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APPELL'S FUNCTION F_1 AND EXTON'S TRIPLE HYPERGEOMETRIC FUNCTION X_9

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Abstract. In the theory of hypergeometric functions of one or several variables, a remarkable amount of mathematicians's concern has been given to develop their transformation formulas and summation identities. Here we aim at presenting explicit expressions (in a single form) of the following weighted Appell's function F_1:

$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c+j\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)\quad (j=0,\,\pm1,\,\ldots,\,\pm5)$$

in terms of Exton's triple hypergeometric X_9. The results are derived with the help of generalizations of Kummer's second theorem very recently provided by Kim et al. A large number of very interesting special cases including Exton's result are also given.

1. Introduction and Preliminaries

The *generalized hypergeometric series* ${}_pF_q$ is defined by (see [12, p. 73]):

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} = {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z),$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \quad \left(n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\};\ \mathbb{N} := \{1,\,2,\,3,\,\cdots\}\right),$$

which is defined (for $\lambda,\,\nu\in\mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(1.2)\quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases}$$

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it being *understood conventionally* that $(0)_0 := 1$ and $\mathbb{C}$ the set of complex numbers. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is, that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q), \tag{1.3}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates in view of the known identity (see, for example, [15, p. 7]):

$$(-n)_k = \begin{cases} \dfrac{(-1)^k\, n!}{(n-k)!} & (0 \leqq k \leqq n;\ k,\, n \in \mathbb{N}_0)\,, \\ 0 & (k > n). \end{cases} \tag{1.4}$$

The special case of (1.1) when $p = 2$ and $q = 1$ is usually called Gauss's hypergeometric function or series. It should also be remarked here that whenever hypergeometric and generalized hypergeometric functions reduce to express in terms of Gamma functions, the results are very important from the applicative point of view. Therefore, the classical summation theorems such as those of Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$ and Dixon, Watson, Whipple and Saalschütz for the series ${}_3F_2$ and their rather recent extensions and generalizations (see [7], [8], [9], [10] and [11]) play an important role in the theory of hypergeometric and generalized hypergeometric series. For applications of the above-mentioned classical summation theorems, we refer to [2], [5], [6], [10], [11], [12] and [13].

Moreover, it is well known that, the product of two hypergeometric series can be expressed as a hypergeometric series with argument x, then the coefficient of x^n in the product must be expressible in terms of Gamma function. With this theory, Bailey [2] obtained the following Kummer's second theorem (see, for example, [12]):

$$e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{matrix} \alpha\,; \\ 2\alpha\,; \end{matrix}\ x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ \alpha + \frac{1}{2}\,; \end{matrix}\ \frac{x^2}{16}\right] \tag{1.5}$$

by using Gauss's second summation theorem. It is noted that Rathie and Choi [14] derived the result (1.5) by making use of Gauss's summation theorem. Very recently Kim et al. [5] generalized the Kummer's second theorem to give explicit expressions for

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\alpha\,;\\ 2\alpha+j\,;\end{matrix}\;x\right] \quad (j=0,\,\pm1,\,\ldots,\,\pm5), \tag{1.6}$$

which will be given in Section 3.

On the other hand, just as Gauss function ${}_2F_1$ was naturally extended to ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator and the enormous success of the theory of hypergeometric series in single variable has stimulated the development of a corresponding theory in two and more variables. Thus the four Appell functions in two variables were introduced. Here we are interested in the following Appell function F_1 (see [1]):

$$F_1\left(a,\,b,\,b'\,;\,c\,;\,x,\,y\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \quad \left(\max\{|x|,\,|y|\}<1\right). \tag{1.7}$$

