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HILBERT 2-CLASS FIELD TOWERS OF INERT IMAGINARY QUADRATIC FUNCTION FIELDS

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Abstract. In this paper we study the infiniteness of Hilbert 2-class field towers of inert imaginary quadratic function fields over $\mathbb{F}_q(T)$, where q is a power of an odd prime number.

1. Introduction and Statement of the Results

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$ of q elements, $\infty = (1/T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. For any finite separable extension F of k, write $\mathcal{O}_F$ for the integral closure of $\mathbb{A}$ in F and H_F for the Hilbert class field of F with respect to $\mathcal{O}_F$ (see [4]). Let ℓ be a prime number. Let $F_0^{(\ell)} = F$ and $F_{n+1}^{(\ell)}$ be the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 0$, i.e., $F_{n+1}^{(\ell)}$ is the maximal extension of $F_n^{(\ell)}$ inside $H_{F_n^{(\ell)}}$ whose degree over $F_n^{(\ell)}$ is a power of ℓ. The sequence of fields

$$F = F_0^{(\ell)} \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$$

is called the *Hilbert ℓ-class field tower of F*. We say that the Hilbert ℓ-class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. Let $\mathcal{C}l_F$ and $\mathcal{O}_F^*$ be the ideal class group and the group of units of $\mathcal{O}_F$, respectively. For any multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$ for the ℓ-rank of A. Schoof [6] has shown that if $r_\ell(\mathcal{C}l_F) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}$, then the Hilbert ℓ-class field tower of F is infinite. This is a function field analog of Golod-Shafarevich.

Now, we assume that q is odd with $q \equiv 3 \bmod 4$. In [2, 3], we study the infiniteness of Hilbert 2-class field towers of ramified imaginary or real quadratic function fields over k. The aim of this paper is to study the infiniteness of Hilbert 2-class field towers of inert imaginary quadratic function fields over k. Let F be an *inert imaginary*

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quadratic function field over k, i.e., F is a quadratic extension of k in which ∞ is inert. Fix a generator γ of $\mathbb{F}_q^*$. Let $\mathcal{P}$ be the set of monic irreducible polynomials in $\mathbb{A}$. Then F can be written as $F = k(\sqrt{D})$ with $D = \gamma P_1 \cdots P_t$, $P_i \in \mathcal{P}$ for $1 \leq i \leq t$ and $2 | \deg D$. Here, D is uniquely determined by F and write $D_F = D$. We say that D_F is *special* if $2 | \deg P_i$ for all $1 \leq i \leq t$. For $0 \neq N \in \mathbb{A}$, write $\omega(N)$ for the number of distinct monic irreducible divisors of N. It is known ([1, Corollary 3.5]) that $r_2(\mathcal{C}l_F)$ is equal to $\omega(D_F)$ or $\omega(D_F) - 1$ according as D_F is special or non-special. Since $\mathcal{O}_F^* = \mathbb{F}_q^*$ (i.e., $r_2(\mathcal{O}_F^*) = 1$), by Schoof's Theorem, F has infinite Hilbert 2-class field tower if $r_2(\mathcal{C}l_F) \geq 5$.

For an inert imaginary quadratic function field F over k, let s_F be the number of monic irreducible divisors of D_F of odd degree. Since $\deg D_F$ is even, s_F is a nonnegative even integer. Let ϑ_F be 1 or 0 according as $s_F = 0$ or $s_F \geq 2$. Let $r_4(\mathcal{C}l_F) = r_2(\mathcal{C}l_F^2)$ be the 4-rank of $\mathcal{C}l_F$. Then we have

Theorem 1.1. *Assume that $q \equiv 3 \bmod 4$. Let F be an inert imaginary quadratic function field over k. If $r_4(\mathcal{C}l_F) \geq 3 + \vartheta_F$, then the Hilbert 2-class field tower of F is infinite.*

For any positive even integer n and integers r, s with $0 \leq s \leq r$, let $Z_{r;n}$ be the set of inert imaginary quadratic function fields F with $r_2(\mathcal{C}l_F) = r$ and $\deg(D_F) = n$, and $Z_{r,s;n}$ be the subset of $Z_{r;n}$ consisting of $F \in Z_{r;n}$ with $r_4(\mathcal{C}l_F) = s$. Let $Z_{r,s;n}^*$ be the subset of $Z_{r,s;n}$ consisting of $F \in Z_{r,s;n}$ whose Hilbert 2-class field tower is infinite. We define a density $\varrho_{r,s}^*$ by

$$\varrho_{r,s}^* = \liminf_{\substack{n \to \infty \\ n:\text{even}}} \frac{|Z_{r,s;n}^*|}{|Z_{r;n}|}.$$

