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ASYMPTOTIC BEHAVIORS OF ALTERNATIVE JENSEN FUNCTIONAL EQUATIONS-REVISITED

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Abstract. In this paper, using an efficient change of variables we refine the Hyers-Ulam stability of the alternative Jensen functional equations of J. M. Rassias and M. J. Rassias and obtain much better bounds and remove some unnecessary conditions imposed in the previous result. Also, viewing the fundamentals of what our method works, we establish an abstract version of the result and consider the functional equations defined in restricted domains of a group and prove their stabilities.

1. Introduction

The Hyers-Ulam stability problems of functional equations was originated by S. M. Ulam in 1940 when he proposed the following question [13]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

Then does there exist a group homomorphism h and $\delta_\delta > 0$ such that

$$d(f(x),\, h(x)) \le \delta_\delta$$

for all $x \in G_1$?

One of the early theorems to be obtained is the following result, essentially due to Hyers [5], that gives an answer for the question of Ulam.

Theorem 1.1. *Suppose that S is an additive semigroup, Y is a Banach space, $\delta \ge 0$, and $f : S \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1.1}$$

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for all $x, y \in S$. *Then there exists a unique function* $A : S \to Y$ *satisfying*

$$A(x+y) = A(x) + A(y) \tag{1.2}$$

for which

$$\|f(x) - A(x)\| \le \delta \tag{1.3}$$

for all $x \in S$.

In 1949-1951 this result was generalized by the authors T. Aoki [1] and D.G. Bourgin [2, 3]. In 1978 Th. M. Rassias generalized the Hyers' result to a new approximately linear mappings [10]. Since then the stability problems have been investigated in various directions for many other functional equations. Among the results, the stability problem in a restricted domain was investigated by F. Skof, who proved the stability problem of the inequality (1.1) in a restricted domain [11] (see [4, 6, 7, 8, 9, 12] for related results). In this paper we refine the result in [7] in both the conditions and the bounds. Also, we consider the functional equations defined in general restricted domains in a group and prove their stabilities.

2. Refined Results

Throughout this paper we denote by X, Y a real normed space and a Banach space, respectively. Throughout this paper, the terminology *domain* means a subset of $X \times X$. Let $f : X \to Y$, $d > 0$ and $\delta \ge 0$. In [7] the authors considered the alternative Jensen functional inequalities

$$\|f(x+y) + f(x-y) + 2f(-x)\| \le \delta \tag{2.1}$$

$$\|f(x+y) - f(x-y) + 2f(-y)\| \le \delta \tag{2.2}$$

$$\left\|2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)\right\| \le \delta \tag{2.3}$$

$$\left\|2f\left(-\frac{x-y}{2}\right) + f(x) - f(y)\right\| \le \delta \tag{2.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge d$.

As main results they obtained the following Hyers-Ulam stability of the inequalities (2.1)∼(2.4) in restricted domains $\{(x, y) : \|x\| + \|y\| \ge d\}$ under some of the additional conditions

$$\|f(-x) + f(x)\| \le \delta^* \tag{2.5}$$

$$\|f(2x) - 2f(x)\| \le \delta^* \tag{2.6}$$

for all $x \in X$ with $\|x\| \geq d$, where δ^* will be given later.

Theorem 2.1. *Suppose that* $f : X \to Y$ *satisfies the inequality* (2.1) *and* (2.5) *with* $\delta^* = \delta/2$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq 8\delta + \|f(0)\| \tag{2.7}$$

for all $x \in X$.

Theorem 2.2. *Suppose that* $f : X \to Y$ *satisfies the inequality* (2.2) *and* (2.5) *with* $\delta^* = \delta$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq 21\delta + \|f(0)\| \tag{2.8}$$

for all $x \in X$.

Theorem 2.3. *Suppose that* $f : X \to Y$ *satisfies the inequality* (2.3), (2.5) *with* $\delta^* = \delta/2$ *and* (2.6) *with* $\delta^* = 2\delta + \|f(0)\|$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq 20\delta + 7\|f(0)\| \tag{2.9}$$

for all $x \in X$.