For more details about these Appell functions and their generalization, we refer to the extensive work by Srivastava and Karlsson [16]. Also, in the course of manipulating certain integral representations of Lauricella functions of three variables and Saran's functions (see [16]), Exton [4] encountered a number of triple hypergeometric functions of the second order whose series representations involve such product as $(a)_{2m+2n+p}$ and $(a)_{2m+n+p}$. In fact, Exton [4] investigated the generalizations of the Horn's functions H_3 and H_4 and introduced a set of twenty triple hypergeometric functions X_1 to X_{20}. In the same paper [4], Exton gave certain integral representations of Laplace type for these functions together with some elementary properties and some transformation and reduction formulas. Here, in this paper, we are interested in the following triple hypergeometric function X_9 defined by

$$\begin{gathered} X_9\,(a,\,b\,;\,c\,;\,x,\,y,\,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}\,(b)_{n+2p}}{(c)_{m+n+p}}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!} \\ \left(\left\{|x|<\tfrac{1}{4},\;|z|<\tfrac{1}{4},\;|y|<\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-4|x|)(1-4|z|)}\right\}\right) \end{gathered} \tag{1.8}$$

and its Laplace type integral is given by

$$
\begin{aligned}
&X_9\,(a,\, b\,;\, c\,;\, x,\, y,\, z) \\
&= \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t}\, s^{a-1}\, t^{b-1}\, {}_0F_1 \left[\begin{matrix} -\,; \\ c\,; \end{matrix}\, x\, s^2 + y\, s\, t + z\, t^2 \right] ds\, dt.
\end{aligned} \tag{1.9}
$$

This paper is organized as follows. In Section 3, we establish explicit expressions (in a single form of general results) of (1.6) obtained earlier by Kim et al. [5] by using a different method. In Section 4, we present explicit expressions (in a single form of general results) of

$$
(1+2x)^{-a}\,(1+2z)^{-b}\, F_1\left(c,\, a,\, b\,;\, 2c+j\,;\, \frac{4x}{1+2x},\, \frac{4z}{1+2z}\right) \quad (j = 0,\, \pm 1,\, \ldots,\, \pm 5).
$$

The results will be derived with the help of those given in Section 2. In Section 4, we express a large number of Appell functions F_1 in terms of Exton's triple hypergeometric function X_9. The results easily established here are simple, interesting and (potentially) useful.

2. Results Required

In our present investigation, we require the following results [5]:

(2.1)

$$
\begin{aligned}
&{}_2F_1 \left[\begin{matrix} -2n,\, \alpha\,; \\ 2\alpha + j\,; \end{matrix}\, 2 \right] \\
&= \mathcal{A}_j(\alpha, n)\, \frac{\Gamma(\alpha)\,\Gamma(1-\alpha)\, \left(\frac{1}{2}\right)_n \left(\alpha + \left[\frac{j+1}{2}\right]\right)_n}{\Gamma\left(\alpha + \frac{j}{2} + \frac{1}{2}|j|\right)\, \Gamma\left(1 - \alpha - \left[\frac{j+1}{2}\right]\right) \left(\alpha + \frac{j}{2}\right)_n \left(\alpha + \frac{j}{2} + \frac{1}{2}\right)_n}
\end{aligned}
$$

and

(2.2)

$$
\begin{aligned}
&{}_2F_1 \left[\begin{matrix} -2n-1,\, \alpha\,; \\ 2\alpha + j\,; \end{matrix}\, 2 \right] \\
&= \frac{\mathcal{B}_j(\alpha, n)}{2\alpha + j}\, \frac{\Gamma(-\alpha)\,\Gamma(\alpha+1)\, \left(\frac{3}{2}\right)_n \left(1 + \alpha + \left[\frac{j}{2}\right]\right)_n}{\Gamma\left(\alpha + \frac{j}{2} + \frac{1}{2}|j|\right)\, \Gamma\left(-\alpha - \left[\frac{j}{2}\right]\right) \left(\alpha + \frac{j}{2} + \frac{1}{2}\right)_n \left(\alpha + \frac{j}{2} + 1\right)_n},
\end{aligned}
$$

where $n \in \mathbb{N}_0$, $j = 0,\, \pm 1,\, \ldots,\, \pm 5$, $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$, and the coefficients $\mathcal{A}_j(\alpha, n)$ and $\mathcal{B}_j(\alpha, n)$ are given in the following table.

