J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2012.19.4.423 ISSN(Online) 2287-6081
Volume 19, Number 4 (November 2012), Pages 423–435

WEAK BLOCH FUNCTIONS, ϕ-UNIFORM AND ϕ-JOHN DOMAINS

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Abstract. We give some properties of weak Bloch functions and also give some properties of ϕ-uniform domains and ϕ-John domains in terms of moduli of continuity of Bloch functions and weak Bloch functions.

1. Introduction

Suppose that D is a domain in euclidean n-space $\mathbb{R}^n, n \geq 2$. Let $\ell(\gamma)$ denote the euclidean length of an arc γ and $\operatorname{dist}(A, B)$ denote the euclidean distance from A to B for two sets $A, B \subset \mathbb{R}^n$.

A domain D in $\mathbb{R}^n$ is said to be *b-uniform* if there is a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc γ in D which satisfies

$$\ell(\gamma) \leq b|x_1 - x_2|$$

and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \leq b \operatorname{dist}(x, \partial D) \tag{1}$$

for each $x \in \gamma$, where $\gamma(x_j, x)$ is the part of γ between x_j and x. Uniform domains arise in many areas of function theory (see [1], [4], [5]).

We say that a domain D in $\mathbb{R}^n$ is *b-inner uniform* if there is a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc γ in D which satisfies (1) and

$$\ell(\gamma) \leq b\lambda_D(x_1, x_2),$$

where $\lambda_D(x_1, x_2) = \inf \ell(\alpha)$ and infimum is taken over all rectifiable arcs α which join x_1 and x_2 in D. Obviously $|x_1 - x_2| \leq \lambda_D(x_1, x_2)$.

Received by the editors September 21, 2012. Revised November 5, 2012. Accepted Nov. 5, 2012.
2010 *Mathematics Subject Classification.* 30C65, 30F45.
Key words and phrases. the quasihyperbolic metric, Bloch functions, weak Bloch functions, ϕ-uniform and ϕ-John domains.

A domain D in $\mathbb{R}^n$ is said to be *b-John* if there is a constant $b \geq 1$ such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc γ in D with (1).

John domains arise naturally in distortion problems of conformal and quasiconformal mappings [3], [4], [5], [7]. An inner uniform domain is a domain intermediate between a uniform domain and a John domain. In [1, Theorem 1.11], they showed that inner uniform domains in $\mathbb{R}^n$ are Gromov hyperbolic. See also [9], [11] for inner uniform domains.

For each pair of x_1, $x_2 \in D \subset \mathbb{R}^n$, we define the *quasihyperbolic metric* k_D in D by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)},$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. A *quasihyperbolic geodesic* is an arc γ along which the above infimum is obtained.

We have some important bounds for quasihyperbolic metric and the bounds involve metrics j_D and j'_D (see [1], [2], [5], [9], [10], [11]). We define

$$j_D(x_1, x_2) = \frac{1}{2} \log(1 + r_D(x_1, x_2))$$

and

$$j'_D(x_1, x_2) = \frac{1}{2} \log(1 + r'_D(x_1, x_2))$$

for x_1, $x_2 \in D \subset \mathbb{R}^n$, where

$$r_D(x_1, x_2) = \frac{|x_1 - x_2|}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}$$

and

$$r'_D(x_1, x_2) = \frac{\lambda_D(x_1, x_2)}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}.$$

For any proper subdomain D of $\mathbb{R}^n$ we have

$$j_D(x_1, x_2) \leq j'_D(x_1, x_2) \leq k_D(x_1, x_2) \tag{2}$$

for x_1, $x_2 \in D$ [1, (2.4)], [6]. These bounds may be reversed exactly if the domain is uniform (or inner uniform) as follows([2, Theorem 5.3.5], [5], [9], [11]).

Theorem 1.1. *A domain D in $\mathbb{R}^n$ is b-uniform, (b-inner uniform), if and only if there is a constant a such that*

$$k_D(x_1, x_2) \leq a j_D(x_1, x_2),$$

$$(k_D(x_1, x_2) \leq a j'_D(x_1, x_2)),$$

for all $x_1, x_2 \in D$, respectively. Here a and b depend only on each other.

