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PROPERTIES OF HYPERHOLOMORPHIC FUNCTIONS ON DUAL TERNARY NUMBERS

Hyun Sook Jung [a] and Kwang Ho Shon [b,*]

Abstract. We research properties of ternary numbers with values in $\Lambda(2)$. Also, we represent dual ternary numbers in the sense of Clifford algebras of real six dimensional spaces. We give generation theorems in dual ternary number systems in view of Clifford analysis, and obtain Cauchy theorems with respect to dual ternary numbers.

1. Introduction

In quaternion algebras, Brackx [1], Deavours [3] and Sudbery [17] obtained some results of quaternion variables. Brackx, Delanghe and Sommen [2] and Gürsey and Tze [5] researched general theories of quaternions. Naser [12] investigated hyper-conjugate harmonic functions on Clifford analysis. And Nôno [13, 15] obtained properties of hyperholomorphic functions of a quaternion variable and domains of holomorphy by the existence of hyper-conjugate harmonic functions. Nôno [14] represented one quaternion variable $z = x_0 + ix_1 + jx_2 + kx_3$ by a pair $z = z_1 + z_2 j$ of two complex variables $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$, and established the identification $\mathcal{T} \cong \mathbf{C}^2$ between the quaternion field $\mathcal{T}$ and the complex space $\mathbf{C}^2$. And Nôno [16] introduced the notion of hyperholomorphy of quaternion valued functions $f(z) = f_1(z)+f_2(z)j$ of a quaternion variable $z = z_1+z_2 j$ in $\mathbf{C}^2$ and proved that any complex valued harmonic function $f_1(z)$ in a domain of holomorphy D in $\mathbf{C}^2$ has a hyper-conjugate harmonic function $f_2(z)$ such that $f(z)$ is hyperholomorphic in D. In 2006, Gürlebeck and Viet [4] obtained some results for complete systems of monogenic rational functions. And Kula and Yayli [8] investigated properties of dual split quaternions and screw motion in Minkowski three-space. We [6, 7] obtained

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*Corresponding author.

some properties of regeneration in complex, quaternion and Clifford analysis and researched the properties of solutions of inhomogeneous Cauchy-Riemann system of quaternions and Clifford analysis in ellipsoid. In 2012, we [9] investigated properties of hyperholomorphic functions in Clifford analysis. In 2013, we [10, 11] researched properties of hyperholomorphic functions and hyper-conjugate harmonic functions of octonion variables, and regularities of functions with values in $\mathbf{C}(n)$ of matrix algebras. The main purpose of the present paper is to give generation theorems in ternary numbers and dual ternary numbers in view of Clifford analysis. A dual ternary number is described as a ternary number with dual numbers as coefficients. Dual numbers are written in the form $z = a + \varepsilon b$, where ε is the dual identity that commutes with e_1, e_2 and has the property $\varepsilon^2 = 0$ (ε is nilpotent).

2. Preliminaries

The skew field $T \cong \mathbf{R}^3$ of ternary numbers $a = \sum_{j=0}^{2} e_j x_j$ is a three dimensional non-commutative real field generated by three bases $e_0 = id.$, e_1 and e_2 are as follows:

$$e_1^2 = e_2^2 = -1.$$

We identify the element e_1 with the imaginary unit $\sqrt{-1}$ in the complex number system. The algebra

$$\Lambda(2) := \{a = \sum_{j=0}^{2} e_j x_j \mid x_j \in \mathbf{R}; j = 0, 1, 2\} \cong T$$

is a non commutative subalgebra of $\mathbf{R}^3$. We can identify $\Lambda(2)$ with $\mathbf{R}^3$. We let

$$a = \sum_{j=0}^{2} e_j x_j \text{ and } b = \sum_{j=0}^{2} e_j y_j. \tag{1}$$

Let Ω be an open subset of $\mathbf{R}^3$, $U(a) = \Sigma_{j=0}^{2} e_j u_j(a)$ and $V(b) = \Sigma_{j=0}^{2} e_j v_j(b)$ be functions defined on Ω with values in $\Lambda(2)$, where $a = (x_0, x_1, x_2)$, $b = (y_0, y_1, y_2)$ in (1), $u_j(a)$ and $v_j(b)$ $(j = 0, 1, 2)$ are real valued functions with variables x_0, x_1, x_2, y_0, y_1 and y_2. We consider that the numbers y_0, y_1 and y_2 are related numbers with respect to the numbers x_0, x_1 and x_2, respectively. And also, we consider the functions v_0, v_1 and v_2 are related functions with respect to the functions u_0, u_1 and u_2, respectively.