TABLE

j	$\mathcal{A}_j(\alpha,n)$	$\mathcal{B}_j(\alpha,n)$
5	$-4(1-\alpha-2n)^2+2(1-\alpha)(1-\alpha-2n)$ $+(1-\alpha)^2+22(1-\alpha-2n)$ $-13(1-\alpha)-20$	$4(\alpha+2n)^2-2(1-\alpha)(\alpha+2n)$ $+(1-\alpha)^2+34(\alpha+2n)$ $+(1-\alpha)+62$
4	$2(\alpha+1+2n)(\alpha+3+2n)-\alpha(\alpha+3)$	$4(\alpha+2n+3)$
3	$-\alpha-4n-2$	$-3\alpha-4n-6$
2	$-\alpha-1-2n$	-2
1	-1	1
0	1	0
-1	1	1
-2	$1-\alpha-2n$	2
-3	$1-\alpha-4n$	$3-3\alpha-4n$
-4	$2(1-2\alpha-n)(3-\alpha-2n)-(1-\alpha)(4-\alpha)$	$4(1-\alpha-2n)$
-5	$4(1-\alpha-2n)^2-2(1-\alpha)(1-\alpha-2n)$ $-(1-\alpha)^2+8(1-\alpha-2n)+7\alpha-7$	$4(\alpha+2n)^2+2(1-\alpha)(\alpha+2n)$ $-(1-\alpha)^2-16(\alpha+2n)+\alpha-1$

3. Generalizations of Kummer's Second Theorem

In this section we establish the following generalizations of the Kummer's second theorem (1.5):

$$e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c+j\,; \end{matrix}\; 4x\right] = \frac{\Gamma(c)\,\Gamma(1-c)}{\Gamma\left(c+\frac{j}{2}+\frac{1}{2}|j|\right)\,\Gamma\left(1-c-\left[\frac{j+1}{2}\right]\right)} \cdot \sum_{m=0}^{\infty} \mathcal{A}_j(c,m)\,\frac{\left(c+\left[\frac{j+1}{2}\right]\right)_m\,x^{2m}}{m!\,\left(c+\frac{j}{2}\right)_m\left(c+\frac{j}{2}+\frac{1}{2}\right)_m}$$
$$-\frac{2x}{2c+j}\,\frac{\Gamma(-c)\,\Gamma(1+c)}{\Gamma\left(c+\frac{j}{2}+\frac{1}{2}|j|\right)\,\Gamma\left(-c-\left[\frac{j}{2}\right]\right)} \cdot \sum_{m=0}^{\infty} \mathcal{B}_j(c,m)\,\frac{\left(1+c+\left[\frac{j}{2}\right]\right)_m\,x^{2m}}{m!\,\left(c+\frac{j}{2}+\frac{1}{2}\right)_m\left(c+\frac{j}{2}+1\right)_m}, \tag{3.1}$$

where $m\in\mathbb{N}_0$, $j=0,\,\pm1,\,\ldots,\,\pm5$, $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$, and the coefficients $\mathcal{A}_j(c,m)$ and $\mathcal{B}_j(c,m)$ are given by replacing α and n in the Table by c and m, respectively.

Proof. Denoting the left-hand side of (3.1) by S and expressing both functions as series, after a simplification, we have

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m\,(c)_n\,2^{m+2n}\,x^{m+n}}{(2c+j)_n\,m!\,n!}.$$

Using the following well known formal manipulation of double series (see [12, p. 56] and [3, Eq. (1.4)]; for other ones, see also [3]):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n), \tag{3.2}$$

after a little simplification, we obtain

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{m-n}\,(c)_n\,2^{m+n}\,x^{m}}{(2c+j)_n\,(m-n)!\,n!}.$$

Using the result (1.4) for $(m-n)!$, after a little simplification, we get

$$S=\sum_{m=0}^{\infty}\frac{(-2)^m\,x^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n\,(c)_n}{(2c+j)_n}\,\frac{2^n}{n!}.$$

Expressing the inner series as in (1.1), we have

$$S=\sum_{m=0}^{\infty}\frac{(-2)^m\,x^m}{m!}\,{}_2F_1\left[\begin{matrix}-m,\,c\,;\\2c+j\,;\end{matrix}\;2\right].$$