A function f analytic in $D \subset \mathbb{R}^2$ is said to be a *Bloch* function, or $f \in \mathcal{B}(D)$, if

$$||f||_{\mathcal{B}(D)} = \sup_{z \in D} |f'(z)| \mathrm{dist}(z, \partial D) < \infty.$$

A real valued harmonic function u in $D \subset \mathbb{R}^n$ is said to be a *Bloch* function, or $u \in \mathcal{B}_h(D)$, if

$$||u||_{\mathcal{B}_h(D)} = \sup_{x \in D} |\nabla u(x)| \mathrm{dist}(x, \partial D) < \infty.$$

If $f \in \mathcal{B}(D)$, then

$$|f'(z)| \leq ||f||_{\mathcal{B}(D)} \frac{1}{\mathrm{dist}(z, \partial D)},$$

and thus

$$|f(x_1) - f(x_2)| \leq ||f||_{\mathcal{B}(D)} \int_\gamma \frac{ds}{\mathrm{dist}(x, \partial D)},$$

where γ is a rectifiable arc joining x_1 to x_2 in D. Hence we generalize Bloch function in terms of quasihyperbolic metric as follows (see [2], [9]).

For $p \geq 1$, a function $f : D \to \mathbb{R}^p$ in $D \subset \mathbb{R}^n$ is said to be a *weak Bloch* function, or $f \in \mathcal{B}_w(D)$, if there is a constant $m > 0$ such that

$$|f(x_1) - f(x_2)| \leq m k_D(x_1, x_2), \, \forall x_1, x_2 \in D.$$

Let

$$||f||_{\mathcal{B}_w(D)} = \inf\{m > 0 \, : \, |f(x_1) - f(x_2)| \leq m k_D(x_1, x_2), \, \forall x_1, x_2 \in D\}.$$

In section 2, we characterize weak Bloch functions and give some properties of them.

Next we define two distance functions δ_D and η_D on a simply connected proper domain $D \subset \mathbb{R}^2$ by

$$\delta_D(z_1, z_2) = \sup_f |f(z_1) - f(z_2)|,$$

$$\eta_D(z_1, z_2) = \sup_u |u(z_1) - u(z_2)|$$

for all $z_1, z_2 \in D$, where the suprema are taken over all $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}(D)} \leq 1$ and $u \in \mathcal{B}_h(D)$ with $||u||_{\mathcal{B}_h}(D) \leq 1$, respectively. In [2, 5.3] we have that for every $f \in \mathcal{B}(D)$

$$|f(z_1) - f(z_2)| \leq ||f||_{\mathcal{B}(D)} \delta_D(z_1, z_2), \, \forall z_1, z_2 \in D.$$

In [2] and [10], the following relations of the distance functions k_D, δ_D and η_D on a domain D are given.

Theorem 1.2. *In a simply connected proper domain $D \subset \mathbb{R}^2$,*

$$\eta_D(z_1, z_2) = \delta_D(z_1, z_2) \le k_D(z_1, z_2) \le c_0\delta_D(z_1, z_2) = c_0\eta_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, where c_0 is an absolute constant.

In [2] and [10], a simply connected uniform (or John) domain $D \subset \mathbb{R}^2$ is charaterized by moduli of continuity of Bloch functions with respect to j_D (or j'_D) as follows. See also [9], [11].

Theorem 1.3. *Let $D \subset \mathbb{R}^2$ be a simply connected proper domain. Then the followings are equivalent.*

(i) *D is c-uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}(D)$*

$$|f(z_1) - f(z_2)| \le c||f||_{\mathcal{B}(D)} j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

(iii) *There is a constant c such that for $u \in \mathcal{B}_h(D)$*

$$|u(z_1) - u(z_2)| \le c||u||_{\mathcal{B}_h(D)} j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

Theorem 1.4. *Let $D \subset \mathbb{R}^2$ be a simply connected proper domain. Then the followings are equivalent.*

(i) *D is c-John.*

(ii) *D is c-inner uniform.*

(iii) *There is a constant c such that for $f \in \mathcal{B}(D)$*

$$|f(z_1) - f(z_2)| \le c||f||_{\mathcal{B}(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

(iv) *There is a constant c such that for $u \in \mathcal{B}_h(D)$*

$$|u(z_1) - u(z_2)| \le c||u||_{\mathcal{B}_h(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

In [9], there are higher dimensional versions of Theorem 1.3 and Theorem 1.4 as follows.