We consider the following differential operators:

$$\frac{\partial}{\partial A} := \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}, \ \frac{\partial}{\partial B} := \frac{\partial}{\partial y_0} - e_1 \frac{\partial}{\partial y_1} - e_2 \frac{\partial}{\partial y_2},$$

$$\frac{\partial}{\partial A^*}=\sum_{j=0}^{2} e_j\frac{\partial}{\partial x_j},\ \frac{\partial}{\partial B^*}=\sum_{j=0}^{2} e_j\frac{\partial}{\partial y_j}.$$

Then, the operator

$$\frac{\partial^2}{\partial A\partial A^*}=\sum_{j=0}^{2}\frac{\partial^2}{\partial {x_j}^2}$$

is the usual real Laplacian Δ_a, which is a typical elliptic differential operator with constant coefficients on $\mathbf{R}^3$ with respect to the variables x_0, x_1 and x_2. And consider differential operators:

$$D := \frac{\partial}{\partial x_0}-e_1\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_2}+\varepsilon(\frac{\partial}{\partial y_0}-e_1\frac{\partial}{\partial y_1}-e_2\frac{\partial}{\partial y_2})=\frac{\partial}{\partial A}+\varepsilon\frac{\partial}{\partial B},$$
$$D^* = \frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+\varepsilon(\frac{\partial}{\partial y_0}+e_1\frac{\partial}{\partial y_1}+e_2\frac{\partial}{\partial y_2})=\frac{\partial}{\partial A^*}+\varepsilon\frac{\partial}{\partial B^*}.$$

Then, we have the following for dual ternary operators:

$$DD^* = D^*D=\sum_{j=0}^{2}\frac{\partial^2}{\partial {x_j}^2}+2\varepsilon(\sum_{j=0}^{2}\frac{\partial^2}{\partial x_j\partial y_j})=\sum_{j=0}^{2}\frac{\partial^2}{\partial {\tau_j}^2},$$

where

$$\frac{\partial}{\partial \tau_j}=\frac{\partial}{\partial x_j}+\varepsilon\frac{\partial}{\partial y_j}\ (j=0,1,2).$$

Definition 1. Let Ω_1 be an open subset of $\mathbf{R}^3$. A function $U(a)=\sum_{j=0}^{2} e_j u_j(a)$ is said to be *hyperholomorphic* in Ω_1 if

(a) $u_j(a)$ $(j=0,1,2)$ are continuously differentiable in Ω_1;
(b) $\frac{\partial}{\partial A^*}\odot U(a)=0$ in Ω_1.

Equation (b) of Definition 1 operates on $U(a)$ as follows:

$$\begin{aligned}\frac{\partial}{\partial A^*}\odot U &= (\sum_{j=0}^{2} e_j\frac{\partial}{\partial x_j})\odot(\sum_{j=0}^{2} e_j u_j)\\ &:= \frac{\partial u_0}{\partial x_0}+e_1\frac{\partial u_1}{\partial x_0}+e_2\frac{\partial u_2}{\partial x_0}+e_1\frac{\partial u_0}{\partial x_1}-\frac{\partial u_1}{\partial x_1}+e_2\frac{\partial u_0}{\partial x_2}-\frac{\partial u_2}{\partial x_2}\\ &= \frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}+e_1(\frac{\partial u_1}{\partial x_0}+\frac{\partial u_0}{\partial x_1})+e_2(\frac{\partial u_2}{\partial x_0}+\frac{\partial u_0}{\partial x_2}).\end{aligned}$$