Theorem 2.4. *Suppose that* $f : X \to Y$ *satisfies the inequality* (2.4), (2.5) *with* $\delta^* = \delta$ *and* (2.6) *with* $\delta^* = 3\delta + \|f(0)\|$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq 24\delta + 4\|f(0)\| \tag{2.10}$$

for all $x \in X$.

In the followings, we can see that the conditions (2.5) and (2.6) are superfluous and the bounds are much smaller than those in the above results.

We call the function A satisfying (1.2) an additive function. We first prove the stability of the inequality (2.1) which refines Theorem 2.1.

Theorem 2.5. *Let* $d > 0$ *and* $\delta \geq 0$ *be fixed. Suppose that* $f : X \to Y$ *satisfies the inequality* (2.1) *for all* $x, y \in X$, *with* $\|x\| + \|y\| \geq d$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \leq 2\delta \tag{2.11}$$

for all $x \in X$.

Proof. For given $x, y \in X$, choose a $z \in X$ such that

$$\|z\| \geq d + \|x - y\|.$$

Then by the triangle inequality we have

$$\| - x - y\| + \|x - y + z\| \geq d \tag{2.12}$$

$$\| - x\| + \|x + z\| \geq d \tag{2.13}$$

$$\| - y\| + \| - y + z\| \geq d \tag{2.14}$$

$$\|z\| \geq d. \tag{2.15}$$

Replacing in (2.1), $x \to -x - y, y \to x - y + z; x \to -x, y \to x + z; x \to -y, y \to -y + z; x \to 0, y \to z$, respectively, we have in view of (2.12) $\sim$ (2.15),

$$\|f(-2y + z) + f(-2x - z) + 2f(x + y)\| \leq \delta \tag{2.16}$$

$$\|f(z) + f(-2x - z) + 2f(x)\| \leq \delta \tag{2.17}$$

$$\|f(-2y + z) + f(-z) + 2f(y)\| \leq \delta \tag{2.18}$$

$$\|f(z) + f(-z) + 2f(0)\| \leq \delta. \tag{2.19}$$

From (2.16)$\sim$(2.19), using the triangle inequality and dividing the result by 2, we have

$$\|f(x + y) - f(x) - f(y) + f(0)\| \leq 2\delta. \tag{2.20}$$

From (2.20), using Theorem 1.1, we get the result. □

We now refine Theorem 2.2.

Theorem 2.6. *Let $d > 0$ and $\delta \geq 0$ be fixed. Suppose that $f : X \to Y$ satisfies the inequality* (2.2) *for all $x, y \in X$, with $\|x\| + \|y\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{3}{2}\delta \tag{2.21}$$

for all $x \in X$.

Proof. For given $x, y \in X$, choose a $z \in X$ such that

$$\|z\| \geq d + \|x - y\|.$$

Then by the triangle inequality we have

$$\|x - y + z\| + \| - x - y\| \geq d \tag{2.22}$$

$$\|x + z\| + \| - x\| \geq d \tag{2.23}$$

$$\| - y + z\| + \| - y\| \geq d. \tag{2.24}$$

Replacing in (2.2), $x \to x-y+z, y \to -x-y; x \to x+z, y \to -x; x \to -y+z, y \to -y$, respectively, we have

$$\|f(-2y + z) - f(2x + z) - 2f(x + y)\| \leq \delta \tag{2.25}$$

$$\|f(z) - f(2x + z) - 2f(x)\| \leq \delta \tag{2.26}$$

$$\|f(-2y + z) - f(z) - 2f(y)\| \leq \delta. \tag{2.27}$$

From (2.25)∼(2.27), using the triangle inequality and dividing the result by 2, we have

$$\|f(x + y) - f(x) - f(y)\| \leq \frac{3}{2}\delta. \tag{2.28}$$

Now by Theorem 1.1, we get the result. □

Finally we refine Theorem 2.3, Theorem 2.4.

