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COMMON FIXED POINT THEOREM FOR MULTIMAPS ON MENGER $\mathcal{L}$-FUZZY METRIC SPACE

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Abstract. In this paper, we obtain a common fixed point theorem for multivalued mappings in a complete Menger $\mathcal{L}$-fuzzy metric space. $\mathcal{L}$-fuzzy metric space is a generalization of fuzzy metric spaces and intuitionistic fuzzy metric spaces. We extend and generalize the results of Kubiaczyk and Sharma [24], Sharma, Kutukcu and Rathore [34].

1. Introduction

Fuzzy sets theory was formalised by Zadeh [49] in 1965 and developed a basic frame work too. It is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and enginereing. Many researchers studied its properties and applications.

There has been a great effort to obtain fuzzy analogues of classical theories. Among other fields a progressive developments is made in the field of fuzzy topology. The concept of fuzzy topology has important applications in quantum particle physics. In particular in connections with both string and $e^{(\infty)}$ theory, (Naschie([10], [11], [12], [13]). The fuzzy topology proves to be very useful tool to deal with such situations where the use of classical theories breaks down. Some most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space, intuitionistic fuzzy metric space, $\mathcal{L}$-fuzzy metric space. Fuzzy metric space has been investigated by many authors from different points of view. Recently, George and Veeramani [17] improved the notion of fuzzy metric spaces introduced by Kramosil and Michalek [25]. Many authors have studied the fixed point theory in these fuzzy metric spaces ([1], [3], [5], [14], [15], [19], [20], [22], [23], [27], [28], [35]-[41]).

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Intutionistic fuzzy metric notion is useful in modeling some phenomena where it is necesary to study the relationship between two probability functions as observed in Gregori, Romaguera and Veeramani [16]. Fixed point theory in intuitionistic fuzzy metric spaces have been studied by ([7], [32], [42]-[46]) and many others.

The concept of $\mathcal{L}$-fuzzy metric space is introduced by Saadati et al. [33] which is a generalization of fuzzy metric spaces [17] and intuitionistic fuzzy metric spaces ([29], [30], [43], [44], [45]). Saadati [32] proved fuzzy Banach and Edelstein fixed point theorems in $\mathcal{L}$-fuzzy metric spaces for modified definition of Cauchy sequence in George and Veeramani's sense [17].

Fixed point theorem and coincidence point theorem for multimaps in different spaces have been studied by Kubiaczyk and Sharma [24], Sharma and Kubiaczyk [47], Sharma, Kutukcu and Rathor [46], Krzyska and Kubiaczyk [26] and Deshpande [6].

In this paper, we prove a common fixed point theorem for multimaps in the settings of Menger $\mathcal{L}$-fuzzy Metric Spaces. We extend and generalize the results of Kubiaczyk and Sharma [24], Sharma, Kutukcu and Rathore [46].

2. PRELIMINARIES

Definition 1 ([18])**.** Let $\mathcal{L}$=$(L, \leq_L)$ be a complete lattice and U a non-empty set called universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (*in* L) to which u satisfies $\mathcal{A}$.

Lemma 1 ([8])**.** *Consider the set* L^* *and the operation* $\leq_{L^*}$ *defined by*

$$L^* = \left\{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \ \text{ and } \ x_1 + x_2 \leq 1\right\}$$

and $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$ *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, \ x) = x$ for all $x \in [0,1]$. These definitions can be straight forwardly extended to any lattice $\mathcal{L} = (L, \ \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 2 ([33])**.** A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)\,(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$(boundary condition)
(ii) $\left(\forall (x,y) \in L^2\right)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$(commutativity)
(iii) $\left(\forall (x,y,z) \in L^3\right)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z))$(associativity)

(iv) $\left(\forall\left(x,\ x',y,\ y'\right)\in L^4\right)$ $\left(x\leq_L x' \text{ and } y\leq_L y'\Rightarrow \mathcal{T}\left(x,y\right)\leq_L \mathcal{T}\left(x',y'\right)\right)$ (monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be *continuous* if for every $x,y\in\mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converges to x and y we have

$$\lim_{n\to\infty}\mathcal{T}(x_n,y_n)=\mathcal{T}(x,y).$$

For example, $\mathcal{T}\ (x,y)=\min(x,y)$ and $\mathcal{T}\ (x,y)=xy$ are two continuous t-norm on $[0,1]$. A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n\in\mathbb{N})$ by $\mathcal{T}^1=\mathcal{T}$ and

$$\mathcal{T}^n\left(x_{(1)},x_{(2)},\cdots,x_{(n+1)}\right)=\mathcal{T}(\mathcal{T}^{n-1}\left(x_{(1)},x_{(2)},\cdots,x_{(n)}\right),x_{(n+1)})$$

for $n\geq 2$ and $x_{(i)}\in L$.