The above equation (b) of Definition 1 for $U(a)$ is equivalent to the following system of equations:

$$\frac{\partial u_0}{\partial x_0}=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},\ \frac{\partial u_1}{\partial x_0}=-\frac{\partial u_0}{\partial x_1},\ \frac{\partial u_2}{\partial x_0}=-\frac{\partial u_0}{\partial x_2}. \tag{2}$$

Theorem 1. *Let $U(a)$ be a hyperholomorphic function in a domain Ω_1 of $\mathbf{R}^3$ and let*

$$k_1 = dx_1 \wedge dx_2 - e_1(dx_0 \wedge dx_2) + e_2(dx_0 \wedge dx_1).$$

Then for any domain $G \subset \Omega_1$ with smooth boundary bG have

$$\int_{bG} k_1 \odot U(a) = 0,$$

where $k_1 \odot U(a)$ is the dot product of ternary numbers of the form k_1 on the function $U(a)$.

Proof. By the rule of the multiplication of ternary numbers, we have

$$\begin{aligned} k_1 \odot U &= (dx_1 \wedge dx_2 - e_1(dx_0 \wedge dx_2) + e_2(dx_0 \wedge dx_1))(u_0 + e_1u_1 + e_2u_2) \\ &= u_0dx_1 \wedge dx_2 + e_1u_1dx_1 \wedge dx_2 + e_2u_2dx_1 \wedge dx_2 - e_1u_0dx_0 \wedge dx_2 \\ &\quad +u_1dx_0 \wedge dx_2 + e_2u_0dx_0 \wedge dx_1 - u_2dx_0 \wedge dx_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(k_1 \odot U) &= (\frac{\partial}{\partial x_0}dx_0 + \frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2)(k_1 \odot U) \\ &= \frac{\partial u_0}{\partial x_0}dx_0 \wedge dx_1 \wedge dx_2 + e_1\frac{\partial u_1}{\partial x_0}dx_0 \wedge dx_1 \wedge dx_2 + e_2\frac{\partial u_2}{\partial x_0}dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad - e_1\frac{\partial u_0}{\partial x_1}dx_1 \wedge dx_0 \wedge dx_2 + \frac{\partial u_1}{\partial x_1}dx_1 \wedge dx_0 \wedge dx_2 + e_2\frac{\partial u_0}{\partial x_2}dx_2 \wedge dx_0 \wedge dx_1 \\ &\quad - \frac{\partial u_2}{\partial x_2}dx_2 \wedge dx_0 \wedge dx_1 \\ &= (\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2})\, dx_0 \wedge dx_1 \wedge dx_2 + e_1(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1})dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad + e_2(\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2})dx_0 \wedge dx_1 \wedge dx_2. \end{aligned}$$

By the equation (2), we have $d(k_1 \odot U(a)) = 0$. By Stoke's theorem, we have

$$\int_{bG} k_1 \odot U(a) = \int_G d(k_1 \odot U(a)) = 0.$$

□

3. Properties of Dual Ternary Numbers

Dual ternary numbers are constructed as the Clifford algebra of real six dimensional space with a degenerate quadratic form. The base elements of dual ternary numbers can be identified with e_1 and e_2 of the ternary numbers and the dual identity ε. Then the dual ternary number $z = x_0 + e_1x_1 + e_2x_2 + \varepsilon(y_0 + e_1y_1 + e_2y_2)$ is written as

$$z = \sum_{j=0}^{2} e_j x_j + \varepsilon(\sum_{j=0}^{2} e_j y_j) = a + \varepsilon b.$$

The conjugate number z^*, the absolute value $|z|$ and the inverse z^{-1} of $z = a + \varepsilon b$ are given by the following:

$$z^* = x_0 - e_1x_1 - e_2x_2 + \varepsilon(y_0 - e_1y_1 - e_2y_2) = a^* + \varepsilon b^*,$$

$$|z|^2 = zz^* = \sum_{j=0}^{2} {x_j}^2 + 2\varepsilon \sum_{j=0}^{2} x_j y_j = \sum_{j=0}^{2} {\xi_j}^2, z^{-1} = \frac{z^*}{|z|^2},$$

where $\xi_j = x_j + \varepsilon y_j$, a^* and b^* are conjugate numbers of a and b, respectively.

