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ON ARCWISE CONNECTEDNESS IM KLEINEN IN HYPERSPACES

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Abstract. Let X be a space and $2^X(C(X), \mathcal{K}(X), C_K(X))$ denote the hyperspace of nonempty closed subsets(connected closed subsets, compact subsets, subcontinua) of X with the Vietoris topology. We investigate the relationships between the space X and its hyperspaces concerning the properties of connectedness im kleinen. We obtained the following : Let X be a locally compact Hausdorff space. Let $x \in X$. Then the following statements are equivalent: (1) X is connected im kleinen at x. (2) 2^X is arcwise connected im kleinen at $\{x\}$. (3) $\mathcal{K}(X)$ is arcwise connected im kleinen at $\{x\}$. (4) $C_K(X)$ is arcwise connected im kleinen at $\{x\}$. (5) $C(X)$ is arcwise connected im kleinen at $\{x\}$.

0. Introduction

Let X be a topological space. Let $2^X = \{E \subset X : E$ is nonempty and closed$\}$, $\mathcal{F}_n(X) = \{E \in 2^X : E$ has at most n elements$\}$, $\mathcal{F}(X) = \{E \in 2^X : E$ is finite$\}$, $\mathcal{K}(X) = \{E \in 2^X : E$ is compact$\}$, $C(X) = \{E \in 2^X : E$ is connected$\}$, $C_K(X) = \mathcal{K}(X) \cap C(X)$.

In 1998, Goodykoontz[3] proved that a Hausdorff space X is connected im kleinen at $x \in X$ if and only if $2^X(\mathcal{K}(X),\ C_K(X))$ is connected im kleinen at $\{x\}$(Result 2.D) and a locally compact Hausdorff space X is connected im kleinen at $x \in X$ if and only if $2^X(\mathcal{K}(X),\ C_K(X),\ C(X))$ is connected im kleinen at $\{x\}$(Result 2.F). In this paper, we will prove that a Hausdorff space X is connected im kleinen at $x \in X$ if and only if $\mathcal{F}_n(X)(\mathcal{F}(X))$ is connected im kleinen at $\{x\}$(Theorem 2.1) and a locally compact Hausdorff space X is connected im kleinen at $x \in X$ if and only if $2^X(\mathcal{K}(X),\ C_K(X),\ C(X))$ is arcwise connected im kleinen at $\{x\}$(Theorem 2.2).

For notational purposes, small letters will denote elements of X, capital letters

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will denote subsets of X and elements of 2^X, and script letters are reserved for subsets of 2^X. If $\mathcal{B} \subset 2^X$, $\cup\mathcal{B} = \{A : A \in \mathcal{B}\}$. If $A \subset X$, $\overline{A}$, $Int(A)$, $Bd(A)$ will denote the closure, interior, boundary of A in X respectively.

1. Preliminaries

Let U_1, U_2,..., U_n be a collection of subsets of X. Let $< U_1, U_2, ..., U_n >$ to be the set of all $E \in 2^X$ such that $E \subset \cup_{i=1}^{i=n} U_i$ and $E \cap U_i \neq \emptyset$ for each $i = 1, 2, ..., n$.

If $(X, \mathcal{T})$ is a topological space, then the Vietoris (finite) topology $\mathcal{T}_v$ on 2^X is the one generated by the collection of the form $< U_1, ..., U_n >$ with U_1, U_2,... ,U_n open subsets of X.

The topology on each of $\mathcal{F}_n(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$, $C(X)$, $C_K(X)$ is the subspace topology induced by the Vietoris topology on 2^X.

Result 1.A ([5]). *Let $(X, \mathcal{T})$ be a topological space. then the collection of the form $< U_1, ..., U_n >$ with U_1, U_2,..., U_n open in X, form a basis for the finite topology on 2^X.*

Result 1.B ([5]). *Let $(X, \mathcal{T})$ be a toplogical space. Then:*

(a) *$\{E \in 2^X : E \subset A\}$ is closed in 2^X if $A \subset X$ is closed.*
(b) *$\{E \in 2^X : E \cap A \neq \emptyset\}$ is closed in 2^X if $A \subset X$ is closed.*