Theorem 1.5. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is c-uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \le c||f||_{\mathcal{B}_w(D)} j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

(iii) *There is a constant c such that for $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \leq c||u||_{\mathcal{B}_w(D)} j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

Theorem 1.6. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is c-inner uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \leq c||f||_{\mathcal{B}_w(D)} j'_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

(iii) *There is a constant c such that for $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \leq c||u||_{\mathcal{B}_w(D)} j'_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

In [8] we obtained results on moduli of continuity of conjugate harmonic functions in uniform domains as follows.

Theorem 1.7. *Let $D \subset \mathbb{R}^2$ be a simply connected proper b-uniform domain. If $f = u + iv$ is an analytic function in D with $u \in \mathcal{B}_h(D)$ then*

$$|f(z_1) - f(z_2)| \leq m j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constant m depends only on b and $||u||_{\mathcal{B}_h(D)}$.

Theorem 1.8. *Let $D \subset \mathbb{R}^2$ be a simply connected proper b-uniform domain. If $f = u + iv$ is an analytic function in D with*

$$|u(z_1) - u(z_2)| \leq m j_D(z_1, z_2),\ \forall z_1, z_2 \in D,$$

then

$$|f(z_1) - f(z_2)| \leq m_1 j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constant m_1 depends only on b and m.

In section 3 and 4, we give analogous results of above theorems for much larger class of domains such as ϕ-uniform domains and ϕ-John domains with center x_0.

Suppose that ϕ is an increasing self-homeomorphism of the non-negative reals $[0, \infty]$. A domain D in $\mathbb{R}^n$ is said to be *ϕ-uniform* if

$$k_D(x_1, x_2) \leq \phi(r_D(x_1, x_2))$$

for all x_1, $x_2 \in D$. We say that D is *ϕ-John with center x_0* if

$$k_D(x_0, x) \le \phi(r_D(x_0, x))$$

for all $x \in D$ and for some fixed $x_0 \in D$ (see [7]).

Remark 1.9.

(i) The domains which are ϕ-uniform with $\phi(t) = c\log(1+t)$ for some constant c are precisely uniform [5], [7].

(ii) The domains which are ϕ-John with center x_0 with $\phi(t) = c\log(1+t)$ for some constant c have been characterized as domains which satisfy a quasihyperbolic boundary condition [3], [4]. A special subclass of these domains is the class of John domains [4], [7].

2. Properties of Weak Bloch Functions

Theorem 2.1. *Let $f : D \to \mathbb{R}^p$, $p \ge 1$, be a function in $D \subset \mathbb{R}^n$. Then the followings are equivalent.*

(i) $f \in \mathcal{B}_w(D)$.

(ii) *There is a constant m such that*

$$|f(x_1) - f(x_2)| \le m j_D(x_1, x_2),$$

for all $x_1, x_2 \in D$ with $|x_1 - x_2| < \mathrm{dist}(x_1, \partial D)$.

(iii)

$$\sup_{x\in D} |\partial f(x)| \mathrm{dist}(x, \partial D) < \infty,$$

where

$$|\partial f(x)| = \limsup_{|h|\to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

Here all constants depend only on each other.

