As mentioned above, in previous studies of stability problem of (1.3), they [13] attempted to get stability theorems by handling the odd and even part of f, respectively. According to our proposal in this paper, we can take the desired approximate solution F at once.

2. Fuzzy Stability of the Functional Equation (1.3)

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the mixed type quadratic and cubic functional equation in the fuzzy normed linear space.

Definition 2.1 ([2])**.** Let X be a real linear space. A function $N: X\times\mathbb{R}\to[0,1]$ (the so-called fuzzy subset) is said to be *a fuzzy norm on* X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N1) $N(x,c)=0$ for $c\le 0$;
(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
(N3) $N(cx,t)=N(x,t/|c|)$ if $c\neq 0$;
(N4) $N(x+y,s+t)\ge\min\{N(x,s),N(y,t)\}$;

(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X, N) is called *a fuzzy normed linear space.* Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called *a fuzzy Banach space.*

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x,y) := f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 3f(y) + 3f(-y)$$

for all $x, y \in X$. For given $q > 0$, the mapping f is called *a fuzzy q-almost mixed-type quadratic and cubic mapping,* if

$$N'(Df(x,y), t+s) \geq \min\{N(x, s^q), N(y, t^q)\} \tag{2.1}$$

for all $x, y \in X$ and all $s, t \in (0,\infty)$. Now we get the general stability result in the fuzzy normed linear space.

Theorem 2.2. *Let q be a positive real number with $q \neq \frac{1}{2}, \frac{1}{3}$. And let f be a fuzzy q-almost mixed-type quadratic and cubic mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N'). Then there is a unique quadratic and cubic mapping $F : X \to Y$ such that*

$$N'(F(x) - f(x), t) \geq \begin{cases} \sup_{s<t} \left\{N\left(x, (4-2^p)^q s^q\right)\right\} & \text{if} \quad q > \frac{1}{2}, \\ \sup_{s<t} \left\{N\left(x, \left(\frac{(8-2^p)(2^p-4)}{4}\right)^q s^q\right)\right\} & \text{if} \quad \frac{1}{3} < q < \frac{1}{2}, \\ \sup_{s<t} \left\{N\left(x, (2^p-8)^q s^q\right)\right\} & \text{if} \quad 0 < q < \frac{1}{3} \end{cases} \tag{2.2}$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. We will prove the theorem in three cases, $q > \frac{1}{2}$, $\frac{1}{3} < q < \frac{1}{2}$, and $0 < q < \frac{1}{3}$.

Case 1. Let $q > \frac{1}{2}$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x) - 2f(0)\right) + 8^{-n}\left(f(2^n x) - f(-2^n x)\right)\right) + f(0)$$

for all $x \in X$ and $n \in \mathbb{N} \cup \{0\}$. Notice that $J_0 f(x) = f(x)$ and

$$J_jf(x) - J_{j+1}f(x) = -\frac{2^{j+1}-1}{2^{3j+4}}Df(0,-2^jx) - \frac{2^{j+1}+1}{2^{3j+4}}Df(0,2^jx) \tag{2.3}$$

for all $x \in X$ and $j \geq 0$. Together with (N3), (N4) and (2.1), this equation implies that if $n+m > m \geq 0$ then

$$\begin{aligned}
&N'\bigg(J_mf(x) - J_{n+m}f(x), \sum_{j=m}^{n+m-1}\left(\frac{2^p}{4}\right)^j \frac{t}{4}\bigg) \\
&\geq N'\left(\sum_{j=m}^{n+m-1}(J_jf(x) - J_{j+1}f(x)), \sum_{j=m}^{n+m-1}\left(\frac{2^p}{4}\right)^j \frac{t}{4}\right) \\
&\geq \min \bigcup_{j=m}^{n+m-1}\left\{N'\left(J_jf(x) - J_{j+1}f(x), \left(\frac{2^p}{4}\right)^j \frac{t}{4}\right)\right\} \\
&\geq \min \bigcup_{j=m}^{n+m-1}\left\{\min\left\{N'\left(-\frac{2^{j+1}+1}{2^{3j+4}}Df(0,2^jx), \frac{(2^{j+1}+1)2^{jp}t}{2^{3j+4}}\right),\right.\right. \\
&\qquad\left.\left. N'\left(-\frac{2^{j+1}-1}{2^{3j+4}}Df(0,-2^jx), \frac{(2^{j+1}-1)2^{jp}t}{2^{3j+4}}\right)\right\}\right\} \\
&\geq \min \bigcup_{j=m}^{n+m-1}\{N(0, 2^j(t-s)^q), N(2^jx, 2^js^q)\} \\
&= N(x, s^q)
\end{aligned}$$