Now, separating the summation into even and odd powers of x and making use of the following identities:

$$(2m)!=2^{2m}\left(\frac{1}{2}\right)_m m!\quad\text{and}\quad(2m+1)!=2^{2m}\left(\frac{3}{2}\right)_m m!,$$

we obtain

$$S=\sum_{m=0}^{\infty}\frac{x^{2m}}{\left(\frac{1}{2}\right)_m\,m!}\,{}_2F_1\left[\begin{matrix}-2m,\,c\,;\\2c+j\,;\end{matrix}\;2\right]$$
$$-2x\sum_{m=0}^{\infty}\frac{x^{2m}}{\left(\frac{3}{2}\right)_m\,m!}\,{}_2F_1\left[\begin{matrix}-2m-1,\,c\,;\\2c+j\,;\end{matrix}\;2\right].$$

Finally using the results (2.1) and (2.2), after a little simplification, we easily arrive at the right-hand side of (3.1). This completes the proof of (3.1). □

Special Cases of (3.1). Here we illustrate some of the very interesting special cases of (3.1) already-presented by Kim et al. [5] (in a slightly changed form) which will be used in the subsequent section:

$$e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c\,; \end{matrix}\; 4x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{1}{2}\,; \end{matrix}\; x^2\right]; \tag{3.3}$$

$$e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c+1\,; \end{matrix}\; 4x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{1}{2}\,; \end{matrix}\; x^2\right] - \frac{2x}{2c+1}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{3}{2}\,; \end{matrix}\; x^2\right]; \tag{3.4}$$

$$e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c-1\,; \end{matrix}\; 4x\right] = {}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{1}{2}\,; \end{matrix}\; x^2\right] + \frac{2x}{2c-1}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{1}{2}\,; \end{matrix}\; x^2\right]; \tag{3.5}$$

$$\begin{aligned} e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c+2\,; \end{matrix}\; 4x\right] =& {}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{3}{2}\,; \end{matrix}\; x^2\right] - \frac{2x}{c+1}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{3}{2}\,; \end{matrix}\; x^2\right] \\ &+ \frac{4x^2}{(c+1)(2c+3)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{5}{2}\,; \end{matrix}\; x^2\right]; \end{aligned} \tag{3.6}$$

$$\begin{aligned} e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c-2\,; \end{matrix}\; 4x\right] =& {}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{1}{2}\,; \end{matrix}\; x^2\right] + \frac{2x}{c-1}\,{}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{1}{2}\,; \end{matrix}\; x^2\right] \\ &+ \frac{4x^2}{(c-1)(2c-1)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{1}{2}\,; \end{matrix}\; x^2\right]; \end{aligned} \tag{3.7}$$

(3.8)

$$\begin{aligned} e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c+3\,; \end{matrix}\; 4x\right] &= {}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{3}{2}\,; \end{matrix}\; x^2\right] - \frac{6x}{2c+3}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{5}{2}\,; \end{matrix}\; x^2\right] \\ &+ \frac{8x^2}{(c+2)(2c+3)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{5}{2}\,; \end{matrix}\; x^2\right] - \frac{16x^3}{(c+2)(2c+3)(2c+5)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{7}{2}\,; \end{matrix}\; x^2\right]; \end{aligned}$$

(3.9)

$$\begin{aligned} e^{-2x}\,{}_1F_1\left[\begin{matrix} c\,; \\ 2c-3\,; \end{matrix}\; 4x\right] &= {}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{3}{2}\,; \end{matrix}\; x^2\right] + \frac{6x}{2c-3}\,{}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{1}{2}\,; \end{matrix}\; x^2\right] \\ &+ \frac{8x^2}{(c-1)(2c-3)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c-\frac{1}{2}\,; \end{matrix}\; x^2\right] + \frac{16x^3}{(c-1)(2c-1)(2c-3)}\,{}_0F_1\left[\begin{matrix} -\,; \\ c+\frac{1}{2}\,; \end{matrix}\; x^2\right]. \end{aligned}$$

Remark 1. The identity (3.3) is the well known Kummer's second theorem (see (1.5)) and the results (3.4) to (3.9) are closely related to it.