Definition 2. Let Ω be an open subset of $\mathbf{C}^3$. A function $W(a,b) = U(a)+\varepsilon V(b) = \sum_{j=0}^{2} e_j u_j(a) + \varepsilon(\sum_{j=0}^{2} e_j v_j(b))$ is said to be *hyperholomorphic* in Ω if

(a) $u_j(a)$ and $v_j(b)(j = 0, 1, 2)$ are continuously differentiable in Ω;
(b) $D^* \odot W(a, b) = 0$ in Ω.

Equation (b) of Definition 2 operates on $W(a, b)$ as follows:

$$\begin{aligned}
D^* \odot W &= (\sum_{j=0}^{2} e_j \frac{\partial}{\partial x_j} + \varepsilon \sum_{j=0}^{2} e_j \frac{\partial}{\partial y_j}) \odot (\sum_{j=0}^{2} e_j u_j + \varepsilon \sum_{j=0}^{2} e_j v_j) \\
&= (\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \varepsilon(\frac{\partial}{\partial y_0} + e_1 \frac{\partial}{\partial y_1} + e_2 \frac{\partial}{\partial y_2})) \\
&\quad (u_0 + e_1u_1 + e_2u_2 + \varepsilon(v_0 + e_1v_1 + e_2v_2)) \\
&= \frac{\partial u_0}{\partial x_0} + e_1 \frac{\partial u_1}{\partial x_0} + e_2 \frac{\partial u_2}{\partial x_0} + e_1 \frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_1} + e_2 \frac{\partial u_0}{\partial x_2} - \frac{\partial u_2}{\partial x_2} \\
&\quad +\varepsilon(\frac{\partial v_0}{\partial x_0} + e_1 \frac{\partial v_1}{\partial x_0} + e_2 \frac{\partial v_2}{\partial x_0} + e_1 \frac{\partial v_0}{\partial x_1} - \frac{\partial v_1}{\partial x_1} + e_2 \frac{\partial v_0}{\partial x_2} - \frac{\partial v_2}{\partial x_2} \\
&\quad + \frac{\partial u_0}{\partial y_0} + e_1 \frac{\partial u_1}{\partial y_0} + e_2 \frac{\partial u_2}{\partial y_0} + e_1 \frac{\partial u_0}{\partial y_1} - \frac{\partial u_1}{\partial y_1} + e_2 \frac{\partial u_0}{\partial y_2} - \frac{\partial u_2}{\partial y_2}) \\
&= \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + e_1(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1}) + e_2(\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2}) \\
&\quad +\varepsilon(\frac{\partial v_0}{\partial x_0} - \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} + \frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} \\
&\quad +e_1(\frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1}) + e_2(\frac{\partial v_2}{\partial x_0} + \frac{\partial v_0}{\partial x_2} + \frac{\partial u_2}{\partial y_0} + \frac{\partial u_0}{\partial y_2})).
\end{aligned}$$

Therefore, the equation (b) of Definition 2 for $W(a,b)$ is equivalent to the following system of equations:

$$\frac{\partial u_0}{\partial x_0}=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},\ \frac{\partial u_1}{\partial x_0}=-\frac{\partial u_0}{\partial x_1},\ \frac{\partial u_2}{\partial x_0}=-\frac{\partial u_0}{\partial x_2},$$

$$\frac{\partial v_0}{\partial x_0}+\frac{\partial u_0}{\partial y_0}=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial u_1}{\partial y_1}+\frac{\partial u_2}{\partial y_2},$$

$$\frac{\partial v_1}{\partial x_0}+\frac{\partial v_0}{\partial x_1}+\frac{\partial u_1}{\partial y_0}+\frac{\partial u_0}{\partial y_1}=0,\ \frac{\partial v_2}{\partial x_0}+\frac{\partial v_0}{\partial x_2}+\frac{\partial u_2}{\partial y_0}+\frac{\partial u_o}{\partial y_2}=0. \tag{3}$$