for all $x \in X$ and $t > 0$, where $0 < s < t$. Hence we have the inequality

$$N'\bigg(J_mf(x) - J_{n+m}f(x), \sum_{j=m}^{n+m-1}\left(\frac{2^p}{4}\right)^j \frac{t}{4}\bigg) \geq \sup_{0<s<t}\{N(x, s^q)\} \tag{2.4}$$

for all $x \in X$and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} N(x,t) = 1$, there is $t_0 > 0$ such that

$$N(x, t_0) \geq 1 - \varepsilon.$$

We observe that for some $\tilde{t}$ with $\tilde{t}^q > t_0$, the series $\sum_{j=0}^{\infty}\left(\frac{2^p}{4}\right)^j \frac{\tilde{t}}{4}$ converges for $p = \frac{1}{q} < 2$. It guarantees that, for an arbitrary given $c > 0$, there exists $n_0 \geq 0$ such that

$$\sum_{j=m}^{n+m-1}\left(\frac{2^p}{4}\right)^j \frac{\tilde{t}}{4} < c$$

for each $m \geq n_0$ and $n > 0$. By (N5) and (2.4), we have

$$\begin{aligned} &N'(J_m f(x) - J_{n+m} f(x), c) \\ &\geq N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{2^p}{4}\right)^j \frac{\tilde{t}}{4}\right) \\ &\geq \sup_{0<s<\tilde{t}} \{N(x, s^q)\} \geq N(x, t_0) \geq 1 - \varepsilon. \end{aligned}$$

for all $x \in X$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N'), and so we can define a mapping $F : X \to Y$ by

$$F(x) := N' - \lim_{n\to\infty} J_n f(x).$$

Moreover, if we put $m = 0$ in (2.4), we have

$$N'(f(x) - J_n f(x), t) \geq \sup_{0<s<t} \left\{ N\left(x, \frac{4^q s^q}{\left(\sum_{j=0}^{n-1} \left(\frac{2^p}{4}\right)^j\right)^q}\right)\right\} \tag{2.5}$$

for all $x \in X$.
Next we will show that F is a quadratic and cubic mapping. Using (N4), we have

$$\begin{aligned} N'(DF(x,y), t) \geq \min\Bigg\{ &N'\left((F - J_n f)(x+2y), \frac{t}{12}\right), \\ &N'\left(3(J_n f - F)(x+y), \frac{t}{12}\right), N'\left(3(F - J_n f)(x), \frac{t}{12}\right), \\ &N'\left((J_n f - F)(x-y), \frac{t}{12}\right), N'\left(3(F - J_n f)(-y), \frac{t}{12}\right), \\ &N'\left(3(J_n f - F)(y), \frac{t}{12}\right), N'\left(DJ_n f(x,y), \frac{t}{2}\right)\Bigg\} \end{aligned} \tag{2.6}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. The first six terms on the right hand side of (2.6) tend to 1 as $n \to \infty$ by the definition of F and (N2), and the last term satisfies the inequality

$$\begin{aligned} N'\left(DJ_n f(x,y), \frac{t}{2}\right) \geq \min\Bigg\{ &N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), \\ &N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 8^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 8^n}, \frac{t}{8}\right)\Bigg\} \end{aligned}$$

for all $x, y \in X$. By (N3) and (2.1), we obtain

$$
\begin{aligned}
N'\left(\frac{Df(\pm 2^n x, \pm 2^n y))}{2\cdot 4^n}, \frac{t}{8}\right) =& N'\left(Df(\pm 2^n x, \pm 2^n y), \frac{4^n t}{4}\right) \\
\geq& \min\left\{N\left(2^n x, \left(\frac{4^n t}{8}\right)^q\right), N\left(2^n y, \left(\frac{4^n t}{8}\right)^q\right)\right\} \\
\geq& \min\left\{N\left(x, \frac{2^{(2q-1)n}}{2^{3q}} t^q\right), N\left(y, \frac{2^{(2q-1)n}}{2^{3q}} t^q\right)\right\}
\end{aligned}
$$