A t-norm $\mathcal{T}$ is said to be of *Hadzic type* if the family $\{\mathcal{T}^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x=1_{\mathcal{L}}$, that is,

$\forall\epsilon\in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}\exists\delta\in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}:\ a>_L\mathcal{N}(\delta)\Longrightarrow\mathcal{T}^n(a)>_L\mathcal{N}\left(\epsilon\right)\quad(n\geq 1).$

$\mathcal{T}_M$ is a trivial example of a t-norm of Hadzic type, but there exists t-norm of Hadzic type weaker than $\mathcal{T}_M$ [20] where

$$\mathcal{T}_M(x,y)=\begin{cases} x & \text{if} \quad x\leq_L y,\\ y & \text{if} \quad y\leq_L x.\end{cases}$$

Definition 3 ([33])**.** A negation on $\mathcal{L}$ is any strictly decreasing mapping $\mathcal{N}:L\to L$ satisfying $\mathcal{N}(0_{\mathcal{L}})=1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}})=0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}\left(x\right))=x$ for all $x\in L$, then $\mathcal{N}$ is called an *involutive negation*. The negation N_s on $\left(\left[0,1\right],\leq\right)$ defined as, for all $x\in\left[0,1\right]$, $N_s\left(x\right)=1-x$ is called the *standard negation* on $\left(\left[0,1\right],\leq\right)$.

Definition 4 ([33])**.** The 3−tuple $(X,\mathcal{M},\mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times(0,+\infty)$ satisfying the following conditions for every $x\ y,z$ in X and t,s in $(0,+\infty)$:

(i) $\mathcal{M}\ (x,y,t)>_L 0_{\mathcal{L}}$;

(ii) $\mathcal{M}\left(x,y,t\right)=1_{\mathcal{L}}$ for all $t>0$ if and only if $x=y$;

(iii) $\mathcal{M}\left(x,y,t\right)=\mathcal{M}\left(y,x,t\right)$;

(iv) $\mathcal{T}\left(\mathcal{M}\left(x,y,s\right),\mathcal{M}\left(y,z,t\right)\right)\leq_L\mathcal{M}\left(x,z,s+t\right)$;

(v) $\mathcal{M}\left(x,y,.\right):(0,\infty)\to L$ is continuous.

If the $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ satisfies the condition:

(vi) $\lim_{n\to\infty}\mathcal{M}(x,y,t)=1_{\mathcal{L}},$

then $(X, \mathcal{M}, \mathcal{T})$ is said to be *Menger $\mathcal{L}$-fuzzy metric space* or for short a $M\mathcal{L}$-fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in (0, +\infty)$, we define the open ball $B(x, r, t)$ with centre $x \in X$ and radius $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$.

Example 1 ([31])**.** Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let $\mathcal{M}$ and $\mathcal{N}$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{1}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)} \right).$$

Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 2. Let $X = N$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), (a_2 + b_2 - a_2 b_2))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let $\mathcal{M}(x, y, t)$ on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}(x, y, t) = \begin{cases} (\frac{x}{y}, \frac{y-x}{x}) & \text{if} \quad x \leq y, \\ (\frac{y}{x}, \frac{x-y}{x}) & \text{if} \quad y \leq x. \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, \mathcal{T})$ is an $\mathcal{L}$-fuzzy metric space.