Proposition 1.1. *Let X be a T_1 space. Then the natural map $i : X \to 2^X$ defined by $i(x) = \{x\}$ for each $x \in X$ is continuous. Since X is a T_1 space, each singleton set $\{x\}$ is a member of 2^X. And any neighborhood of $\{x\}$ in 2^X has the form $< U >$, where U is open in X so that $i(U) \subset < U >$. Hence i is continuous. Furthermore, i is a homeomorphism between X and the subspace $\mathcal{F}_1(X)$.*

Result 1.C ([5]). *Let X be a T_1 space. Then*

(a) *$< U_1, ..., U_n > \subset < V_1, ..., V_m >$ if and only if $\cup_{i=1}^{n} U_i \subset \cup_{i=1}^{m} V_i$, and for each V_i there exists a U_j such that $U_j \subset V_i$.*
(b) *$\overline{< U_1, ..., U_n >} = < \overline{U_1}, ..., \overline{U_n} >$.*
(c) *If $\{U_\alpha\}_{\alpha \in A}$ is a neighborhood basis at $x \in X$, then $\{< U_\alpha >\}_{\alpha \in A}$ is a neighborhood basis at $\{x\}$ in 2^X.*
(d) *If $\mathcal{O}$ is an open set in 2^X, then $\cup\mathcal{O}$ is open in X.*

On the other hand, if U is open in X, then $2^U = < U >$ is open in 2^X.

Proposition 1.2. *Let X be a T_1 space. If $\mathcal{U}$ is an open set in the subspace $\mathcal{K}(X)$, then $\cup\mathcal{U}$ is open in X.*

Proposition 1.3. *Let X be a T_1 space. If $\mathcal{U}$ is an open subset of the subspace $\mathcal{F}(X)$, then $\cup\mathcal{U}$ is open in X.*

Proposition 1.4. *Let X be a T_1 space. If $\mathcal{U}$ is an open set in $\mathcal{F}_n(X)$, then $\cup\mathcal{U}$ is open in X.*

A T_1 space X is call *regular* if each $x \in X$ and closed set A not containing x have disjoint neighborhoods.

Proposition 1.5. (a) *Suppose X is a locally connected regular space. If $\mathcal{U}$ is an open set in the subspace $C(X)$, then $\cup\mathcal{U}$ is open in X.*

(b) *Suppose X is a locally compact and locally connected Hausdorff space. If $\mathcal{O}$ is an open subset of the subspace $C_K(X)$, then $\cup\mathcal{O}$ is open in X.*

Result 1.D ([5])**.** (a) *Let X be a T_1 space. Then $\mathcal{F}(X)$ is dense in 2^X.*

(b) *If X is Hausdorff, the $\mathcal{F}_n(X)$ is closed in 2^X for each n.*

(c) *Let X be a T_1 space. Then the natural map $f : X^n \to \mathcal{F}_n(X)$, defined by $f((x_1, ..., x_n)) = \{x_1, ..., x_n\}$, is continuous, surjective, and open.*

Remark 1.D'. (1) *Assertion (c) is false for infinite product.*

(2) *There is a Hausdorff space X such that no $\mathcal{F}_n(X)$ is dense in 2^X. Let X be an infinite Hausdorff space. For a fixed n, let $U_1, ..., U_{n+1}$ be pairwise disjoint nonempty open subsets of X. Then $\mathcal{F}_n(X) \cap < U_1, ..., U_{n+1} >= \emptyset$. Hence for a Hausdorff space X, X is finite if and only if $\mathcal{F}_n(X)$ is dense in 2^X for some n.*

(3) *A space X is discrete if and only if 2^X is discrete.*

Proposition 1.6. *Let X be a T_1 space. Then $\mathcal{K}(X)$ is dense in 2^X.*

Result 1.E ([5])**.** (a) *Let X be a regular space. Then $\cup\mathcal{B} = \cup\{E : E \in \mathcal{B}\} \in 2^X$, for each $\mathcal{B} \in \mathcal{K}(2^X)$.*

(b) *Let X be a space (no separation axiom is assumed). Then $\cup\mathcal{B} \in \mathcal{K}(X)$, for each $\mathcal{B} \in \mathcal{K}(\mathcal{K}(X))$.*

Result 1.F ([5])**.** *Let X be a topological space. If $\mathcal{B}$ is a connected subset of 2^X which also contains at least one connected element, then $\cup\mathcal{B}$ is connected in X.*

Proposition 1.7. *Let X be a topological space. If $\mathcal{B}$ is a connected subset of $\mathcal{F}_n(X)$ ($\mathcal{F}(X)$, $\mathcal{K}(X)$) which contains a connected element, then $\cup\mathcal{B}$ is connected in X.*

In particular, if $\mathcal{B}$ is a connected subset of $C_K(X)$ (or $C(X)$), then $\cup\mathcal{B}$ is connected.