and

$$
N'\left(\frac{Df(\pm 2^n x, \pm 2^n y))}{2\cdot 8^n}, \frac{t}{8}\right) \geq \min\left\{N\left(x, \frac{2^{(3q-1)n}}{2^{3q}} t^q\right), N\left(y, \frac{2^{(3q-1)n}}{2^{3q}} t^q\right)\right\}
$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $q > \frac{1}{2}$, together with (N5), we can deduce that the last term of (2.6) also tends to 1 as $n \to \infty$. It follows from (2.6) that

$$
N'(DF(x,y),t) = 1
$$

for each $x, y \in X$ and $t > 0$. By (N2), this means that $DF(x,y) = 0$ for all $x, y \in X$. Next we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that

$$
N'\left(F(x) - J_n f(x), t - t'\right) \geq 1 - \varepsilon.
$$

By (2.5), we have

$$
\begin{aligned}
N'(F(x) - f(x), t) \geq& \min\left\{N'\left(F(x) - J_n f(x), t - t'\right), N'\left(J_n f(x) - f(x), t'\right)\right\} \\
\geq& \min\left\{1 - \varepsilon, \sup_{0<s<t'}\left\{N\left(x, \frac{4^q s^q}{\left(\sum_{j=0}^{n-1}\left(\frac{2^p}{4}\right)^j\right)^q}\right)\right\}\right\} \\
\geq& \min\left\{1 - \varepsilon, N\left(x, (4-2^p)^q t'^q\right)\right\}.
\end{aligned}
$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (2.2) in this case. Finally, to prove the uniqueness of the quadratic and cubic mapping F, assume that there exists a quadratic and cubic mapping F' which satisfies (2.2). Then by (2.3), we get

$$
\left\{
\begin{aligned}
F(x) - J_n F(x) &= \sum_{j=0}^{n-1}(J_j F(x) - J_{j+1}F(x)) = 0 \\
F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1}\left(J_j F'(x) - J_{j+1}F'(x)\right) = 0
\end{aligned}
\right.
\tag{2.7}
$$

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.2), this implies that

$$
\begin{aligned}
&N'(F(x)-F'(x),t)\\
&=N'(J_nF(x)-J_nF'(x),t)\\
&\geq \min\left\{N'\left(J_nF(x)-J_nf(x),\ \frac{t}{2}\right),\ N'\left(J_nf(x)-J_nF'(x),\ \frac{t}{2}\right)\right\}\\
&\geq \min\left\{N'\left(\frac{(F-f)(2^nx)}{2\cdot 4^n},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(2^nx)}{2\cdot 4^n},\ \frac{t}{8}\right),\right.\\
&\qquad N'\left(\frac{(F-f)(-2^nx)}{2\cdot 4^n},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(-2^nx)}{2\cdot 4^n},\ \frac{t}{8}\right),\\
&\qquad N'\left(\frac{(F-f)(2^nx)}{2\cdot 8^n},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(2^nx)}{2\cdot 8^n},\ \frac{t}{8}\right),\\
&\qquad \left.N'\left(\frac{(F-f)(-2^nx)}{2\cdot 8^n},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(-2^nx)}{2\cdot 8^n},\ \frac{t}{8}\right)\right\}\\
&\geq \sup_{s<t}\left\{N\left(x,2^{(2q-1)n-2q}(4-2^p)^qs^q\right)\right\}
\end{aligned}
$$

for all $x\in X$ and $n\in\mathbb{N}$. Observe that, for $q=\frac{1}{p}>\frac{1}{2}$, the last term of the above inequality tends to 1 as $n\to\infty$ by (N5). This implies that

$$N'(F(x)-F'(x),t)=1.$$

Hence we conclude that

$$F(x)=F'(x)$$

for all $x\in X$ by (N2).

Case 2. Let $\frac{1}{3}<q<\frac{1}{2}$ and let $J_nf: X\to Y$ be a mapping defined by

$$J_nf(x)=\frac{1}{2}\left(8^{-n}\big(f(2^nx)-f(-2^nx)\big)+4^n\left(f\left(\frac{x}{2^n}\right)+f\left(-\frac{x}{2^n}\right)-2f(0)\right)\right)+f(0)$$

for all $x\in X$. Then we have $J_0f(x)=f(x)$ and

$$
\begin{aligned}
J_jf(x)-J_{j+1}f(x)=&-\frac{1}{2^{3j+4}}Df(0,2^jx)+\frac{1}{2^{3j+4}}Df(0,-2^jx)\\
&+2^{2j-1}Df\left(0,\frac{x}{2^{j+1}}\right)+2^{2j-1}Df\left(0,\frac{-x}{2^{j+1}}\right)
\end{aligned}
$$