Lemma 2 ([17])**.** *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}(x, y, t)$ is non-decreasing with respect to t for all x, y in X.*

Definition 5 ([33])**.** A sequence $\{x_n\}_{n \in \mathbb{N}}$ in an $\mathcal{L}-$fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a *Cauchy sequence* if for each $\varepsilon \in L \backslash \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be *convergent* to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ $\left(\text{denoted by } x_n \xrightarrow{\mathcal{M}} x\right)$ if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ whenever $n \to +\infty$ for every $t > 0$. An $\mathcal{L}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Definition 6 ([28])**.** Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be *continuous* on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times (0, \infty)$ converges to a point $(x, y, t) \in X \times X \times (0, \infty)$, that is,

$$\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = \lim_{n\to\infty} \mathcal{M}(y_n, y, t) = 1_{\mathcal{L}},$$

and $\lim_{n\to\infty} \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 3 ([33])**.** *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X \times X \times (0, \infty)$.*

Lemma 4. *Let $\{y_n\}$ be a sequence in an $M\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$. Suppose*

$$\mathcal{M}(y_n, y_{n+1}, kt) \geq_L \mathcal{M}(y_{n-1}, y_n, t) \tag{2.1}$$

for all $t > 0$ and $n = 1, 2, \cdots$ and $k \in (0, 1)$. Then $\{y_n\}$ is a Cauchy sequence.

Proof. To prove that $\{y_n\}$ is a Cauchy sequence, we prove that (2.1) holds for all $n \geq n_0$ and for every $m \in \mathbb{N}$.

$$\mathcal{M}(y_n, y_{n+m}, t) \geq \mathcal{N}(\epsilon). \tag{2.2}$$

Here we use induction method. From (2.1), we have

$$\begin{aligned} \mathcal{M}(y_n, y_{n+1}, t) &\geq_L \mathcal{M}\left(y_{n-1}, y_n, \tfrac{t}{k}\right) \\ &\geq_L \mathcal{M}\left(y_{n-2}, y_{n-1}, \tfrac{t}{k^2}\right) \\ &\geq_L \cdots \geq_L \mathcal{M}\left(y_0, y_1, \tfrac{t}{k^n}\right) \to 1_{\mathcal{L}} \text{ as } n \to \infty. \end{aligned}$$

Thus for $t > 0$ and $\{\epsilon \in L/\{0_{\mathcal{L},}, 1_{\mathcal{L},}\}\}$ we can choose $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}(y_n, y_{n+1}, t) \geq_L \mathcal{N}(\epsilon).$$

Thus (2.2) holds for $m = 1$. Suppose that (2.2) holds for m. Then we shall show that it also holds for $m + 1$.

$$\begin{aligned} \mathcal{M}(y_n, y_{n+m+1}, t) &\geq \mathcal{T}\left(\mathcal{M}\left(y_n, y_{n+m}, \frac{t}{2}\right), \mathcal{M}\left(y_{n+m}, y_{n+m+1}, \frac{t}{2}\right)\right) \\ &> \mathcal{N}(\epsilon). \end{aligned}$$

Hence (2.2) holds for $m + 1$. This completes the proof. □

Lemma 5. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $M\mathcal{L}$-fuzzy metric space. If there exists $k \in (0, 1)$ such that*

$$\mathcal{M}(x, y, kt) \geq_L \mathcal{M}(x, y, t),$$

then $x = y$.

Proof. Since $\mathcal{M}(x, y, kt) \geq_L \mathcal{M}(x, y, t)$, we have

$$\mathcal{M}(x, y, t) \geq_L \mathcal{M}\left(x, y, k^{-1}t\right).$$

By applying the above inequality repeatedly we have

$$\begin{aligned}\mathcal{M}(x, y, t) &\geq_L \mathcal{M}\left(x, y, k^{-1}t\right) \geq_L \mathcal{M}\left(x, y, k^{-2}t\right) \geq_L \mathcal{M}\left(x, y, k^{-3}t\right) \\ &\geq_L \cdots \geq_L \mathcal{M}\left(x, y, k^{-n}t\right) \cdots (n \in \mathbb{N}),\end{aligned}$$

which tends to $1_{\mathcal{L},}$ as $n \to \infty$. Thus $\mathcal{M}(x, y, t) = 1_{\mathcal{L},}$ for all $t > 0$ and we get $x = y$. This completes the proof. □

We introduce the following concepts for multivalued mappings in $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$:

Definition 7. We denote by $CB(X)$, the set of all nonempty bounded and closed subsets of X. We have

$$\mathcal{M}^{\bigtriangledown}(y, B, t) = \sup_{b \in B} \{\mathcal{M}(y, b, t)\},$$

$$\mathcal{M}_{\bigtriangledown}(A, B, t) = \inf\left\{\inf_{a \in A}\left\{\mathcal{M}^{\bigtriangledown}(a, B, t)\right\}, \inf_{b \in B}\left\{\mathcal{M}^{\bigtriangledown}(A, b_x, t)\right\}\right\}$$

for all A, B in $CB(X)$ and $t > 0$.