Proof. The equivalence of (i) and (ii) was proved in [9, Theorem 3.1]. Now we show the equivalence of (i) and (iii). Suppose that (i) holds and fix $x \in D$ and $|h| < \mathrm{dist}(x, \partial D)$. Let γ be the quasihyperbolic geodesic joining x to $x + h$ in D. Then by (i) there is a constant $m > 0$ such that

$$\begin{aligned} |f(x+h) - f(x)| &\le m\int_\gamma \frac{ds}{\mathrm{dist}(w, \partial D)} \le m\int_{[x,\,x+h]} \frac{ds}{\mathrm{dist}(w, \partial D)} \\ &\le m\int_0^{|h|} \frac{ds}{\mathrm{dist}(x, \partial D) - |h|} \\ &\le m\frac{|h|}{\mathrm{dist}(x, \partial D) - |h|}. \end{aligned}$$

Thus

$$|\partial f(x)| = \limsup_{|h|\to 0} \frac{|f(x+h) - f(x)|}{|h|} \le m\mathrm{dist}(x, \partial D)^{-1}.$$

Next suppose that $|\partial f(x)| \le m\mathrm{dist}(x, \partial D)^{-1}$ for all $x \in D$ for some constant $m > 0$. Fix $x_1, x_2 \in D$ and choose a rectifiable curve γ joining x_1, x_2 in D. Let s denote arclength measured along γ from x_1, let $x(s)$ denote the corresponding representation for γ and set $g(s) = f(x(s))$. Then

$$|\partial g(s)| = \limsup_{|h|\to 0} \frac{|g(s+h) - g(s)|}{|h|} \le |\partial f(x(s))|.$$

For $0 < s < \ell = \ell(\gamma)$, g is absolutely continuous and

$$\begin{aligned} |f(x_1) - f(x_2)| = |g(\ell) - g(0)| &\le \int_0^\ell |\partial g(s)| ds \le \int_0^\ell |\partial f(x((s))| ds \\ &\le m \int_0^\ell \mathrm{dist}(x(s), \partial D)^{-1} ds. \end{aligned}$$

Thus

$$|f(x_1) - f(x_2)| \le m k_D(x_1, x_2),$$

for all $x_1, x_2 \in D$. Hence $f \in \mathcal{B}_w(D)$. □

Remark 2.2.

(i) For a domain $D \subset \mathbb{R}^2$, $\mathcal{B}(D)$ is the intersection of $\mathcal{B}_w(D)$ with the class of analytic functions in D [9, Remark 1.2].

(ii) For a domain $D \subset \mathbb{R}^n$, $\mathcal{B}_h(D)$ is the intersection of $\mathcal{B}_w(D)$ with the class of real valued harmonic functions in D [2, Example 5.4.9 (i)].

Proof of Remark 2.2 (ii). (ii) is in [2, Example 5.4.9 (i)] without proof. Here we prove it by Theorem 2.1. If $u \in \mathcal{B}_h(D)$, then

$$|u(x_1) - u(x_2)| \le \int_\gamma |\nabla u| ds \le ||u||_{\mathcal{B}_h(D)} \int_\gamma \frac{ds}{\mathrm{dist}(x, \partial D)}$$

for all $x_1, x_2 \in D$, where γ is a rectifiable curve joining x_1 to x_2 in D. Thus

$$|u(x_1) - u(x_2)| \le ||u||_{\mathcal{B}_h(D)} k_D(x_1, x_2)$$

for all $x_1, x_2 \in D$ and hence $u \in \mathcal{B}_w(D)$.

Suppose that u is a real harmonic function in D and $u \in \mathcal{B}_w(D)$. Then by Theorem 2.1 there is a constant $m > 0$ such that

$$|\partial u(x)| = |\nabla u(x)| = \lim_{|h|\to 0} \frac{|u(x+h) - u(x)|}{|h|} \le m\mathrm{dist}(x, \partial D)^{-1}$$

and hence $u \in \mathcal{B}_h(D)$. □

Lemma 2.3. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. Then $u \in \mathcal{B}_w(D)$ and $||u||_{\mathcal{B}_w(D)} \leq 1$. In general, u is not harmonic.*

Proof. The first part is Lemma 3.6 in [9]. For the second part, suppose that u is harmonic in a proper subdomain $D \subset \mathbb{R}^2$. Since $u(z_0) = 0$ and $u(z) \geq 0$ in D, by the minimum principle for harmonic functions u must be a constant function and thus $u(z) = 0$ for all $z \in D$. It is a contradiction. □