4. Main Result

The main result of this paper to be established is as follows:

$$
\begin{aligned}
&(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c+j\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)\\
&=\frac{1}{\Gamma(a)\,\Gamma(b)}\int_0^\infty\int_0^\infty e^{-s-t}\,s^{a-1}\,t^{b-1}\left\{\frac{\Gamma(c)\,\Gamma(1-c)}{\Gamma\left(c+\frac{j}{2}+\frac{1}{2}|j|\right)\,\Gamma\left(1-c-\left[\frac{j+1}{2}\right]\right)}\right.\\
&\quad\cdot\sum_{m=0}^{\infty}\mathcal{A}_j(c,m)\,\frac{\left(c+\left[\frac{j+1}{2}\right]\right)_m\,(xs+zt)^{2m}}{m!\,\left(c+\frac{j}{2}\right)_m\,\left(c+\frac{j}{2}+\frac{1}{2}\right)_m}\\
&\quad-\frac{2(xs+zt)}{2c+j}\,\frac{\Gamma(-c)\,\Gamma(1+c)}{\Gamma\left(c+\frac{j}{2}+\frac{1}{2}|j|\right)\,\Gamma\left(-c-\left[\frac{j}{2}\right]\right)}\\
&\quad\left.\cdot\sum_{m=0}^{\infty}\mathcal{B}_j(c,m)\,\frac{\left(1+c+\left[\frac{j}{2}\right]\right)_m\,(xs+zt)^{2m}}{m!\,\left(c+\frac{j}{2}+\frac{1}{2}\right)_m\,\left(c+\frac{j}{2}+1\right)_m}\right\}ds\,dt,
\end{aligned}
\tag{4.1}
$$

where $m\in\mathbb{N}_0$, $j=0,\,\pm1,\,\ldots,\,\pm5$, $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$, and the coefficients $\mathcal{A}_j(c,m)$ and $\mathcal{B}_j(c,m)$ are given by replacing α and n in the Table by c and m, respectively.

Proof. Denoting the left-hand side of (4.1) by L and expressing the Appell's function F_1 as a series given in (1.7), after a little simplification, we have

$$
L=\frac{1}{\Gamma(a)\,\Gamma(b)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c)_{m+n}\,4^{m+n}\,x^m\,z^n}{(2c+j)_{m+n}\,m!\,n!}\,\frac{\Gamma(m+a)\,\Gamma(n+b)}{(1+2x)^{m+a}\,(1+2z)^{n+b}}.
$$

Using the Euler's Gamma function $\Gamma(z)$ defined by

$$
\Gamma(z):=\int_0^\infty e^{-t}\,t^{z-1}\,dt\quad(\Re(z)>0),
\tag{4.2}
$$

we get

$$
\int_0^\infty e^{-ax}\,x^{n-1}\,dx=\frac{\Gamma(a)}{a^n}.
$$

Applying this formula to L, we obtain

$$
\begin{aligned}
L=&\frac{1}{\Gamma(a)\,\Gamma(b)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c)_{m+n}\,4^{m+n}\,x^m\,z^n}{(2c+j)_{m+n}\,m!\,n!}\\
&\cdot\int_0^\infty\int_0^\infty e^{-s(1+2x)-t(1+2z)}\,s^{m+a-1}\,t^{n+b-1}\,ds\,dt.
\end{aligned}
$$

Using (3.2) and changing the order of summation and integration, after a little simplification, we get

$$L = \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s(1+2x)-t(1+2z)}\, s^{a-1}\, t^{b-1} \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(c)_m\, 4^m}{(2c+j)_m} \frac{(xs)^m}{(m-n)!\, n!} \left(\frac{zt}{xs}\right)^n ds\, dt.$$

Using the result (1.4) for $(m-n)!$, after a little simplification, we have

$$L = \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s(1+2x)-t(1+2z)}\, s^{a-1}\, t^{b-1} \cdot \sum_{m=0}^{\infty} \frac{(c)_m\, 4^m}{(2c+j)_m\, m!} (xs)^m \sum_{n=0}^{m} \frac{\left(-\frac{zt}{xs}\right)^n (-m)_n}{n!} ds\, dt.$$

Using binomial theorem for the inner sum, we obtain

$$\begin{aligned} L =& \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s(1+2x)-t(1+2z)}\, s^{a-1}\, t^{b-1} \cdot \sum_{m=0}^{\infty} \frac{(c)_m\, 4^m}{(2c+j)_m\, m!} (xs)^m \left(1 + \frac{zt}{xs}\right)^m ds\, dt \\ =& \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t}\, s^{a-1}\, t^{b-1}\, e^{-2(xs+zt)} \cdot \sum_{m=0}^{\infty} \frac{(c)_m}{(2c+j)_m} \frac{4^m}{m!} (xs+zt)^m\, ds\, dt \end{aligned}$$