3. Main Results

Theorem 1. *Let* $(X, \mathcal{M}, \mathcal{T})$ *be a complete* $M\mathcal{L}$*-fuzzy metric space. Let* $S, T : X \to CB(X)$ *satisfying*

$$\begin{aligned}\mathcal{M}_{\bigtriangledown}(Sx, Ty, kt) \geq_L \inf\{&\mathcal{M}(x, y, t), \mathcal{M}^{\bigtriangledown}(x, Sx, t), \mathcal{M}^{\bigtriangledown}(y, Ty, t), \\ &\mathcal{M}^{\bigtriangledown}(x, Ty, (2-\alpha)t), \mathcal{M}^{\bigtriangledown}(y, Sx, t)\}.\end{aligned} \tag{3.1}$$

Then S *and* T *have a common fixed point.*

Proof. Let x_0 be an arbitrary point in X and $x_1 \in X$ be such that $x_1 \in Sx_0$ and

$$\mathcal{M}(x_0, x_1, kt) \geq_L \mathcal{M}^{\bigtriangledown}(x_0, Sx_0, kt) - \varepsilon.$$

Let $x_2 \in X$ be such that $x_2 \in Tx_1$ and

$$\mathcal{M}(x_1, x_2, kt) \geq_L \mathcal{M}^{\bigtriangledown}(x_1, Tx_1, kt) - \frac{\varepsilon}{2}.$$

Inductively, $x_{2n+1} \in X$ is such that $x_{2n+1} \in Sx_{2n}$,

$$\mathcal{M}(x_{2n}, x_{2n+1}, kt) \geq_L \mathcal{M}^{\bigtriangledown}(x_{2n}, Sx_{2n}, kt) - \frac{\varepsilon}{2^{2n}},$$

and $x_{2n+2} \in X$ is such that $x_{2n+2} \in Tx_{2n+1}$ and

$$\mathcal{M}\left(x_{2n+1}, x_{2n+2}, kt\right) \geq_L \mathcal{M}^{\triangledown}\left(x_{2n+1}, Tx_{2n+1}, kt\right) - \frac{\varepsilon}{2^{2n+1}}.$$

Now we show that $\{x_n\}$ is a Cauchy sequence. By (3.1) for all $t \geq 0$ and $\alpha = 1-q$ with $q \in (0,1)$, we write

$$\begin{aligned}
&\mathcal{M}\left(x_{2n+1}, x_{2n+2}, kt\right)\\
&\geq_L \mathcal{M}^{\triangledown}\left(x_{2n+1}, Tx_{2n+1}, kt\right) - \frac{\varepsilon}{2^{2n+1}}\\
&\geq_L \mathcal{M}_{\triangledown}\left(Sx_{2n}, Tx_{2n+1}, kt\right) - \frac{\varepsilon}{2^{2n+1}}\\
&\geq_L \inf\{\mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}^{\triangledown}\left(x_{2n}, Sx_{2n}, t\right), \mathcal{M}^{\triangledown}\left(x_{2n+1}, Tx_{2n+1}, t\right),\\
&\qquad \mathcal{M}^{\triangledown}\left(x_{2n}, Tx_{2n+1}, (2-\alpha)\, t\right), \mathcal{M}^{\triangledown}\left(x_{2n+1}, Sx_{2n}, t\right)\} - \frac{\varepsilon}{2^{2n+1}}\\
&\geq_L \inf\{\mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}\left(x_{2n+1}, x_{2n+2}, t\right),\\
&\qquad \mathcal{M}\left(x_{2n}, x_{2n+2}, (1+q)\, t\right), \mathcal{M}\left(x_{2n+1}, x_{2n+1}, t\right)\} - \frac{\varepsilon}{2^{2n+1}}\\
&\geq_L \inf\{\mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}\left(x_{2n+1}, x_{2n+2}, t\right),\\
&\qquad \mathcal{T}(\mathcal{M}\left(x_{2n}, x_{2n+1}, t\right)), \mathcal{M}\left(x_{2n+1}, x_{2n+1}, t\right), 1_{\mathcal{L}}\} - \frac{\varepsilon}{2^{2n+1}}.
\end{aligned}$$