Theorem 2.7. *Let $d > 0$ and $\delta \geq 0$ be fixed. Suppose that $f, g, h : X \to Y$ satisfy*

$$\|f(x + y) - g(x) - h(y)\| \leq \delta \tag{2.29}$$

for all $x, y \in X$, with $\|x\| + \|y\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 4\delta \tag{2.30}$$

for all $x \in X$.

Proof. For given $x, y \in X$, choose a $z \in X$ such that

$$\|z\| \geq \frac{1}{2}(d + \|x\| + \|y\|).$$

Then by the triangle inequality we have

$$\|x - z\| + \|y + z\| \geq d \tag{2.31}$$

$$\|x - z\| + \|z\| \geq d \tag{2.32}$$

$$\| - z\| + \|y + z\| \geq d \tag{2.33}$$

$$\| - z\| + \|z\| \geq d. \tag{2.34}$$

Replacing in (2.29), $x \to x-z, y \to y+z; x \to x-z, y \to z; x \to -z, y \to y+z; x \to -z, y \to z$, respectively, using (2.3) and the triangle inequality we have

$$\begin{aligned} \|f(x+y)-f(x)-f(y)+f(0)\| &\le \|f(x+y)-g(x-z)-h(y+z)\| \\ &\quad + \|-f(x)+g(x-z)+h(z)\| \\ &\quad + \|-f(y)+g(-z)+h(y+z)\| \\ &\quad + \|f(0)-g(-z)-h(z)\| \\ &\le 4\delta. \end{aligned} \tag{2.35}$$

Now by Theorem 1.1, there exists a unique additive function $A : X \to Y$ such that

$$\|f(x)-A(x)-f(0)\| \le 4\delta$$

for all $x \in X$. This completes the proof. □

Replacing $f(x)$ by $2f\left(-\frac{x}{2}\right)$ and both $g(x), h(x)$ by $-f(x)$ in (2.29) we obtain the following.

Corollary 2.8. *Let $d > 0$, $\delta \ge 0$. Suppose that $f : X \to Y$ satisfies*

$$\left\|2f\left(-\frac{x+y}{2}\right)+f(x)+f(y)\right\| \le \delta \tag{2.36}$$

for all $x, y \in X$, with $\|x\|+\|y\| \ge d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x)-A(x)-f(0)\| \le 2\delta$$

for all $x \in X$.

Replacing $f(x)$ by $2f\left(-\frac{x}{2}\right)$, $g(x)$ by $-f(x)$ and $h(y)$ by $f(-y)$ in (2.29) we obtain the following.

Corollary 2.9. *Let $d > 0$, $\delta \ge 0$. Suppose that $f : X \to Y$ satisfies*

$$\left\|2f\left(-\frac{x-y}{2}\right)+f(x)-f(y)\right\| \le \delta \tag{2.37}$$

for all $x, y \in X$, with $\|x\|+\|y\| \ge d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x)-A(x)-f(0)\| \le 2\delta$$

for all $x \in X$.

3. Asymptotic Behaviors

In this section we consider the behaviors of the functions $f : X \to Y$ satisfying each of the following conditions

$$\|f(x+y) + f(x-y) + 2f(-x)\| \to 0, \tag{3.1}$$

$$\|f(x+y) - f(x-y) + 2f(-y)\| \to 0, \tag{3.2}$$

$$\left\|2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)\right\| \to 0, \tag{3.3}$$

$$\left\|2f\left(-\frac{x-y}{2}\right) + f(x) - f(y)\right\| \to 0, \tag{3.4}$$

for all $x, y \in X$ as $\|x\| + \|y\| \to \infty$.

As a consequence of Theorem 2.5 we have the following.

Theorem 3.1. *Suppose that $f : X \to Y$ satisfies the asymptotic condition* (3.1) *for all $x, y \in X$ as $\|x\| + \|y\| \to \infty$. Then there exists a unique additive function $A : X \to Y$ such that*

$$f(x) = A(x) + f(0) \tag{3.5}$$

for all $x \in X$.