Using the definition (1.1) for the inner sum, we get

$$L = \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t}\, s^{a-1}\, t^{b-1}\, e^{-2(xs+zt)}\, {}_1F_1\left[\begin{matrix} c\,; \\ 2c+j\,; \end{matrix}\; 4(xs+zt)\right] ds\, dt.$$

Finally, the use of (3.1) for the ${}_1F_1$ leads to the right-hand side of (4.1). This completes the proof of (4.1). □

Special Cases of (4.1). Here we illustrate some of the very interesting special cases of our main result (4.1) as follows:

$$\begin{aligned} &(1+2x)^{-a}\,(1+2z)^{-b}\, F_1\left(c,\, a,\, b\,;\, 2c\,;\, \frac{4x}{1+2x},\, \frac{4z}{1+2z}\right) \\ &\quad = X_9\left(a,\, b\,;\, c+\frac{1}{2}\,;\, x^2,\, 2xz,\, z^2\right); \end{aligned} \tag{4.3}$$

(4.4)

$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c+1\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$= X_9\left(a,\,b\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right) - \frac{2ax}{2c+1}\,X_9\left(a+1,\,b\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$- \frac{2bz}{2c+1}\,X_9\left(a,\,b+1\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

(4.5)

$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c-1\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$= X_9\left(a,\,b\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right) + \frac{2ax}{2c-1}\,X_9\left(a+1,\,b\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+ \frac{2bz}{2c-1}\,X_9\left(a,\,b+1\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c+2\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$= X_9\left(a,\,b\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right) - \frac{2ax}{c+1}\,X_9\left(a+1,\,b\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right) \tag{4.6}$$
$$- \frac{2bz}{c+1}\,X_9\left(a,\,b+1\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right)$$

$$+ \frac{4a(a+1)x^2}{(c+1)(2c+3)}\,X_9\left(a+2,\,b\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+ \frac{8abxz}{(c+1)(2c+3)}\,X_9\left(a+1,\,b+1\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+ \frac{4b(b+1)z^2}{(c+1)(2c+3)}\,X_9\left(a,\,b+2\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c-2\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$= X_9\left(a,\,b\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right) + \frac{2ax}{c-1}\,X_9\left(a+1,\,b\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right) \tag{4.7}$$
$$+ \frac{2bz}{c-1}\,X_9\left(a,\,b+1\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$

$$+\frac{4a(a+1)x^2}{(c-1)(2c-1)}X_9\left(a+2,\,b\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+\frac{8abxz}{(c-1)(2c-1)}X_9\left(a+1,\,b+1\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+\frac{4b(b+1)z^2}{(c-1)(2c-1)}X_9\left(a,\,b+2\,;\,c+\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

(4.8)
$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c+3\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$=X_9\left(a,\,b\,;\,c+\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right)-\frac{6ax}{2c+3}X_9\left(a+1,\,b\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$-\frac{6bz}{2c+3}X_9\left(a,\,b+1\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$

$$+\frac{8a(a+1)x^2}{(c+2)(2c+3)}X_9\left(a+2,\,b\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+\frac{16abxz}{(c+2)(2c+3)}X_9\left(a+1,\,b+1\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+\frac{8b(b+1)z^2}{(c+2)(2c+3)}X_9\left(a,\,b+2\,;\,c+\frac{5}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

$$-\frac{16a(a+1)(a+2)x^3}{(c+1)(2c+3)(2c+5)}X_9\left(a+3,\,b\,;\,c+\frac{7}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$-\frac{48ab(a+1)x^2z}{(c+1)(2c+3)(2c+5)}X_9\left(a+2,\,b+1\,;\,c+\frac{7}{2}\,;\,x^2,\,2xz,\,z^2\right)$$

$$-\frac{48ab(b+1)xz^2}{(c+1)(2c+3)(2c+5)}X_9\left(a+1,\,b+2\,;\,c+\frac{7}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$-\frac{16b(b+1)(b+2)z^3}{(c+1)(2c+3)(2c+5)}X_9\left(a,\,b+3\,;\,c+\frac{7}{2}\,;\,x^2,\,2xz,\,z^2\right);$$