for all $x\in X$ and $j\geq 0$. If $n+m>m\geq 0$, then we have

$$N'\biggl(J_mf(x)-J_{n+m}f(x),\sum_{j=m}^{n+m-1}\left(\frac{1}{8}\left(\frac{2^p}{8}\right)^j+\frac{1}{2^p}\left(\frac{4}{2^p}\right)^j\right)t\biggr)$$

$$\begin{aligned}
&\geq \min \bigcup_{j=m}^{n+m-1} \Big\{ N'\left(-\frac{Df(0,2^jx)}{2^{3j+4}}, \frac{2^{jp}t}{2^{3j+4}}\right),\\
&\qquad N'\left(\frac{Df(0,-2^jx)}{2^{3j+4}}, \frac{2^{jp}t}{2^{3j+4}}\right),\\
&\qquad N'\left(2^{2j-1}Df\left(0,\frac{x}{2^{j+1}}\right), \frac{2^{2j-1}t}{2^{(j+1)p}}\right),\\
&\qquad N'\left(2^{2j-1}Df\left(0,-\frac{x}{2^{j+1}}\right), \frac{2^{2j-1}t}{2^{(j+1)p}}\right)\Big\}\\
&\geq \min \bigcup_{j=m}^{n+m-1} \Big\{ N(2^jx, 2^js^q), N(0, 2^j(t-s)^q),\\
&\qquad N\left(\frac{x}{2^{j+1}}, \frac{s^q}{2^{j+1}}\right), N\left(0, \frac{(t-s)^q}{2^{j+1}}\right)\Big\}\\
&= N(x, s^q)
\end{aligned}$$

for all $x \in X$ and $t > 0$, where $0 < s < t$. In the similar argument following (2.4) of the previous case, we can define the limit $F(x) := N' - \lim_{n\to\infty} J_nf(x)$ of the Cauchy sequence $\{J_nf(x)\}$ in the Banach fuzzy space Y. Moreover, putting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_nf(x), t) \geq \sup_{s<t}\left\{ N\left(x, \frac{s^q}{\left(\sum_{j=0}^{n-1}\left(\frac{1}{8}\left(\frac{2^p}{8}\right)^j + \frac{1}{2^p}\left(\frac{4}{2^p}\right)^j\right)\right)^q}\right)\right\} \tag{2.8}$$

for each $x \in X$ and $t > 0$. To prove that F is a quadratic and cubic mapping, we need to show that the last term of (2.6) in Case 1 tends to 1 as $n \to \infty$. It is from (N3) and (2.1) that

$$\begin{aligned}
&N'\left(DJ_nf(x,y), \frac{t}{2}\right)\\
&\quad\geq \min\Big\{ N'\left(\frac{Df(2^nx,2^ny)}{2\cdot 8^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^nx,-2^ny)}{2\cdot 8^n}, \frac{t}{8}\right),\\
&\qquad N'\left(2^{2n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{2n-1}Df\left(\frac{-x}{2^n},\frac{-y}{2^n}\right), \frac{t}{8}\right)\Big\}\\
&\quad\geq \min\Big\{ N(x, 2^{(3q-1)n-3q}t^q), N(y, 2^{(3q-1)n-3q}t^q),\\
&\qquad N(x, 2^{(1-2q)n-3q}t^q), N(y, 2^{(1-2q)n-3q}t^q)\Big\}
\end{aligned}$$

for each $x, y \in X$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $\frac{1}{3} < q < \frac{1}{2}$. Hence, together with the similar argument after (2.6), we can say that $DF(x, y) = 0$ for all $x, y \in X$. Recall that the inequality (2.2) follows from (2.5) in Case 1. By the same reasoning, we get (2.2) from (2.8) in this case. Now to prove the uniqueness of F, let F' be another quadratic and cubic mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.7), we have