Example 1.7.1. (1) We give an example of a connected subset $\mathcal{B}$ of 2^X which contains no connected element and $\cup\mathcal{B}$ is not connected.
Let X be the space of reals. Let U_1 and U_2 be connected open set such that $U_1 \cap U_2 = \emptyset$. Let $\mathcal{B} = < U_1, U_2 >$ (or $\mathcal{B} = < U_1, U_2 > \cap \mathcal{F}(X)$. Then by Proposition 4.11 in [5], $\mathcal{B}$ is connected and $\cup\mathcal{B} = U_1 \cup U_2$.

(2) Here is an example of a connected subset $\mathcal{B}$ of $\mathcal{F}_2(X)$ which contains no connected element such that $\cup\mathcal{B}$ is disconnected. Let X be the space of the reals. Let U_1 and U_2 be disjoint connected open sets in X. Let $\mathcal{B} = < U_1, U_2 > \cap \mathcal{F}_2(X)$. Then $\mathcal{B}$ is the continuous image of the connected set $U_1 \times U_2$ by Result 1.D.(c). Hence $\mathcal{B}$ is connected. And $\cup\mathcal{B} = U_1 \cup U_2$.

(3) This is an example of a disconnected subset $\mathcal{B}$ of 2^X which contains no connected element such that $\cup\mathcal{B}$ is connected. Let X be the space of reals. Let $A = [0,1] \cup \{2\}$, $B = [1,2] \cup \{0\}$, and $\mathcal{B} = \{A, B\}$. Then $\cup\mathcal{B} = [0,2]$.

Result 1.G ([2]). *Let X be a T_1 space. Let U be a connected open set and $U_1, \cdots, U_n$ be nonempty open sets such that $U = \cup_{i=1}^n U_i$. Then $< U_1, .., U_n >$ is connected in 2^X.*

Result 1.H ([4]). *If X is a compact connected Hausdorff space, then 2^X and $C(X)$ are (arcwise) connected.*

2. CONNECTEDNESS IM KLEINEN AND ARCWISE CONNECTEDNESS IM KLEINEN

Definition 2.1. The space X is *locally connected* at $x \in X$ provided that for each neighborhood U of x there is a connected neighborhood V of x such that $V \subset U$. The space X is *connected im kleinen* at x provided for each neighborhood U of x there is a component of U which contains x in its interior. The space X is *locally connected* provided that X is locally connected at each of its points. If a space X is connected im kleinen at each of its points, then X is locally connected.

Definition 2.2. The space X is *locally arcwise connected at x* provided that for each neighborhood U of x there is an arcwise connected neighborhood V of x such that $V \subset U$. The space X is *locally arcwise connected* provided that X is locally arcwise connected at each of its points. The space X is *arcwise connected im kleinen at x* provided for each neighborhood U of x there is an arcwise connected, component of U which contains x in its interior. If a space X is arcwise connected im kleinen at each of its points, then X is locally arcwise connected.

Result 2.A ([6])**.** *Let X be a locally compact Hausdorff space. Let $A_1, A_2 \in \mathcal{U} =< U_1, \cdots, U_n > \cap K(X), U_i$ open. Then there exists an arc in $\mathcal{U}$ between A_1 and A_2 if and only if there exists an element $B \in \mathcal{U}$ such that $A_i \subset B, i = 1, 2$ and each component of B intersects $A_i, i = 1, 2$.*

Proposition 2.1. *X is connected im kleinen at x if and only if for each open set U containing x, there exists an open set V containing x such that for each $y \in V$, there exists a connected subset C_y of U containing x and y.*

Proof. Suppose that X is connected im kleinen at x. Let U be an open set containing x. Then there exists a component M of U containing x in its interior and so $V = Int(M)$.