(4.9)
$$(1+2x)^{-a}\,(1+2z)^{-b}\,F_1\left(c,\,a,\,b\,;\,2c-3\,;\,\frac{4x}{1+2x},\,\frac{4z}{1+2z}\right)$$
$$=X_9\left(a,\,b\,;\,c-\frac{3}{2}\,;\,x^2,\,2xz,\,z^2\right)+\frac{6ax}{2c-3}X_9\left(a+1,\,b\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$
$$+\frac{6bz}{2c-3}X_9\left(a,\,b+1\,;\,c-\frac{1}{2}\,;\,x^2,\,2xz,\,z^2\right)$$

$$
\begin{aligned}
&+ \frac{8a(a+1)x^2}{(c-1)(2c-3)}\, X_9\left(a+2,\, b\, ;\, c-\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right)\\
&+ \frac{16abxz}{(c-1)(2c-3)}\, X_9\left(a+1,\, b+1\, ;\, c-\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right)\\
&+ \frac{8b(b+1)z^2}{(c-1)(2c-3)}\, X_9\left(a,\, b+2\, ;\, c-\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right);\\
&+ \frac{16a(a+1)(a+2)x^3}{(c-1)(2c-1)(2c-3)}\, X_9\left(a+3,\, b\, ;\, c+\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right)\\
&+ \frac{48ab(a+1)x^2z}{(c-1)(2c-1)(2c-3)}\, X_9\left(a+2,\, b+1\, ;\, c+\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right)\\
&+ \frac{48ab(b+1)xz^2}{(c-1)(2c-1)(2c-3)}\, X_9\left(a+1,\, b+2\, ;\, c+\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right)\\
&+ \frac{16b(b+1)(b+2)z^3}{(c-1)(2c-1)(2c-3)}\, X_9\left(a,\, b+2\, ;\, c+\frac{1}{2}\, ;\, x^2,\, 2xz,\, z^2\right).
\end{aligned}
$$

Proof. Here we choose to prove only (4.6). The same argument will establish the other results. Setting $j = 2$ in (4.1), we have

$$
\begin{aligned}
T &:= (1+2x)^{-a}\,(1+2z)^{-b}\, F_1\left(c,\, a,\, b\, ;\, 2c+2\, ;\, \frac{4x}{1+2x},\, \frac{4z}{1+2z}\right)\\
&= \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t}\, s^{a-1}\, t^{b-1}\, e^{-2(xs+zt)}\, {}_1F_1\left[\begin{matrix} c\, ; \\ 2c+2\, ; \end{matrix}\; 4(xs+zt)\right] ds\, dt.
\end{aligned}
$$

Using (3.6) for the ${}_1F_1$, we obtain

$$
\begin{aligned}
T = \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t}\, s^{a-1}\, t^{b-1} \Bigg\{ &{}_0F_1\left[\begin{matrix} -\, ; \\ c+\frac{3}{2}\, ; \end{matrix}\; (xs+zt)^2\right]\\
&- \frac{2(xs+zt)}{c+1}\, {}_0F_1\left[\begin{matrix} -\, ; \\ c+\frac{3}{2}\, ; \end{matrix}\; (xs+zt)^2\right]\\
&+ \frac{4(xs+zt)^2}{(c+1)(2c+3)}\, {}_0F_1\left[\begin{matrix} -\, ; \\ c+\frac{5}{2}\, ; \end{matrix}\; (xs+zt)^2\right] \Bigg\}\, ds\, dt.
\end{aligned}
$$

Now, separating the integral into six integrals and interpreting each of those integrals in terms of X_9 in (1.9), we easily arrive at the right-hand side of (4.6). This completes the proof of (4.6). □

Remark 2. The result (4.3) is a known identity due to Exton [4] and the identities (4.4) to (4.9) are closely related to it. We conclude by noting that if set $z = x$ in (4.3) to (4.9), we also get seven (presumably new) interesting identities.

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