$$\begin{aligned}
&N'(F(x) - F'(x), t) \\
&\quad = N'(J_nF(x) - J_nF'(x), t) \\
&\quad \geq \min\left\{N'\left(J_nF(x) - J_nf(x), \frac{t}{2}\right), N'\left(J_nf(x) - J_nF'(x), \frac{t}{2}\right)\right\} \\
&\quad \geq \min\left\{N'\left(\frac{(F-f)(2^nx)}{2\cdot 8^n}, \frac{t}{8}\right), \left(\frac{(f-F')(2^nx)}{2\cdot 8^n}, \frac{t}{8}\right),\right. \\
&\qquad\quad N'\left(\frac{(F-f)(-2^nx)}{2\cdot 8^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^nx)}{2\cdot 8^n}, \frac{t}{8}\right), \\
&\qquad\quad N'\left(2^{2n-1}\left((F-f)\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right), N'\left(2^{2n-1}\left((f-F')\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right), \\
&\qquad\quad \left.N'\left(2^{2n-1}\left((F-f)\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right), N'\left(2^{2n-1}\left((f-F')\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right)\right\} \\
&\quad \geq \min\left\{\sup_{s<t}\left\{N\left(x, 2^{(3q-1)n-2q}\left(\frac{(8-2^p)(2^p-4)}{4}\right)^q s^q\right)\right\},\right. \\
&\qquad\quad \left.\sup_{s<t}\left\{N\left(x, 2^{(1-2q)n-2q}\left(\frac{(8-2^p)(2^p-4)}{4}\right)^q s^q\right)\right\}\right\}
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} 2^{(3q-1)n-2q} = \lim_{n\to\infty} 2^{(1-2q)n-2q} = \infty$ in this case, both terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x),\ t) = 1$ and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < \frac{1}{3}$ and define $J_nf : X \to Y$ by

$$J_nf(x) = \frac{1}{2}\left(4^n\left(f(2^{-n}x) + f(-2^{-n}x) - 2f(0)\right) + 8^n\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right) + f(0)$$

for all $x \in X$. Then we have $J_0f(x) = f(x)$ and

$$\begin{aligned}
J_jf(x) - J_{j+1}f(x) =& (2^{3j-1} + 2^{2j-1})Df\left(0, \frac{x}{2^{j+1}}\right) \\
&- (2^{3j-1} - 2^{2j-1})Df\left(0, \frac{-x}{2^{j+1}}\right)
\end{aligned}$$

which implies that if $n + m > m \geq 0$ then

$$
\begin{aligned}
&N'\Bigg(J_mf(x) - J_{n+m}f(x), \sum_{j=m}^{n+m-1} \left(\frac{8}{2^p}\right)^j \frac{t}{2^p}\Bigg) \\
&\geq \min \bigcup_{j=m}^{n+m-1} \Bigg\{ N'\left((2^{3j-1}+2^{2j-1})Df\left(0, \frac{x}{2^{j+1}}\right), \frac{(2^{3j-1}+2^{2j-1})t}{2^{(j+1)p}}\right), \\
&\qquad N'\left(-(2^{3j-1}-2^{2j-1})Df\left(0, -\frac{x}{2^{j+1}}\right), \frac{(2^{3j-1}-2^{2j-1})t}{2^{(j+1)p}}\right) \Bigg\} \\
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ N\left(\frac{x}{2^{j+1}}, \frac{s^q}{2^{j+1}}\right), N\left(0, \frac{(t-s)^q}{2^{j+1}}\right)\right\} \\
&= N(x, s^q)
\end{aligned}
$$

for all $x \in X$ and $t > 0$, where $0 < s < t$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{n\to\infty} J_nf(x)$. Putting $m = 0$ in the above inequality, we have

$$
N'(f(x) - J_nf(x), t) \geq \sup_{s<t} \left\{ N\left(x, \frac{s^q}{\left(\frac{1}{2^p}\sum_{j=0}^{n-1}\left(\frac{8}{2^p}\right)^j\right)^q}\right)\right\} \tag{2.9}
$$

for all $x \in X$ and $t > 0$. Notice that

$$
\begin{aligned}
&N'\left(DJ_nf(x,y), \frac{t}{2}\right) \\
&\geq \min\Bigg\{ N'\left(\frac{4^n}{2}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(\frac{4^n}{2}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right), \\
&\qquad N'\left(2^{3n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{3n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right)\Bigg\} \\
&\geq \min\Big\{ N\left(x, 2^{(1-2q)n-3q}t^q\right), N\left(y, 2^{(1-2q)n-3q}t^q\right), \\
&\qquad N\left(x, 2^{(1-3q)n-3q}t^q\right), N\left(y, 2^{(1-3q)n-3q}t^q\right)\Big\}
\end{aligned}
$$