Conversely, let U be an open set containing x. Then there exists an open neighborhood V of x such that for each $y \in V$, there exists a connected subset C_y of U containing x and y. Then $M = \cup_{y \in V} C_y$, $x \in Int(M) \subset U$ and M is connected. □

Result 2.B ([3])**.** *Let X be a Hausdorff space. Then the following are equivalent:*

(1) *X is connected;*
(2) *$\mathcal{F}_n(X)$ is connected;*
(3) *$\mathcal{F}(X)$ is connected:*
(4) *$C_K(X)$ is connected;*
(5) *$\mathcal{K}(X)$ is connected;*
(6) *2^X is connected.*

Result 2.C ([3])**.** *Let X be a Hausdorff space. Let $x \in X$. Then X is connected im kleinen at x if and only if $C_K(X)$ is connected im kleinen at $\{x\}$.*

Result 2.D ([3])**.** *Let X be a Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*

(1) *X is connected im kleinen at x*
(2) *2^X is connected im kleinen at $\{x\}$*
(3) *$\mathcal{K}(X)$ is connected im kleinen at $\{x\}$*
(4) *$C_K(X)$ is connected im kleinen at $\{x\}$.*

Remark 2.D'. *The equivalence of (1), (2), and (3) above when "connected im kleinen" is replaced by "locally connected" is given in* [1].

Theorem 2.1. *Let X be a Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*

(1) *X is connected im kleinen at x*
(2) *$\mathcal{F}_n(X)$ is connected im kleinen at $\{x\}$*
(3) *$\mathcal{F}(X)$ is connected im kleinen at $\{x\}$.*

Proof. (1)⇒(2). Suppose that X is connected im kleinen at x. Let $< U > \cap \mathcal{F}_n(X)$ be a basic open set containing $\{x\}$. Since X is connected im kleinen at x and $x \in U$, U has the component C containing x in its interior. Let V be an open set containing x such that $x \in V$ and $V \subset Int(C)$. Then $\{x\} \in < V > \cap \mathcal{F}_n(X) \subset < C > \cap \mathcal{F}_n(C) \subset \mathcal{F}_n(C) \subset < U > \cap \mathcal{F}_n(X)$ and $\mathcal{F}_n(C)$ is connected by Result 2.B. Hence $< U > \cap \mathcal{F}_n(X)$ has a component containing $\{x\}$ in its interior.
(2)⇒(1). Suppose that $\mathcal{F}_n(X)$ is connected im kleinen at $\{x\}$. Let U be an open set such that $x \in U$. Since $\mathcal{F}_n(X)$ is connected im kleinen at $\{x\}$, there is a component $\mathcal{L}$ of $< U > \cap \mathcal{F}_n(X)$ containing $\{x\}$ in its interior. Let V be an open set such that $\{x\} \in < V > \cap \mathcal{F}_n(X) \subset \mathcal{L} \subset < U > \cap \mathcal{F}_n(X)$. Then $x \in V \subset \cup\mathcal{L} \subset U$. Since $\cup\mathcal{L}$ is connected by Proposition 1.7, U contains a component containing x in its interior. Hence X is connected im kleinen at x. Proof for (1)⇔(3) is very much similar to above. □

Result 2.E ([2])**.** *Let X be a locally compact Hausdorff space. Let $x \in X$. Then X is connected im kleinen at x if and only if $C(X)$ is connected im kleinen at $\{x\}$.*

Proof. Suppose that X is connected im kleinen at x. Let $< U > \cap C(X)$ be a basic open set in $C(X)$ containing $\{x\}$. Let V be a neighborhood of x with compact closure such that $\overline{V} \subset U$. Then there exists a component M of V which contains x in its interior. Let $W = Int(M)$. Then $\{x\} \in < W > \cap C(X) \subset < V > \cap C(X) \subset < U > \cap C(X)$. If $E \in < W > \cap C(X)$, then $E \in C_K(X)$. Then $C(E)$ is a connected subset of $< V > \cap C(X)$ by Result 1.H. Since $C(E) \cap \mathcal{F}_1(M) \neq \emptyset$, $C(E) \cup \mathcal{F}_1(M)$ is connected and is contained in $< V > \cap C(X)$. It follows that there is a connected subset of $< V > \cap C(X)$ which contains $\{x\}$ in its interior.

