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APPROXIMATING COMMON FIXED POINTS OF A SEQUENCE OF ASYMPTOTICALLY QUASI-f-g-NONEXPANSIVE MAPPINGS IN CONVEX NORMED VECTOR SPACES

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Abstract. In this paper, we introduce asymptotically quasi-f-g-nonexpansive mappings in convex normed vector spaces and consider approximating common fixed points of a sequence of asymptotically quasi-f-g-nonexpansive mappings in convex normed vector spaces.

1. Introduction and Preliminaries

Now we introduce asymptotically quasi-f-g-nonexpansive mappings and asymptotically f-g-nonexpansive mappings with convex normed vector spaces.

Definition 1.1. Let $(X, \|\cdot\|)$ be a normed vector space, $T : (X, \|\cdot\|) \to (X, \|\cdot\|)$ be a self-mapping and $f,\ g : (X, \|\cdot\|) \to (0, \infty)$ be functions.

(i) T is said to be asymptotically f-g-nonexpansive if there exist two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in X such that

$$\lim_{n\to\infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = 0$$

satisfying

$$\|T^n x - T^n y\| \le f(x_n) \cdot \|x - y\| + g(y_n) \quad \text{for} \quad x, y \in X$$

(ii) T is said to be asymptotically quasi-f-g-nonexpansive if there exist two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in X such that

$$\lim_{n\to\infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = 0$$

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satisfying

$$\|T^n x - p\| \le f(x_n) \cdot \|x - p\| + g(y_n) \quad \text{for } p \in F(T) \quad \text{and} \quad x \in X,$$

where $F(T)$ is the set of fixed points of T.

Example 1.1. Let $(X, \|\cdot\|)$ be the 2-dimensional Euclidean normed vector space $(\mathbb{R}^2, \|\cdot\|)$, $T : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be a self-mapping defined by $T((x_1, x_2)) = (\frac{1}{2}x_1, \frac{1}{3}x_2)$ for $(x_1, x_2) \in \mathbb{R}^2$ and f, $g : (\mathbb{R}^2, \|\cdot\|) \to (0, \infty)$ be two functions defined by $f((x_1, x_2)) = \frac{1}{x_1^2 + x_2^2}$, $g((x_1, x_2)) = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Take two sequences $\langle x_n \rangle = \langle (x_{1n}, x_{2n}) \rangle$ and $\langle y_n \rangle = \langle (y_{1n}, y_{2n}) \rangle$ in $\mathbb{R}^2$ such that $x_{1n} = \frac{1}{\sqrt{3}}$, $x_{2n} = \frac{\sqrt{2}}{\sqrt{3}}$ and $y_{1n} = \frac{1}{n}$, $y_{2n} = \frac{1}{2n}$ for $n \in \mathbb{N}$, respectively. Then $F(T) = \{(0, 0)\}$. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $p = (0, 0) \in F(T)$, we have

$$\begin{aligned} \|T^n x - p\| &= \left\| \left(\frac{1}{2^n} x_1, \frac{1}{3^n} x_2 \right) \right\| \\ &= \sqrt{\frac{1}{2^{2n}} x_1^2 + \frac{1}{3^{2n}} x_2^2}, \end{aligned}$$

$$\begin{aligned} f(x_n) \cdot \|x - p\| + g(y_n) &= \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} \cdot \sqrt{x_1^2 + x_2^2} + \left(\frac{1}{n^2} + \frac{1}{(2n)^2} \right) \\ &= \sqrt{x_1^2 + x_2^2} + \left(\frac{1}{n^2} + \frac{1}{(2n)^2} \right), \end{aligned}$$

$$\begin{aligned} \lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{1}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} = 1, \\ \lim_{n \to \infty} g(y_n) &= \lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{1}{4n^2} \right) = 0. \end{aligned}$$

Thus, we have

$$\|T^n x - p\| \le f(x_n) \cdot \|x - p\| + g(y_n) \quad \text{for } p \in F(T) \text{ and } x \in X,$$

which shows that the mapping T is asymptotically quasi-f-g-nonexpansive.