3. Weak Bloch Functions, ϕ-uniform and ϕ-John Domains

Theorem 3.1. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is ϕ-uniform.*

(ii) *For $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \leq ||f||_{\mathcal{B}_w(D)}\phi(r_D(x_1, x_2)),\ \forall x_1, x_2 \in D.$$

(iii) *For $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \leq ||u||_{\mathcal{B}_w(D)}\phi(r_D(x_1, x_2)),\ \forall x_1, x_2 \in D.$$

Proof. First we show that (i) implies (ii). Suppose that D is ϕ-uniform. Then for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,

$$|f(x_1) - f(x_2)| \leq ||f||_{\mathcal{B}_w(D)} k_D(x_1, x_2) \leq ||f||_{\mathcal{B}_w(D)}\phi(r_D(x_1, x_2)),$$

for all x_1, $x_2 \in D$. Next obviously (ii) implies (iii), and we need to show that (iii) implies (i). Suppose that (iii) holds. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. Then by Lemma 2.3 and (iii)

$$k_D(x, x_0) = |u(x) - u(x_0)| \leq \phi(r_D(x, x_0))$$

for all $x \in D$. Thus by triangle inequality

$$k_D(x_1, x_2) \leq \phi(r_D(x_1, x_2))$$

for all x_1, $x_2 \in D$ and D is ϕ-uniform. □

In particular, for $n = 2$ we can characterize ϕ-uniform by $\mathcal{B}(D)$ and $\mathcal{B}_h(D)$ instead of $\mathcal{B}_w(D)$.

Theorem 3.2. *Let $D \subset \mathbb{R}^2$ be a simply connected proper domain. Then the followings are equivalent.*

(i) *D is ϕ-uniform.*

(ii) *For $f \in \mathcal{B}(D)$, $f : D \to \mathbb{R}^2$,*

$$|f(z_1) - f(z_2)| \le ||f||_{\mathcal{B}(D)}\phi(r_D(z_1, z_2)),\ \forall z_1, z_2 \in D.$$

(iii) *For $u \in \mathcal{B}_h(D)$, $u : D \to \mathbb{R}$*

$$|u(z_1) - u(z_2)| \le ||u||_{\mathcal{B}_h(D)}\phi(r_D(z_1, z_2)),\ \forall z_1, z_2 \in D.$$

The ϕ's are not necessarily the same, but they depend only on each other.

Proof. By the exactly same argument in the proof of Theorem 3.1, (i) implies (ii) and (iii). To show that (ii) (or (iii)) implies (i), fix $z_1, z_2 \in D$. Then by Theorem 1.2 and (ii) (or (iii))

$$k_D(z_1, z_2) \le c_0\delta_D(z_1, z_2) = c_0\eta_D(z_1, z_2) \le c_0\phi(r_D(z_1, z_2)),$$

for an absolute constant c_0. Thus D is ϕ_1-uniform with $\phi_1 = c_0\phi$. □

Remark 3.3. If $\phi(t) = c\log(1+t)$ for some constant c in Theorem 3.1 and Theorem 3.2, then we have the same results as Theorem 1.5 and Theorem 1.3, respectively.

Theorem 3.4. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is ϕ-John with center x_0.*

(ii) *There is a point $x_0 \in D$ such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \le 2||f||_{\mathcal{B}_w(D)}\phi(r''_D(x_j, x_0)),\ \forall x_1, x_2 \in D.$$

(iii) *There is a point $x_0 \in D$ such that for $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \le 2||u||_{\mathcal{B}_w(D)}\phi(r''_D(x_j, x_0)),\ \forall x_1, x_2 \in D.$$

Here

$$r''_D(x_j, x_0) = \frac{\max_{j=1,2}|x_j - x_0|}{\min_{j=1,2}\{\mathrm{dist}(x_j, \partial D), \mathrm{dist}(x_0, \partial D)\}}$$

The ϕ's are not necessarily the same, but they depend only on each other.