Then we have

Theorem 1.2. *Assume that $q \equiv 3 \bmod 4$. We have $\varrho_{3,s}^* \geq 2^{-3}(2^3 - 1)^{-1}$ for $s = 1, 2$ and $\varrho_{4,s}^* \geq 2^{-6}(2^4 - 1)^{-1}$ for $0 \leq s \leq 3$.*

We remark that Theorem 1.2 means that a positive proportion of inert imaginary quadratic function fields F with $r_2(\mathcal{C}l_F) = r$ have infinite Hilbert 2-class field towers and $r_4(\mathcal{C}l_F) = s$ for $r = 3, s = 1, 2$ or $r = 4, 0 \leq s \leq 3$.

2. Preliminaries

2.1. Rédei-matrix and 4-rank of class group Let F be an inert imaginary quadratic function field over k with $D_F = \gamma P_1 \cdots P_t$. Let $d_i \in \mathbb{F}_2$ be defined by

$d_i \equiv \deg P_i \bmod 2$ for $1 \leq i \leq t$. Let $M_F = (e_{ij})_{1\leq i,j\leq t}$ be a $t \times t$ matrix over $\mathbb{F}_2$, where $e_{ij} \in \mathbb{F}_2$ is defined by $(\frac{P_i}{P_j}) = (-1)^{e_{ij}}$ for $1 \leq i \neq j \leq t$ and the diagonal entries $e_{ii} \in \mathbb{F}_2$ are defined to satisfy the relation $d_i = \sum_{j=1}^{t} e_{ij}$. We associate a $(t+1) \times t$ matrix R_F over $\mathbb{F}_2$ to F as follows:

- If D_F is non-special, R_F is the $(t+1) \times t$ matrix obtained from M_F by adding $(d_1 \ \cdots \ d_t)$ in the last row.
- If D_F is special, R_F is the $(t+1) \times t$ matrix obtained from M_F by adding $(e_{B1} \ \ldots \ e_{Bt})$ in the last row, where B is a monic polynomial in $\mathbb{A}$ such that $(B) = N_{F/k}(\mathfrak{B}), \mathfrak{B}^{\sigma-1} = (x)$ with $N_{F/k}(x) \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$ and $e_{Bi} \in \mathbb{F}_2$ is defined by $(\frac{B}{P_i}) = (-1)^{e_{Bi}}$. (Here, σ is the generator of $Gal(F/k)$.)

Then we have ([1, Corollary 3.8])

$$r_4(\mathcal{C}l_F) = \omega(D_F) - \operatorname{rank} R_F. \tag{2.1}$$

2.2. Some lemmas Let E and K be finite geometric separable extensions of k such that E/K is a cyclic extension of degree ℓ, where ℓ is a prime number not dividing q. Let $S_\infty(K)$ be the set of primes of K lying above ∞. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of primes $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. In [2, Proposition 2.1], we have shown that the Hilbert ℓ-class field tower of E is infinite if

$$\gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell|S_\infty(K)| + (1-\ell)\rho_{E/K} + 1}. \tag{2.2}$$

Now, by using this result, we give some sufficient conditions for an inert imaginary quadratic function field F to have infinite Hilbert 2-class field tower.

Lemma 2.1. *Let F be an inert imaginary quadratic function field over k. If there exists a nonconstant divisor D' of D_F such that either D' or D_F/D' is monic of even degree and $(\frac{D'}{P_i}) = 1$ for monic irreducible divisors P_i $(1 \leq i \leq 4)$ of D_F, then F has infinite Hilbert 2-class field tower.*

Proof. Let $K = k(\sqrt{D'})$ and $E = KF$. Since P_1, P_2, P_3 and P_4 split in K, we have $\gamma_{E/K} \geq 8$. We also have $|S_\infty(K)| = \rho_{E/K} = 2$ or $(|S_\infty(K)|, \rho_{E/K}) = (1, 0)$ according as D' is monic of even degree or D_F/D' is monic of even degree. Then E has infinite Hilbert 2-class field tower. It can be easily shown that E is contained in $F_1^{(2)}$. Thus F also has infinite Hilbert 2-class field tower. □