Definition 1.2 ([1, 3])**.** Let $(X, \|\cdot\|)$ be a normed vector space. A mapping $W : X^3 \times I^3 \to X$ is called a *convex structure* on X, if it satisfies the following condition; For any $(x, y, z) \in X^3$ and $(a, b, c) \in I^3$ with $a + b + c = 1$,

$$\|W(x, y, z; a, b, c) - u\| \le a \cdot \|x - u\| + b \cdot \|y - u\| + c \cdot |z - u\|$$

for all $u \in X$, where $I = [0, 1]$.

A normed vector space $(X, \|\cdot\|)$ with a convex structure W is called a *convex normed vector space* and is denoted as $(X, \|\cdot\|, W)$. A nonempty subset C of a convex normed vector space $(X, \|\cdot\|, W)$ is said to be a *convex subset* of $(X, \|\cdot\|)$, if $W(x, y, z; a, b, c) \in C$ for $(x, y, z) \in C^3$ and $(a, b, c) \in I^3$ with $a + b + c = 1$.

2. Main Results

A convex normed vector space becomes a convex metric space if we define a metric d by $d(x, y) = \|x - y\|$ for x, $y \in X$. When we speak about metric properties in a normed vector space, we referring to this metric. It should be pointed out that each normed vector space is a special example of convex metric space, but there exist some convex metric spaces which can not be embedded into any normed spaces [5].

Now, we introduce a new implicit iteration process;

$$x_{n+1} = W(x_n, T_i^n x_n, T_i^n x_{n+1}; \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}), \tag{2.1}$$

where $T_i : C \to C$ is an asymptotically quasi-f_i-g_i-nonexpansive mapping of a nonempty convex subset C of $(X, \|\cdot\|, W)$ for functions f_i, $g_i : (X, \|\cdot\|) \to (0, \infty)$ $(i \in \mathbb{N})$ and sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in I satisfying $\alpha_n + \beta_n + \gamma_n = 1$ $(n \in \mathbb{N})$.

Now we consider the approximating common fixed points of a sequence of quasi-f-g-nonexpansive mappings in convex normed vector spaces.

Lemma 2.1 ([4])**.** *Let $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle \delta_n \rangle$ be sequences of nonnegative real numbers satisfying the following inequality $a_{n+1} \leq (1 + \delta_n) a_n + b_n$, $n \geq 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.*

Theorem 2.1. *Let C be a nonempty closed convex subset of a real complete convex normed vector space $(X, \|\cdot\|, W)$. Let $\langle T_i : i \in \mathbb{N} \rangle$ be a sequence of asymptotically quasi-f_i-g_i-nonexpansive mappings of C with sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in X such that $\lim_{n\to\infty} f_i(x_n) = 1$, $\lim_{n\to\infty} g_i(y_n) = 0$ and $g_i(y_1) = 0$ for $i \in \mathbb{N}$. Suppose that $F = \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty closed. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in I satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for $n \geq 1$. Starting from an arbitararily given $x_0 \in K$, we define the sequence $\langle x_n \rangle_{n\geq 1}$ by* (2.1)*. Then the following are equivalent,*

(i) *$\langle x_n \rangle$ converges strongly to a common fixed point of the mappings $\langle T_i : i \in \mathbb{N} \rangle$,*
(ii) *$\underline{\lim}\, d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$.*

Proof. Obviously, (ii) implies (i). Now we show that (i) implies (ii). For $p \in F$,

$$\begin{aligned}\|x_{n+1} - p\| &= \|W(x_n, T_i^n x_n, T_i^n x_{n+1}; \alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}) - p\| \\ &\le \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot \|T_i^n x_n - p\| + \gamma_{n+1} \cdot \|T_i^n x_{n+1} - p\| \\ &\le \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot f_i(x_n) \cdot \|x_n - p\| + g_i(y_n) \\ &\quad + \gamma_{n+1} \cdot f_i(x_n) \cdot \|x_{n+1} - p\| + g_i(y_n) \ \text{ for } \ x, y \in C\end{aligned} \tag{2.2}$$