We add the following condition of integrability:

$$\frac{\partial v_0}{\partial x_0}=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2},\ \frac{\partial v_1}{\partial x_0}+\frac{\partial v_0}{\partial x_1}=0,\ \frac{\partial v_2}{\partial x_0}+\frac{\partial v_0}{\partial x_2}=0. \tag{4}$$

We let

$$dxy=dx_0\wedge dx_1\wedge dx_2\wedge dy_0\wedge dy_1\wedge dy_2.$$

Theorem 2. *Under the condition of integrability* (4), *let* $W(a,b)$ *be a hyperholomorphic function on a domain* Ω *of* $\mathbf{C}^3$ *and*

$$k_2=\hat{dy_0}-e_1\hat{dy_1}+e_2\hat{dy_2}+\varepsilon(\hat{dx_0}-e_1\hat{dx_1}+e_2\hat{dx_2}),$$

where $\hat{dx_j}$ *is the* dx_j*-removed form on* dxy*, and* $\hat{dy_j}$ *is the* dy_j*-removed form on* dxy $(j=0,1,2)$*. Then for any domain* $G\subset\Omega$ *with smooth boundary* bG*,*

$$\int_{bG}k_2\odot W(a,b)=0,$$

where $k_2\odot W(a,b)$ *is the dot product of ternary numbers of the form* k_2 *on the function* $W(a,b)$*.*

Proof. By the rule of the multiplication of ternary numbers, we have

$$\begin{aligned}k_2\odot W &= (\hat{dy_0}-e_1\hat{dy_1}+e_2\hat{y_2}+\varepsilon(\hat{dx_0}-e_1\hat{dx_1}+e_2\hat{dx_2}))(u_0+e_1u_1+e_2u_2\\&\quad+\varepsilon(v_0+e_1v_1+e_2v_2))\\&= u_o\hat{dy_0}+e_1u_1\hat{dy_0}+e_2u_2\hat{dy_0}-e_1u_0\hat{dy_1}+u_1\hat{dy_1}+e_2u_0\hat{dy_2}-u_2\hat{dy_2}\\&\quad+\varepsilon(v_0\hat{dy_0}+e_1v_1\hat{dy_0}+e_2v_2\hat{dy_0}-e_1v_0\hat{dy_1}+v_1\hat{dy_1}+e_2v_0\hat{dy_2}-v_2\hat{dy_2}\\&\quad+u_0\hat{dx_0}+e_1u_1\hat{dx_0}+e_2u_2\hat{dx_0}-e_1u_0\hat{dx_1}+u_1\hat{dx_1}+e_2u_0\hat{dx_2}-u_2\hat{dx_2}).\end{aligned}$$

Therefore,

$$
\begin{aligned}
d(k_2 \odot W) &= (\frac{\partial}{\partial x_0}dx_0 + \frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 \\
&\quad +\varepsilon(\frac{\partial}{\partial y_0}dy_0 + \frac{\partial}{\partial y_1}dy_1 + \frac{\partial}{\partial y_2}dy_2))(k_2 \odot W) \\
&= \varepsilon(-\frac{\partial u_0}{\partial y_0}dxy + \frac{\partial u_1}{\partial y_1}dxy + \frac{\partial u_2}{\partial y_2}dxy + \frac{\partial u_0}{\partial x_0}dxy - \frac{\partial u_1}{\partial x_1}dxy - \frac{\partial u_2}{\partial x_2}dxy \\
&\quad +e_1(\frac{\partial u_0}{\partial x_1}dxy + \frac{\partial u_1}{\partial x_0}dxy - \frac{\partial u_1}{\partial y_0}dxy - \frac{\partial u_0}{\partial y_1}dxy) \\
&\quad +e_2(\frac{\partial u_2}{\partial x_0}dxy + \frac{\partial u_0}{\partial x_2}dxy - \frac{\partial u_2}{\partial y_0}dxy - \frac{\partial u_0}{\partial y_2})) \\
&= -\varepsilon((\frac{\partial u_0}{\partial y_0} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} - \frac{\partial u_0}{\partial x_0})dxy \\
&\quad +e_1(\frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} - \frac{\partial u_1}{\partial x_0} - \frac{\partial u_0}{\partial x_1})dxy \\
&\quad +e_2(\frac{\partial u_2}{\partial y_0} + \frac{\partial u_0}{\partial y_2} - \frac{\partial u_2}{\partial x_0} - \frac{\partial u_0}{\partial x_2})dxy).
\end{aligned}
$$