Letting $q \to 1$ and using the facts that $\mathcal{T}$ is a continuous $t-$norm and $\mathcal{M}$ is continuous function on $X \times X \times (0, \infty)$, we obtain

$$\mathcal{M}\left(x_{2n+1}, x_{2n+2}, kt\right) \geq_L \inf\left\{\mathcal{M}\left(x_{2n}, x_{2n+1}, t\right), \mathcal{M}\left(x_{2n+1}, x_{2n+2}, t\right)\right\} - \frac{\varepsilon}{2^{2n+1}}. \tag{3.2}$$

Similarly we can obtain

$$\mathcal{M}\left(x_{2n+2}, x_{2n+3}, kt\right) \geq_L \inf\left\{\mathcal{M}\left(x_{2n+1}, x_{2n+2}, t\right), \mathcal{M}\left(x_{2n+2}, x_{2n+3}, t\right)\right\} - \frac{\varepsilon}{2^{n+2}}. \tag{3.3}$$

Thus from (3.2) and (3.3), it follows that

$$\mathcal{M}\left(x_{n+1}, x_{n+2}, kt\right) \geq_L \inf\left\{\mathcal{M}\left(x_{n}, x_{n+1}, t\right), \mathcal{M}\left(x_{n+1}, x_{n+2}, t\right)\right\} - \frac{\varepsilon}{2^{n+1}}$$

for $n = 1, 2, \cdots$.

Thus for positive integers n and p we have

$$\begin{aligned}
&\mathcal{M}\left(x_{n+1}, x_{n+2}, kt\right)\\
&\geq_L \inf\left\{\mathcal{M}\left(x_{n}, x_{n+1}, t\right), \mathcal{M}\left(x_{n+1}, x_{n+2}, \frac{t}{k^p}\right)\right\} - \frac{\varepsilon}{2^{n+1}}.
\end{aligned}$$

Letting $p \to \infty$ we get

$$\mathcal{M}(x_{n+1}, x_{n+2}, kt) \geq_L \mathcal{M}(x_n, x_{n+1}, t) - \frac{\varepsilon}{2^{n+1}}.$$

Since ϵ is arbitrary making $\epsilon \to 0_{\mathcal{L}}$, we obtain

$$\mathcal{M}(x_{n+1}, x_{n+2}, kt) \geq_L \mathcal{M}(x_n, x_{n+1}, t).$$

Therefore by Lemma 5, $\{x_n\}$ is a Cauchy sequence, so it converges to a point $z \in X$.

Now by (3.1), with $\alpha = 1$, we have

$$\begin{aligned}
&\mathcal{M}^{\triangledown}(x_{2n+2}, Sz, kt) \geq_L \mathcal{M}_{\triangledown}(Sz, Tx_{2n+1}, kt) \\
&\geq_L \inf\{\mathcal{M}(z, x_{2n+1}, t), \mathcal{M}^{\triangledown}(z, Sz, t), \mathcal{M}^{\triangledown}(x_{2n+1}, Tx_{2n+1}, t), \\
&\qquad \mathcal{M}^{\triangledown}(z, Tx_{2n+1}, t), \mathcal{M}^{\triangledown}(x_{2n+1}, Sz, t)\} \\
&\geq_L \inf\{\mathcal{M}(z, x_{2n+1}, t), \mathcal{M}^{\triangledown}(z, Sz, t), \mathcal{M}(x_{2n+1}, Tx_{2n+2}, t), \\
&\qquad \mathcal{M}(z, Tx_{2n+1}, t), \mathcal{M}^{\triangledown}(x_{2n+1}, Sz, t)\}.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\mathcal{M}^{\triangledown}(z, Sz, kt) \geq_L \inf\left\{1_{\mathcal{L}}, \mathcal{M}^{\triangledown}(z, Sz, t), 1_{\mathcal{L}}, 1_{\mathcal{L}}, \mathcal{M}^{\triangledown}(z, Sz, t)\right\}.$$

This gives

$$\mathcal{M}^{\triangledown}(z, Sz, kt) \geq_L \mathcal{M}^{\triangledown}(z, Sz, t),$$

using Lemma 5 we get $z \in Sz$. Similarly we can show that $z \in Tz$. □

Remark. It follows from Definition 7, [2] and [4].