Proof. First we show that (i) implies (ii). Suppose that D is ϕ-John with center x_0. Then for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,

$$\begin{aligned}
|f(x_1) - f(x_2)| &\le ||f||_{\mathcal{B}_w(D)}k_D(x_1, x_2)\\
&\le ||f||_{\mathcal{B}_w(D)}(k_D(x_1, x_0)) + k_D(x_0, x_2))\\
&\le ||f||_{\mathcal{B}_w(D)}(\phi(r_D(x_1, x_0)) + \phi(r_D(x_0, x_2)))\\
&\le 2||f||_{\mathcal{B}_w(D)}\phi(r''_D(x_j, x_0)).
\end{aligned}$$

for all $x_1, x_2 \in D$.

Next obviously (ii) implies (iii), and we need to show that (iii) implies (i). Suppose that (iii) holds. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. Then by Lemma 2.3 and (iii)

$$k_D(x, x_0) = |u(x) - u(x_0)| \le 2\phi(r''_D(x, x_0)) = 2\phi(r_D(x, x_0))$$

for all $x \in D$. Thus D is ϕ_1-John with center x_0 and $\phi_1 = 2\phi$. □

In particular, for $n = 2$ we can characterize ϕ-John with center x_0 by $\mathcal{B}(D)$ and $\mathcal{B}_h(D)$ instead of $\mathcal{B}_w(D)$.

Theorem 3.5. *Let $D \subset \mathbb{R}^2$ be a simply connected proper domain. Then the followings are equivalent.*

(i) *D is ϕ-John with center z_0.*

(ii) *There is a point $z_0 \in D$ such that for $f \in \mathcal{B}(D)$, $f : D \to \mathbb{R}^2$,*

$$|f(z_1) - f(z_2)| \le 2||f||_{\mathcal{B}(D)}\phi(r''_D(z_j, z_0)),\ \forall z_1, z_2 \in D.$$

(iii) *There is a point $z_0 \in D$ such that for $u \in \mathcal{B}_h(D)$, $u : D \to \mathbb{R}$,*

$$|u(z_1) - u(z_2)| \le 2||u||_{\mathcal{B}_h(D)}\phi(r''_D(z_j, z_0)),\ \forall z_1, z_2 \in D.$$

The ϕ's are not necessarily the same, but they depend only on each other.

Proof. By the exactly same argument in the proof of Theorem 3.4 we can prove that (i) implies (ii) and (iii). Now we show that (ii) (or (iii)) implies (i). By Theorem 1.2 and (ii) (or (iii))

$$k_D(z, z_0) \le c_0\delta_D(z, z_0) = c_0\eta_D(z, z_0) \le c_0 2\phi(r''_D(z, z_0)) \le c_0 2\phi(r_D(z, z_0)),$$

for all $z \in D$ and for an absolute constant c_0. Thus D is ϕ_1-John with center x_0 and $\phi_1 = 2c_0\phi$. □

Remark 3.6. If $\phi(t) = c\log(1 + t)$ for some c in Theorem 3.4 and Theorem 3.5, then we have the results for the class of domains which satisfy the quasihyperbolic boundary condition. Thus the results are also for the class of inner uniform domains.

4. MODULI OF CONTINUITY OF CONJUGATE HARMONIC BLOCH FUNCTIONS

In [8] we obtain a result on conjugate harmonic Bloch functions in a domain as follows.

Theorem 4.1. *Let $D \subset \mathbb{R}^2$ be a simply connected proper subdomain. If $f = u + iv$ is an analytic function in D with $u \in \mathcal{B}_h(D)$, then f is in $\mathcal{B}(D)$ with $||f||_{\mathcal{B}(D)} \leq ||u||_{\mathcal{B}_h(D)}$.*

Now we get some results on a certain moduli of continuity of conjugate harmonic functions in a ϕ-uniform domain, a ϕ-John domain with center z_0 and an inner uniform domain.