Proof. The condition (3.1) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(x+y) + f(x-y) + 2f(-x)\| \le \frac{1}{n} \tag{3.6}$$

for all $\|x\| + \|y\| \ge d_n$. By Theorem 2.5, there exists a unique additive function $A_n : X \to Y$ such that

$$\|f(x) - A_n(x) - f(0)\| \le \frac{2}{n} \tag{3.7}$$

for all $x \in X$. Replacing n by positive integers m, k in (3.7) and using the triangle inequality with the results we have

$$\|A_m(x) - A_k(x)\| \le \frac{2}{m} + \frac{2}{k} \le 4 \tag{3.8}$$

for all $x \in X$. From the additivity of A_m, A_k, it follows that $A_m = A_k$ for all $m, k \in \mathbb{N}$. Letting $n \to \infty$ in (3.7), we get the result.

□

Similarly, using Theorem 2.6, Corollary 2.8 and Corollary 2.9 we have the followings.

Theorem 3.2. *Suppose that $f : X \to Y$ satisfies the asymptotic condition* (3.2) *for all $x, y \in X$ as $\|x\| + \|y\| \to \infty$. Then f is an unique additive function.*

Theorem 3.3. *Suppose that $f : X \to Y$ satisfies the asymptotic condition* (3.3) *or* (3.4) *for all $x, y \in X$ as $\|x\| + \|y\| \to \infty$. Then $f(x) - f(0)$ is an additive function.*

4. Stability of the Equation in Restricted Domains of a Group

Throughout this section we denote by G an abelian group, e the identity element of G, Y a Banach space, $f : G \to Y$ and U a subset of $G \times G$ with a certain condition. We consider the alternative Jensen functional inequalities

$$\|f(xy) + f(xy^{-1}) + 2f(x^{-1})\| \le \delta \tag{4.1}$$

$$\|f(xy) - f(xy^{-1}) + 2f(y^{-1})\| \le \delta \tag{4.2}$$

$$\left\|2f(x^{-1}y^{-1}) + f(x^2) + f(y^2)\right\| \le \delta \tag{4.3}$$

$$\left\|2f(x^{-1}y) + f(x^2) - f(y^2)\right\| \le \delta \tag{4.4}$$

for all $(x, y) \in U \subset G \times G$.

We will impose some conditions on U. We denote by $G \times G = \{(a_1, a_2) : a_1, a_2 \in G\}$ the product group, i.e., for $a = (a_1, a_2)$, $b = (b_1, b_2) \in G \times G$ we define $ab = (a_1b_1, a_2b_2)$, $b^{-1} = (b_1^{-1}, b_2^{-1})$. For a subset H of $G \times G$ and $(a_1, a_2) \in G \times G$, we define $(a_1, a_2)H = \{(h_1a_1, h_2a_2) : (h_1, h_2) \in H\}$.

For given $x, y \in G$ we denote by $P_{x,y}$, $Q_{x,y}$ the set of points in $G \times G$,

$$\begin{aligned} P_{x,y} &= \{(e, e), (x, x^{-1}), (y, y), (xy, x^{-1}y)\}, \\ Q_{x,y} &= \{(x, x^{-1}), (y, y), (xy, x^{-1}y)\}, \\ R_{x,y} &= \{(e, e), (x, e), (e, y), (x, y)\}. \end{aligned}$$

The set $P_{x,y}$ and $Q_{x,y}$ can be viewed as vertices of a parallelogram in $G \times G$ and $R_{x,y}$ can be viewed as the vertices of a rectangle in $G \times G$.