(i) $\mathcal{M}^{\triangledown}(A, y, t)$ is nondecreasing with respect to t for all x, y in X.

(ii) $\mathcal{M}^{\triangledown}(A, y, t)$ is left continuous.

(iii) $\mathcal{M}^{\triangledown}(A, y, t) = 1_{\mathcal{L}}$, for all $t > 0$ if and only if $y \in A$.

Theorem 2. *Let $(X, \mathcal{M}, \mathcal{T})$ be a complete $M\mathcal{L}$-fuzzy metric space. Let $T_n : X \to CB(X)\,(n \in \mathbb{N})$ and contnuous mapping $I : X \to X$, be such that $T_n(X) \subset I(X)$ where I commute with T_n for every $n \in \mathbb{N}$ and there exists $k \in (0,1)$ such that*

$$\begin{aligned}
&\mathcal{M}_{\triangledown}(T_i x, T_j y, kt) \geq_L \inf\{\mathcal{M}(Ix, Iy, t), \mathcal{M}^{\triangledown}(Ix, T_i x, t), \mathcal{M}^{\triangledown}(Iy, T_j y, t), \\
&\qquad \mathcal{M}^{\triangledown}(Ix, T_j y, (2-\alpha)t), \mathcal{M}^{\triangledown}(Iy, T_i x, t)\}
\end{aligned} \tag{3.4}$$

for all x, y in X, $\alpha \in (0, 2)$ and $t > 0$ for every $i, j \in \mathbb{N}$ $(i \neq j)$. Then there exists a common coincidence point of T_n and I, that is, there exists a point z in X such that $I_z \in \cap\, T_n z$, $n \in \mathbb{N}$.

Proof. Let x_0 be an arbitrary point in X. Then there exists $x_1 \in X$ such that $Ix_1 \in T_1x_0$. Let $y_1 = Ix_1$. From (3.4) we have

$$\mathcal{M}(x_0, y_1, kt) = \mathcal{M}(x_0, Ix_1, kt) \geq_L \mathcal{M}^{\nabla}(x_0, T_1x_0, kt) - \frac{\epsilon}{2}.$$

Now there exists $x_2 \in X$ is such that $Ix_2 \in T_2x_1$. Let $Y_2 = Ix_2$ and

$$\mathcal{M}(y_1, y_2, kt) = \mathcal{M}(Ix_1, Ix_2, kt) \geq_L \mathcal{M}^{\nabla}(y_1, T_2x_1, kt) - \frac{\varepsilon}{2^2}.$$

Inductively, we construct a sequence $\{y_n\}$ in X such that,

$$\mathcal{M}(y_n, y_{n+1}, kt) = \mathcal{M}(Ix_n, Ix_{n+1}, kt) \geq_L \mathcal{M}^{\nabla}(y_n, T_{n+1}x_n, kt) - \frac{\varepsilon}{2^n}.$$

Now we show that $\{y_n\}$ is a Cauchy sequence. By (3.4) for all $t > 0$ and $\alpha = 1-q$ with $q \in (0,1)$, we write

$$\begin{aligned}
&\mathcal{M}(y_n, y_{n+1}, kt)\\
&\geq_L \mathcal{M}^{\nabla}(y_n, T_{n+1}x_n, kt) - \frac{\varepsilon}{2^n}\\
&\geq_L \mathcal{M}_{\nabla}(T_nx_{n-1}, T_{n+1}x_n, kt) - \frac{\varepsilon}{2^n}\\
&\geq_L \inf\{\mathcal{M}(Ix_{n-1}, Ix_n, t), \mathcal{M}^{\nabla}(Ix_{n-1}, T_nx_{n-1}, t), \mathcal{M}^{\nabla}(Ix_n, T_{n+1}x_n, t),\\
&\qquad \mathcal{M}^{\nabla}(Ix_{n-1}, T_{n+1}x_n, (2-\alpha)t), \mathcal{M}^{\nabla}(Ix_n, T_nx_{n-1}, t)\} - \frac{\varepsilon}{2^n}\\
&\geq_L \inf\{\mathcal{M}(Ix_{n-1}, Ix_n, t), \mathcal{M}(Ix_{n-1}, Ix_n, t), \mathcal{M}(Ix_n, Ix_{n+1}, t),\\
&\qquad \mathcal{M}(Ix_{n-1}, Ix_{n+1}, (1+k)t), \mathcal{M}(Ix_n, Ix_n, t)\} - \frac{\varepsilon}{2^n}\\
&\geq_L \inf\{\mathcal{M}(y_{n-1}, y_n, t), \mathcal{M}(y_{n-1}, y_n, t), \mathcal{M}(y_n, y_{n+1}, t),\\
&\qquad \mathcal{M}(y_{n-1}, y_n, kt), 1_{\mathcal{L}})\} - \frac{\varepsilon}{2^n}.
\end{aligned}$$