Theorem 4.2. *Let $D \subset \mathbb{R}^2$ be a simply connected proper ϕ-uniform domain. If $f = u + iv$ is an analytic function in D with $u \in \mathcal{B}_h(D)$, then*

$$|f(z_1) - f(z_2)| \leq ||u||_{\mathcal{B}_h(D)}\phi(r_D(z_1, z_2)),\ \forall z_1, z_2 \in D.$$

Proof. If $u \in \mathcal{B}_h(D)$, then by Theorem 4.1 $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}(D)} \leq ||u||_{\mathcal{B}_h(D)}$. Since D is ϕ-uniform, by Theorem 3.2

$$|f(z_1) - f(z_2)| \leq ||f||_{\mathcal{B}(D)}\phi(r_D(z_1, z_2)) \leq ||u||_{\mathcal{B}_h(D)}\phi(r_D(z_1, z_2))$$

for all $z_1, z_2 \in D$. □

Corollary 4.3. *Let $D \subset \mathbb{R}^2$ be a simply connected proper ϕ-uniform domain. If $f = u + iv$ is an analytic function in D with*

$$|u(z_1) - u(z_2)| \leq m j_D(z_1, z_2),\ \forall z_1, z_2 \in D,$$

then

$$|f(z_1) - f(z_2)| \leq m\phi(r_D(z_1, z_2)),\ \forall z_1, z_2 \in D.$$

The constants m are not necessarily the same, but they depend only on each other.

Proof. Since u is harmonic in D, by (2)

$$|u(z_1) - u(z_2)| \leq m j_D(z_1, z_2) \leq m k_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Then $u \in \mathcal{B}_w(D)$ and hence by Remark 2.2 (ii) $u \in \mathcal{B}_h(D)$. Then by Theorem 4.2 we get the result. □

Remark 4.4. If $\phi(t) = c\log(1+t)$ for some constant c in Theorem 4.2 and Corollary 4.3, then we have the same results as Theorem 1.7 and Theorem 1.8.

Theorem 4.5. *Let $D \subset \mathbb{R}^2$ be a simply connected proper ϕ-John domain with center z_0. If $f = u + iv$ is an analytic function in D with $u \in \mathcal{B}_h(D)$, then*

$$|f(z_1) - f(z_2)| \leq 2||u||_{\mathcal{B}(D)}\phi(r''_D(z_j, z_0)),\ \forall z_1, z_2 \in D.$$

Proof. If $u \in \mathcal{B}_h(D)$, then by Theorem 4.1 $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}(D)} \leq ||u||_{\mathcal{B}_h(D)}$. Since D is a ϕ-John domain with center z_0, by Theorem 3.5

$$|f(z_1) - f(z_2)| \leq 2||f||_{\mathcal{B}(D)}\phi(r''_D(z_j, z_0)) \leq 2||u||_{\mathcal{B}_h(D)}\phi(r''_D(z_j, z_0))$$

for all $z_1, z_2 \in D$. □

Theorem 4.6. *Let $D \subset \mathbb{R}^2$ be a simply connected proper b-inner uniform domain. If $f = u + iv$ is an analytic function in D with $u \in \mathcal{B}_h(D)$, then*

$$|f(z_1) - f(z_2)| \leq m||u||_{\mathcal{B}_h(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constant m depends only on b.

Proof. If $u \in \mathcal{B}_h(D)$, then by Theorem 4.1 $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}(D)} \leq ||u||_{\mathcal{B}_h(D)}$. Since D is b-inner uniform, by Theorem 1.4 there is a constant m such that

$$|f(z_1) - f(z_2)| \leq m||f||_{\mathcal{B}(D)} j'_D(z_1, z_2) \leq m||u||_{\mathcal{B}_h(D)} j'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. □

Corollary 4.7. *Let $D \subset \mathbb{R}^2$ be a simply connected proper b-inner uniform domain. If $f = u + iv$ is an analytic function in D with*

$$|u(z_1) - u(z_2)| \leq m j'_D(z_1, z_2),\ \forall z_1, z_2 \in D,$$

then

$$|f(z_1) - f(z_2)| \leq m_1||u||_{\mathcal{B}(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constant m_1 depends only on b and m.

Proof. Since u is harmonic in D, by (2)

$$|u(z_1) - u(z_2)| \leq m j'_D(z_1, z_2) \leq m k_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Then $u \in \mathcal{B}_w(D)$ and hence by Remark 2.2 (ii) $u \in \mathcal{B}_h(D)$. Then by Theorem 4.6 we get the result. □

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