for each $x, y \in X$ and $t > 0$. Since $0 < q < \frac{1}{3}$, both terms on the right hand side tend to 1 as $n \to \infty$, which implies that the last term of (2.6) tends to 1 as $n \to \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.9) in this case. To prove the uniqueness of F, let $F' : X \to Y$ be another quadratic and cubic mapping satisfying (2.2). Then by (2.7), we get

$$\begin{aligned}
&N'(F(x)-F'(x),t)\\
&\quad\geq \min\left\{N'\left(J_nF(x)-J_nf(x),\frac{t}{2}\right),N'\left(J_nf(x)-J_nF'(x),\frac{t}{2}\right)\right\}\\
&\quad\geq \min\Bigg\{N'\left(\frac{4^n}{2}\left((F-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}\left((f-F')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad N'\left(\frac{4^n}{2}\left((F-f)\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}\left((f-F')\left(-\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad N'\left(2^{3n-1}\left((F-f)\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{3n-1}\left((f-F')\left(\frac{x}{2^n}\right)\right),\frac{t}{8}\right),\\
&\qquad N'\left(2^{3n-1}\left((F-f)\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right),N'\left(2^{3n-1}\left((f-F')\left(\frac{-x}{2^n}\right)\right),\frac{t}{8}\right)\Bigg\}\\
&\quad\geq \sup_{s<t}N\left\{\left(x,2^{(1-3q)n-2q}(2^p-8)^qs^q\right)\right\}
\end{aligned}$$

for all $x\in X$ and $n\in\mathbb{N}$. Observe that, for $0<q<\frac{1}{3}$, the last term tends to 1 as $n\to\infty$ by (N5). This implies that $N'(F(x)-F'(x),t)=1$ and $F(x)=F'(x)$ for all $x\in X$ by (N2). □

Corollary 2.3. *Let f be an even mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique quadratic mapping $F: X\to Y$ such that*

$$N'(\tilde{F}(x)-f(x)+f(0),t)\geq \sup_{s<t}N\{(x,(|4-2^p|s)^q)\} \tag{2.10}$$

for all $x\in X$ and $t>0$, where $p=1/q$.

Proof. Let J_nf be defined as in Theorem 2.2. Since f is an even mapping, we obtain

$$J_nf(x)=\begin{cases}\frac{f(2^nx)+f(-2^nx)-2f(0)}{2\cdot 4^n}+f(0) & \text{if } 0<q<\frac{1}{2},\\ 2^{2n-1}\left(f(2^{-n}x)+f(-2^{-n}x)-2f(0)\right)+f(0) & \text{if } q>\frac{1}{2}\end{cases}$$

for all $x\in X$. Notice that $J_0f(x)=f(x)$ and

$$J_jf(x)-J_{j+1}f(x)=\begin{cases}\frac{-1}{2^{2j+3}}(Df(0,2^jx)+Df(0,-2^jx)) & \text{if } 0<q<\frac{1}{2},\\ 2^{2j-1}\left(Df\left(0,\frac{x}{2^{j+1}}\right)+Df\left(0,\frac{-x}{2^{j+1}}\right)\right) & \text{if } q>\frac{1}{2}\end{cases}$$

for all $x\in X$ and $j\in\mathbb{N}\cup\{0\}$. From these, using the similar method in Theorem 2.2, we obtain a quadratic and cubic mapping F satisfying

$$N'(F(x)-f(x),t)\geq \sup_{s<t}N\{(x,(|4-2^p|s)^q)\}$$

for all $x \in X$ and $t > 0$. Notice that F is also even, $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ for all $x \in X$, and $DF(x,y) = 0$ for all $x, y \in X$. Put $\tilde{F} = F - f(0)$, then

$$F(x+y) + F(x-y) - 2F(x) - 2F(y) = \frac{1}{6}(DF(2y,x) + 3DF(x,y) + DF(x,-y) - DF(0,x+y) - 3DF(0,2y)) = 0$$

for all $x, y \in X$. This means that $\tilde{F}$ is a quadratic mapping satisfying (2.10). □

Corollary 2.4. *Let f be an odd mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique cubic mapping $F : X \to Y$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{s<t} N\left(x, (|8 - 2^p|s)^q\right) \tag{2.11}$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. Let $J_n f$ be defined as in Theorem 2.2. Since f is an odd mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(2^n x)+f(-2^n x)}{2^{3n+1}} & \text{if } 0 < q < \frac{1}{3}, \\ 2^{3n-1}(f(2^{-n}x) + f(-2^{-n}x)) & \text{if } q > \frac{1}{3} \end{cases}$$