Definition 4.1. Let $U \subset G \times G$. We introduce the following conditions $(C1)$, $C2)$ and $(C3)$ on U: For any $x, y \in G$, there exists a $z \in G$ such that

$(C1)$ $\quad (e, z)P_{x,y} = \{(e, z), (x, x^{-1}z), (y, yz), (xy, x^{-1}yz)\} \subset U,$

$(C2)$ $\quad (z, e)Q_{x,y} = \{(xz, x^{-1}), (yz, y), (xyz, x^{-1}y)\} \subset U,$

$(C3)$ $\quad (z, z^{-1})R_{x,y} = \{(z, z^{-1}), (xz, z^{-1}), (z, yz^{-1}), (xz, yz^{-1})\} \subset U,$

respectively.

The sets $(e,z)P_{x,y}, (z,e)Q_{x,y}$ and $(z,z^{-1})R_{x,y}$ can be understood as the translations of $P_{x,y}, Q_{x,y}$ and $R_{x,y}$ by (e,z) and (z,e) and (z,z^{-1}), respectively.

We refer the reader to [4] for several examples of the sets U satisfying some of the conditions $(C1)$, $(C2)$ and $(C3)$.

Theorem 4.2. *Let $U \subset G \times G$ satisfy the condition $(C1)$ and $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies (4.1) for all $(x,y) \in U$. Then there exists a function $A : G \to Y$ satisfying the Cauchy equation*

$$A(xy) - A(x) - A(y) = 0$$

such that

$$\|f(x) - A(x) - f(e)\| \leq 2\delta \tag{4.5}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that

$$(e,z)P_{x^{-1},y^{-1}} \subset U.$$

Replacing in (4.1), (x,y) by $(x^{-1}y^{-1}, xy^{-1}z), (x^{-1}, xz), (y^{-1}, y^{-1}z), (e,z)$, respectively, we have

$$\|f(y^{-2}z) + f(x^{-2}z^{-1}) + 2f(xy)\| \leq \delta \tag{4.6}$$

$$\|f(z) + f(x^{-2}z^{-1}) + 2f(x)\| \leq \delta \tag{4.7}$$

$$\|f(y^{-2}z) + f(z^{-1}) + 2f(y)\| \leq \delta \tag{4.8}$$

$$\|f(z) + f(z^{-1}) + 2f(e)\| \leq \delta. \tag{4.9}$$

From (4.6)∼(4.9), using the triangle inequality and dividing the result by 2, we have

$$\|f(x+y) - f(x) - f(y) + f(e)\| \leq 2\delta. \tag{4.10}$$

From (4.10), using Theorem 1.1, we get the result. □

Theorem 4.3. *Let $U \subset G \times G$ satisfy the condition $(C2)$ and $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies (4.2) for all $(x,y) \in U$. Then there exists a function $A : G \to Y$ satisfying the Cauchy equation*

$$A(xy) - A(x) - A(y) = 0$$

such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\delta \tag{4.11}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that

$$(e, z)Q_{x,y^{-1}} \subset U.$$

Replacing in (4.2), (x, y) by $(xy^{-1}z, x^{-1}y^{-1}), (xz, x^{-1}), (y^{-1}z, y^{-1})$, respectively, we have

$$\|f(y^{-2}z) - f(x^2z) + 2f(xy)\| \le \delta \tag{4.12}$$

$$\|f(z) - f(x^2z) + 2f(x)\| \le \delta \tag{4.13}$$

$$\|f(y^{-2}z) - f(z) + 2f(y)\| \le \delta. \tag{4.14}$$

From (4.12)∼(4.14), using the triangle inequality and dividing the result by 2, we have

$$\|f(xy) - f(x) - f(y)\| \le \frac{3}{2}\delta. \tag{4.15}$$

From (4.15), using Theorem 1.1, we get the result. □

Theorem 4.4. *Let $U \subset G \times G$ satisfy the condition $(C3)$ and $\delta \ge 0$. Suppose that $f : G \to Y$ satisfies (4.3) for all $(x, y) \in U$. Then there exists a function $A : G \to Y$ satisfying the Cauchy equation*

$$A(xy) - A(x) - A(y) = 0$$

such that

$$\|f(x) - A(x) - f(e)\| \le 2\delta \tag{4.16}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that