For arbitrary positive number ε, take a natural number K so that

$$f_i(x_n) < 1 + \varepsilon, \ g_i(y_n) < \varepsilon, \ \sum_{n=0}^{\infty} s_n < \infty \ \text{ and } \ \sum_{n=0}^{\infty} t_n < \infty,$$

where $s_n = \dfrac{(1 - \alpha_{n+1}) \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)}$ and $t_n = \dfrac{2 \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)}$ for $n \ge K$.

From (2.2) we obtain, for $n \ge K$,

$$\|x_{n+1} - p\| \le \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot (1 + \varepsilon) \cdot \|x_n - p\| + \gamma_{n+1} \cdot (1 + \varepsilon) \cdot \|x_{n+1} - p\| + 2 \cdot \varepsilon.$$

Hence

$$\begin{aligned}(1 - \gamma_{n+1} \cdot (1 + \varepsilon)) \cdot \|x_{n+1} - p\| &\le \alpha_{n+1} \cdot \|x_n - p\| + \beta_{n+1} \cdot (1 + \varepsilon) \cdot \|x_n - p\| + 2 \cdot \varepsilon \\ &= (\alpha_{n+1} + \beta_{n+1} \cdot (1 + \varepsilon)) \cdot \|x_n - p\| + 2 \cdot \varepsilon \text{ for } n \ge K,\end{aligned} \tag{2.3}$$

The inequality (2.3) shows that

$$\begin{aligned}\|x_{n+1} - p\| &\le \left(\frac{\alpha_{n+1} + \beta_{n+1} \cdot (1 + \varepsilon)}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)} \right) \|x_n - p\| + \frac{2 \cdot \varepsilon}{1 - \gamma_{n+1} \cdot (1 + \varepsilon)} \\ &= (1 + s_n)\|x_n - p\| + t_n.\end{aligned}$$

Since $p \in F$ is arbitrary,

$$\|d(x_{n+1}, F)\| \le (1 + s_n)\|d(x_n, F)\| + t_n.$$

Thus by Lemma 2.1, $\lim\limits_{n \to \infty} \|d(x_{n+1}, F)\|$ exists. Since $\varliminf\limits_{n \to \infty} \|d(x_n, F)\| = 0$ and $\sum\limits_{n=0}^{\infty} t_n < \infty$, for arbitrary positive number ε, there exists a natural number $N_0 \in \mathbb{N}$ such that

$$\|d(x_n, F)\| \le \frac{\varepsilon}{4L} \qquad \text{for } n \ge N_0$$

and

$$\sum_{n=N_0}^{\infty} t_n \le \frac{\varepsilon}{2L}, \qquad \text{where } L = e^{\sum_{j=1}^{m} s_{n+m-j}}.$$