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{1}{2^{3j+4}}(Df(0, -2^j x) - Df(0, 2^j x)) & \text{if } 0 < q < \frac{1}{3}, \\ 2^{3j-1}\left(Df\left(0, \frac{x}{2^{j+1}}\right) - Df\left(0, \frac{-x}{2^{j+1}}\right)\right) & \text{if } q > \frac{1}{3} \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.2, we obtain a quadratic and cubic mapping F satisfying (2.11). Notice that F is also odd, $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ for all $x \in X$, and $DF(x,y) = 0$ for all $x, y \in X$. Hence, we get

$$F(x+2y) - 3F(x+y) + 3F(x) - F(x-y) - 6F(y) = DF(x,y) = 0$$

for all $x, y \in X$. This means that F is an cubic maipping. □

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then we can define a fuzzy norm N_X on X by following

$$N_X(x,t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$, see [17]. Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, |||\cdot|||)$ such that

$$|||Df(x,y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 2, 3$. Let N_Y be a fuzzy norm on Y. Then we get

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & s+t \leq |||Df(x,y)||| \\ 1, & s+t > |||Df(x,y)||| \end{cases}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x,y), s+t) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq |||Df(x,y)||| \geq s + t$$

and so either $\|x\|^p \geq s$ or $\|y\|^p \geq t$ in this case. Hence, for $q = \frac{1}{p}$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case, the inequality

$$N_Y(Df(x,y), s+t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}$$

holds. It means that f is a fuzzy q-almost cubic-quadratic mapping, and by Theorem 2.2, we get the following stability result.

Corollary 2.5 (compare with Corollary 3.4 in [13]). *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, |||\cdot|||)$ be a Banach space. If*

$$|||Df(x,y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique quadratic and cubic mapping $F : X \to Y$ such that

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{||x||^p}{4-2^p} & \text{if } 0 < p < 2, \\ \frac{4||x||^p}{(8-2^p)(2^p-4)} & \text{if } 2 < p < 3, \\ \frac{||x||^p}{2^p-8} & \text{if } 3 < p \end{cases}$$

for all $x \in X$.

References

1. T. Aoki: On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2** (1950), 64-66.
2. T. Bag & S.K. Samanta: Finite dimensional fuzzy normed linear spaces. *J. fuzzy Math.* **11** (2003), no. 3, 687-705.
3. S.C. Cheng & J.N. Mordeson: Fuzzy linear operator and fuzzy normed linear spaces. *Bull. Calcutta Math. Soc.* **86** (1994), 429-436.
4. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), 431-436.
5. D. H. Hyers: On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **27** (1941), 222-224.
6. A. K. Katsaras: Fuzzy topological vector spaces II. *Fuzzy Sets and Systems* **12** (1984), 143-154.

7. I. Kramosil & J. Michalek: Fuzzy metric and statistical metric spaces. *Kybernetica* **11** (1975), 326-334.
8. Y.-H. Lee: On the stability of the monomial functional equation. *Bull. Korean Math. Soc.* **45** (2008), 397-403.
9. Y.-H. Lee & K.-W. Jun: On the Stability of Approximately Additive Mappings. *Proc. Amer. Math. Soc.* **128** (2000), 1361-1369.
10. A. K. Mirmostafaee & M. S. Moslehian: Fuzzy versions of Hyers-Ulam-Rassias theorem. *Fuzzy Sets and Systems* **159** (2008), 720-729.
11. M. Mursaleen & S.A. Mohiuddine: On stability of a cubic functional equation in intuitionistic fuzzy normed spaces. *Chaos, Solitons & Fractals* **42** (2009), 2997-3005.
12. Th. M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
13. W. Towanlong & P. Nakmahachalasint: A mixed-type quadratic and cubic functional equation and its stability. *Thai Journal of Mathematics Special Issue (Annual Meeting in Mathematics)* (2010), 61-71.
14. S.M. Ulam: *A Collection of Mathematical Problems*. Interscience, New York (1960).

[a]Department of Mathematics Education, Gongju National University of Education, Gongju 314-711, Republic of Korea
Email address: chjlee@gjue.ac.kr

[b]Department of Mathematics Education, Gongju National University of Education, Gongju 314-711, Republic of Korea
Email address: lyhmzi@gjue.ac.kr