By the equations (3) and (4), we have $d(k_2 \odot W(a,b)) = 0$. By Stoke's theorem, we have

$$\int_{bG} k_2 \odot W(a,b) = \int_G d(k_2 \odot W(a,b)) = 0.$$

□

References

1. F. Brackx: On (k)-monogenic functions of a quaternion variable. *Res. Notes in Math.* **8** (1976), 22-44.
2. F. Brackx, R. Delanghe & F. Sommen: Clifford analysis. *Res. Notes in Math.* **76** (1982), 1-43.
3. C.A. Deavours: The quaternion calculus. *Amer. Math. Monthly* **80** (1973), 995-1008.
4. K. Gürlebeck and T.Q. Viet: *Function spaces in complex and Clifford analysis, On some complete system of monogenic rational functions.* Proc. 14h Inter. Conf. on Finite or Inf. Dimen. Complex Anal. Appl., Advances in Complex Anal. Appl. Vol. 14, Hue Univ. (2008), 156-169.
5. F. Gürsey & H.C. Tze: Complex and Quaternionic Analyticity in Chiral and Gauge Theories I. *Ann. of Physics* **128** (1980), 29-130.
6. J. Kajiwara, X.D. Li & K.H. Shon: *Regeneration in Complex, Quaternion and Clifford analysis.* Proc. 9th Inter. Conf. on Finite or Inf. Dimen. Complex Anal. and Appl., Advances in Complex Anal. Appl. Vol. 2, Kluwer Academic Publishers (2004), 287-298.

7. ______: Function spaces in complex and Clifford analysis, Inhomogeneous Cauchy Riemann system of quaternion and Clifford analysis in ellipsoid. *Proc. 14th Inter. Conf. on Finite or Inf. Dimen. Complex Anal. Appl., Advances in Complex Anal. Appl. Vol. 14, Hue Univ.* (2008), 127-155.
8. L. Kula & Y. Yayli: Dual spilt quaternions and screw motion in Minkowski 3-space. *Iranian J. Sci. Tech, Trans. A.* **30** (2006), 245-258.
9. S.J. Lim & K.H. Shon: Properties of hyperholomorphic functions in Clifford analysis. *East Asian Math. J.* **28** (2012), 553-559.
10. ______: Hyperholomorphic functions and hyper-conjugate harmonic functions of octonion variables. *J. Inequal. Appl.* **77** (2013), 1-8.
11. ______: Regularities of functions with values in C(n) of matrix algebras M(n;C). *submitted in J. Inequal. Appl.* (2013).
12. M. Naser: Hyperholomorphic functions. *Siberian Math. J.* **12** (1971), 959-968.
13. K. Nôno: Hyperholomorphic functions of a quaternion variable. *Bull. Fukuoka Univ. Ed.* **32** (1983), 21-37.
14. ______: On the Quaternion Linearization of Laplacian. *Bull. Fukuoka Univ. Ed.* **35** (1985), 5-10.
15. ______: Characterization of domains of holomorphy by the existence of hyper-conjugate harmonic functions. *Rev. Roumaine Math. Pures Appl.* **31** (1986), no. 2, 159-161.
16. ______: Domains of Hyperholomorphic in $\mathbb{C}^2 \times \mathbb{C}^2$. *Bull. Fukuoka Univ. Ed.* **36** (1987), 1-9.
17. A. Sudbery: Quaternionic analysis. *Math. Proc. Camb. Phil. Soc.* **85** (1979), 199–225.

[a]Department of Mathematics, College of Natural Sciences, Pusan National University, Busan 609-735, Korea
Email address: dream771004@naver.com

[b]Department of Mathematics, College of Natural Sciences, Pusan National University, Busan 609-735, Korea
Email address: khshon@pusan.ac.kr