In particular, there exists a point $p_1 \in F$ and $N_1 > N_0$ such that

$$\|x_{N_1} - p_1\| \leq \frac{\varepsilon}{4L}.$$

On the other hand, from the fact that

$$\|x_{n_1} - p\| \leq (1 - s_n)\|x_n - p\| + t_n$$

and the inequality $1 + x \leq e^x$ for $x \geq 0$, we obtain that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + s_{n+m-1}) \cdot \|x_{n+m-1} - p\| + t_{n+m-1} \\
&\leq e^{s_{n+m-1}}[(1 + s_{n+m-2}) \cdot \|x_{n+m-2} - p\| + t_{n+m-2}] + t_{n+m-1} \\
&\leq e^{\sum\limits_{j=1}^{2} s_{n+m-j}}[(1 + s_{n+m-3}) \cdot \|x_{n+m-3} - p\| + t_{n+m-3}] \\
&\quad + e^{s_{n+m-1}} \cdot t_{n+m-2} + t_{n+m-1} \\
&\leq e^{\sum\limits_{j=1}^{3} s_{n+m-j}}[(1 + s_{n+m-4})\|x_{n+m-4} - p\| + t_{n+m-4}] \\
&\quad + e^{\sum\limits_{j=1}^{2} s_{n+m-j}} \cdot t_{n+m-3} + e^{s_{n+m-1}} \cdot t_{n+m-2} + t_{n+m-1} \\
&= e^{\sum\limits_{j=1}^{4} s_{n+m-j}}\|x_{n+m-4} - p\| + e^{\sum\limits_{j=1}^{3} s_{n+m-j}} \cdot t_{n+m-4} \\
&\quad + e^{\sum\limits_{j=1}^{2} s_{n+m-j}} \cdot t_{n+m-3} + e^{s_{n+m-1}} \cdot t_{n+m-2} + t_{n+m-1} \\
&\leq \\
&\ \vdots \\
&\leq e^{\sum\limits_{j=1}^{m} s_{n+m-j}} \cdot \|x_m - p\| + e^{\sum\limits_{j=1}^{m-1} s_{n+m-j}} \cdot t_n + \cdots + e^{\sum\limits_{j=1}^{3} s_{n+m-j}} \cdot t_{n+m-4} \\
&\quad + e^{\sum\limits_{j=1}^{2} s_{n+m-j}} \cdot t_{n+m-3} + e^{s_{n+m-1}} \cdot t_{n+m-2} + t_{n+m-1} \\
&\leq L \cdot \|x_n - p\| + L \cdot \sum_{j=1}^{n+m-1} t_j.
\end{aligned}$$

Thus for $n > N_1$,

$$\begin{aligned}
\|x_{n+m} - p_1\| &= \|x_{N_1+(n+m-N_1)} - p_1\| \\
&\leq L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n+m-1} t_j
\end{aligned} \tag{2.4}$$

and

$$\|x_n - p_1\| = \|x_{N_1+(n-N_1)} - p_1\| \leq L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n-1} t_j \tag{2.5}$$

Hence from (2.4) and (2.5), we obtain that

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \\ &\leq L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n+m-1} t_j + L \cdot \|x_{N_1} - p_1\| + L \cdot \sum_{j=N_1}^{n-1} t_j \\ &\leq 2L \cdot \|x_{N_1} - p_1\| + L \cdot \left(\sum_{j=N_1}^{n+m-1} t_j + \sum_{j=N_1}^{n-1} t_j \right) \\ &\leq 2L \cdot \frac{\varepsilon}{4L} + 2 \cdot L \cdot \sum_{j=N_1}^{n+m-1} t_j \\ &\leq 2L \cdot \frac{\varepsilon}{4L} + 2 \cdot L \cdot \frac{\varepsilon}{2L} \\ &= \varepsilon\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed convex subset C of a real complete convex normed vector space $(X, \|\cdot\|, W)$. Therefore the sequence $\{x_n\}$ converges to a point of C. Let $\lim_{n\to\infty} x_n = p^*$. Now we will show that $p^* \in F$. Let $\{p_n\}$ be a sequence in F such that $p_n \to p'$. Since

$$\begin{aligned}\|p' - T_i p'\| &\leq \|p' - p_n\| + \|T_i p' - p_n\| \\ &\leq \|p' - p_n\| + |f_i(x_1)| \cdot \|p' - p_n\| + |g_i(y_1)| \to 0 \ \ (\text{as } \ n \to \infty),\end{aligned}$$

$\|p' - T_i p'\| = 0$ for $i \in \mathbb{N}$. Thus $p' \in F$, which means that F is closed. Since

$$d(p^*, F) = \lim_{n\to\infty} d(p_n, F) = 0,$$

we have $p^* \in F$, which completes the proof. □

Remark 2.1. (i) We obtain the same results for asymptotically f-g-nonexpansive mappings as a corollary.
(ii) We obtain the same results for asymptotically (quasi) nonexpansive mappings [2, 4, 6].

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