The proof of the converse is the same as the corresponding proof in Corollary 4 of [2]. □

Result 2.F ([3])**.** *Let X be a locally compact Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*
(1) *X is connected im kleinen at x*
(2) *2^X is connected im kleinen at $\{x\}$*
(3) *$\mathcal{K}(X)$ is connected im kleinen at $\{x\}$*

(4) $C_K(X)$ *is connected im kleinen at* $\{x\}$

(5) $C(X)$ *is connected im kleinen at* $\{x\}$.

Theorem 2.2. *Let* X *be a locally compact Hausdorff space. Let* $x \in X$. *Then the following statements are equivalent:*

(1) X *is connected im kleinen at* x.

(2) 2^X *is arcwise connected im kleinen at* $\{x\}$.

(3) $\mathcal{K}(X)$ *is arcwise connected im kleinen at* $\{x\}$.

(4) $C_K(X)$ *is arcwise connected im kleinen at* $\{x\}$.

(5) $C(X)$ *is arcwise connected im kleinen at* $\{x\}$.

Proof. (1)⇒(2). Suppose that X is connected im kleinen at x. Let $< U >$ be a basic open in 2^X containing $\{x\}$. Since X is connected im kleinen at x and $x \in U$, U has a component M containing x in its interior. Let V be an open set containing x such that $x \in V$, $\overline{V}$ compact, and $V \subset Int(M)$. Let $W = Int(M)$ and $E \in< W > \cap K(\overline{V})$. Then $\{x\} \subset \overline{M}, \overline{M} \in K(\overline{V})$. So by Result 2.A, there exists an arc in $< U > \cap K(\overline{V}) \subset< U >$ between $\{x\}$ and E. Hence 2^X is arcwise connected im kleinen at $\{x\}$.

(1)⇒(3). We can see easily (1)⇒(3) in similar way of (1)⇒(2).

(1)⇒(4). Suppose that X is connected im kleinen at x. Let $< U > \cap C_K(X)$ be a basic open set containing $\{x\}$. Then $x \in U$ and there exists an open set V containing x such that $\overline{V}$ compact and $\overline{V} \subset U$. Since X is connected im kleinen at x, there is a component M of V which contains x in its interior. Let $W = Int(M)$. Then $x \in W \subset M \subset V$, $\{x\} \in< W > \cap C_K(X) \subset< V > \cap C_K(X) \subset< U > \cap C_K(X)$. Let $E \in< W > \cap C_K(X)$. Then $\{x\} \subset \overline{M}, E \subset \overline{M}$, and $\overline{M} \in C_K(X)$. So $\overline{M} \in< V > \cap C_K(X)$. Hence by Result 2.A, there exists an arc in $< V > \cap C_K(X)$ between $\{x\}$ and E. So $C_K(X)$ is arcwise connected im kleinen at $\{x\}$.

(1)⇒(5). Suppose that X is connected im kleinen at x. Let $< U > \cap C(X)$ be a basic open set in $C(X)$ containing $\{x\}$. Then $x \in U$ and there exists an open set V containing x such that $\overline{V}$ compact and $\overline{V} \subset U$. Since X is connected im kleinen at x, there is a component M of V which contains x in its interior. Let $W = Int(M)$. Then $\{x\} \in< W > \cap C(X) \subset< V > \cap C(X) \subset< U > \cap C(X)$. If $E \in< W > \cap C(X)$, then $E \in C_K(X)$. And $C(E)$ is a connected subset of $< V > \cap C(X)$ by Result 1.H. Since $E \in C(\overline{M})$ and $C(\overline{M}) = C_K(\overline{M})$, $\{x\} \subset \overline{M}$ and $E \subset \overline{M}$, there exists an arc in $< W > \cap C(\overline{M}) \subset< U > \cap C(X)$ between $\{x\}$ and E. Thus $C(X)$ is connected im kleinen at $\{x\}$.

(2)⇒(1). If X is arcwise connected im kleinen at $x \in X$, then X is connected im kleinen at x. So if 2^X is arcwise connected im kleinen at $\{x\}$, then 2^X is connected im kleinen at $\{x\}$. Hence by above Result, X is connected im kleinen at x.

We can obtain (3)⇒(1), (4)⇒(1), (5)⇒(1) in the same way of (2)⇒(1). □

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