$$(z, z^{-1})R_{x^{-1},y^{-1}} \subset U.$$

Replacing in (4.3), (x, y) by $(x^{-1}z, y^{-1}z^{-1}), (x^{-1}z, z^{-1}), (z, y^{-1}z^{-1}), (z, z^{-1})$, respectively, we have

$$\|2f(xy) + f(x^{-2}z^2) + f(y^{-2}z^{-2})\| \le \delta \tag{4.17}$$

$$\|2f(x) + f(x^{-2}z^2) + f(z^{-2})\| \le \delta \tag{4.18}$$

$$\|2f(y) + f(z^2) + f(y^{-2}z^{-2})\| \le \delta \tag{4.19}$$

$$\|2f(e) + f(z^2) + f(z^{-2})\| \le \delta. \tag{4.20}$$

From (4.17)∼(4.20), using the triangle inequality and dividing the result by 2, we have

$$\|f(xy) - f(x) - f(y) + f(e)\| \le 2\delta. \tag{4.21}$$

From (4.21), using Theorem 1.1, we get the result. □

Theorem 4.5. *Let $U \subset G \times G$ satisfy the condition $(C3)$ and $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies (4.4) for all $(x, y) \in U$. Then there exists a function $A : G \to Y$ satisfying the Cauchy equation*

$$A(xy) - A(x) - A(y) = 0$$

such that

$$\|f(x) - A(x) - f(e)\| \leq 2\delta \tag{4.22}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that

$$(z, z^{-1})R_{x^{-1},y} \subset U.$$

Replacing in (2.3), (x, y) by $(x^{-1}z, yz^{-1}), (x^{-1}z, z^{-1}), (z, yz^{-1}), (z, z^{-1})$, respectively, we have

$$\|2f(xy) + f(x^{-2}z^2) - f(y^2z^{-2})\| \leq \delta \tag{4.23}$$

$$\|2f(x) + f(x^{-2}z^2) - f(z^{-2})\| \leq \delta \tag{4.24}$$

$$\|2f(y) + f(z^2) - f(y^2z^{-2})\| \leq \delta \tag{4.25}$$

$$\|2f(e) + f(z^2) - f(z^{-2})\| \leq \delta. \tag{4.26}$$

From (4.23)∼(4.26), using the triangle inequality and dividing the result by 2, we have

$$\|f(xy) - f(x) - f(y) + f(e)\| \leq 2\delta. \tag{4.27}$$

From (4.27), using Theorem 1.1, we get the result. □

Let $G = X$ be a normed space and let $U_{p,q} = \{(x, y) : x, y \in X, \|px + qy\| \geq d\}$. Then $U_{p,q}$ satisfies $(C1)$ if $q \neq 0$, $(C2)$ if $p \neq 0$ and $(C3)$ if $p \neq q$. Thus, as direct consequences of above results we have the followings.

Corollary 4.6. *Let $d > 0$, $q \neq 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x + y) + f(x - y) + 2f(-x)\| \leq \delta \tag{4.28}$$

for all $x, y \in X$, with $\|px + qy\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\delta \tag{4.29}$$

for all $x \in X$.

Corollary 4.7. *Let $d > 0$ and $p \neq 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x-y) + 2f(-y)\| \leq \delta \tag{4.30}$$

for all $x, y \in X$, with $\|px + qy\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\delta \tag{4.31}$$

for all $x \in X$.

Corollary 4.8. *Let $d > 0$, $p \neq q$. Suppose that $f : X \to Y$ satisfies*

$$\left\|2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)\right\| \leq \delta \tag{4.32}$$

for all $x, y \in X$, with $\|px + qy\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\delta$$

for all $x \in X$.

Corollary 4.9. *Let $d > 0$, $p \neq q$. Suppose that $f : X \to Y$ satisfies*

$$\left\|2f\left(-\frac{x-y}{2}\right) + f(x) - f(y)\right\| \leq \delta \tag{4.33}$$

for all $x, y \in X$, with $\|px + qy\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\delta$$

for all $x \in X$.

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