Letting $k \to 1$ and using the facts that $\mathcal{T}$ is a continuous t-norm and $\mathcal{M}$ is a continuous function on $X \times X \times (0, \infty)$, we obtain

$$\mathcal{M}(y_n, y_{n+1}, kt) \geq_L \inf\{\mathcal{M}(y_{n-1}, y_n, t), \mathcal{M}(y_n, y_{n+1}, t)\} - \frac{\varepsilon}{2^n} \tag{3.5}$$

for $n = 1, 2, \cdots$ and so, for positive integers n and p and $\epsilon \in L/\{0_{\mathcal{L}},, 1_{\mathcal{L}},\}$, we have

$$\mathcal{M}(y_n, y_{n+1}, kt) \geq_L \inf\left\{\mathcal{M}(y_{n-1}, y_n, t), \mathcal{M}\left(y_n, y_{n+1}, \frac{t}{k^p}\right)\right\} - \frac{\varepsilon}{2^n}.$$

Since ϵ is arbitrary making $\epsilon \to 0_{\mathcal{L}}$ and $\mathcal{M}(y_n, y_{n+1}, \frac{t}{k^p}) \to 1_{\mathcal{L}}$ as $n \to \infty$, we obtain

$$\mathcal{M}(y_n, y_{n+1}, kt) \geq_L \mathcal{M}(y_{n-1}, y_n, t).$$

Therefore by Lemma 4 $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to a point $z \in X$. Now by (3.5) with $\alpha = 1$, we have

$$\mathcal{M}^{\bigtriangledown}\left(IIx_n, T_m z, kt\right) \geq_L \mathcal{M}_{\bigtriangledown}\left(T_n I x_{n-1}, T_m z, kt\right)$$
$$\geq_L \inf\{\mathcal{M}\left(IIx_{n-1}, Tz, t\right), \mathcal{M}^{\bigtriangledown}\left(IIx_{n-1}, T_n I x_{n-1}, t\right), \mathcal{M}^{\bigtriangledown}\left(Tz, T_m z, t\right), \mathcal{M}^{\bigtriangledown}\left(IIx_{n-1}, T_m z, t\right), \mathcal{M}^{\bigtriangledown}\left(Iz, T_n I x_{n-1}, t\right)\}$$
$$\geq_L \inf\{\mathcal{M}\left(IIx_{n-1}, Iz, t\right), \mathcal{M}\left(IIx_{n-1}, IIx_n, t\right), \mathcal{M}\left(Iz, T_m z, t\right), \mathcal{M}\left(IIx_{n-1}, T_m z, t\right), \mathcal{M}^{\bigtriangledown}\left(Iz, I_n I x_n, t\right)\}.$$

Since $\lim_{n\to\infty} \mathcal{M}\left(IIx_{n-1}, Iz, t\right) = 1_{\mathcal{L}}, \lim_{n\to\infty} \mathcal{M}\left(IIx_{n-1}, IIx_n, t\right) = 1_{\mathcal{L}}$, we have

$$\lim_{n\to\infty} \mathcal{M}^{\bigtriangledown}\left(IIx_{n-1}, T_m z, t\right) = \mathcal{M}^{\bigtriangledown}\left(Iz, T_m z, t\right).$$

Hence for any $m \in N$, we write

$$\mathcal{M}^{\bigtriangledown}\left(Iz, T_m z, kt\right) \geq_L \mathcal{M}^{\bigtriangledown}\left(Iz, T_m z, t\right).$$

This implies by Lemma 5, that $Iz \in T_m z$ and therefore $Iz \cap T_n z$, $n \in \mathbb{N}$. □

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