Lemma 2.2. *Let F be an inert imaginary quadratic function field over k. If D_F has two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying*

$(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ *for monic irreducible divisors* P_i $(i = 1, 2)$ *of* D_F, *then* F *has infinite Hilbert 2-class field tower.*

Proof. Let $K = k(\sqrt{D_1}, \sqrt{D_2})$ and $E = KF$. Since P_1, P_2 and ∞ splits completely in K, E is contained in $F_1^{(2)}$. Since $\gamma_{E/K} \geq 8$ and $|S_\infty(K)| = \rho_{E/K} = 4$, we see that E has infinite Hilbert 2-class field tower. Hence, F also has infinite Hilbert 2-class field tower. □

2.3. Some asymptotic results Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_q[T]$. For positive integers n and t, write $\mathcal{P}(n, t)$ for the set of monic square free polynomials $N \in \mathbb{A}$ with $\deg N = n$ and $\omega(N) = t$, and $\mathcal{P}'(n, t)$ for the subset of $\mathcal{P}(n, t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n, t)$ such that $\deg P_i \neq \deg P_j$ for $1 \leq i \neq j \leq t$. Let $\mathcal{P}_2(n, t)$ be the subset of $\mathcal{P}(n, t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n, t)$ such that $\deg P_i$ is even for all $1 \leq i \leq t$ and $\mathcal{P}'_2(n, t) = \mathcal{P}_2(n, t) \cap \mathcal{P}'(n, t)$. As $n \to \infty$, we have

$$|\mathcal{P}(n, t)| = \frac{q^n (\log n)^{t-1}}{(t-1)! n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big), \tag{2.3}$$

$$|\mathcal{P}_2(n, t)| = \frac{q^n (\log n)^{t-1}}{(t-1)! 2^{t-1} n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big), \tag{2.4}$$

$$|\mathcal{P}(n, t) \setminus \mathcal{P}'(n, t)| = o\Big(\frac{q^n (\log n)^{t-1}}{n}\Big). \tag{2.5}$$

For $N = P_1 \cdots P_t, M = Q_1 \cdots Q_t \in \mathcal{P}'(n, t)$, we say that N and M are equivalent if $\deg P_i \equiv \deg Q_i \bmod 2$ for $1 \leq i \leq t$ and $(\frac{P_i}{P_j}) = (\frac{Q_i}{Q_j})$ for $1 \leq i < j \leq t$. Write $\mathcal{N}(N)$ for the set of polynomials in $\mathcal{P}'(n, t)$ which are equivalent to N. Then we have ([2, Proposition 2.9])

$$|\mathcal{N}(N)| = 2^{1 - \frac{(t^2+t)}{2}} \cdot \frac{q^n (\log n)^{t-1}}{(t-1)! n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big) \tag{2.6}$$

as $n \to \infty$.

For a positive even integer n and a positive integer t, let $\mathcal{R}'(n, t) = \mathcal{P}'(n, t) \setminus \mathcal{P}'_2(n, t)$. Then we have ([3, Proposition 2.4])

$$|\mathcal{R}'(n, t)| = (1 - 2^{1-t}) \frac{q^n (\log n)^{t-1}}{(t-1)! n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big). \tag{2.7}$$

3. Proof of Theorem 1.1

Let F be an inert imaginary quadratic function field with $D_F = \gamma P_1 \cdots P_t$. Let s_F be the number of monic irreducible divisors P_i of D_F of odd degree. Since

$\deg D_F$ is even, s_F is an nonnegative even integer. Assume that $\deg P_i$ is odd for $1 \le i \le s_F$. Write $\vec{R}_i$ for the i-th row vector of R_F for $1 \le i \le t$ and $\vec{0}$ for the zero row vector. By Schoof's Theorem, the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}l_F) \ge 5$. Hence, it remains to prove the cases $(r_2(\mathcal{C}l_F), r_4(\mathcal{C}l_F)) = (4,4)$ if $s_F = 0$ and $(r_2(\mathcal{C}l_F), r_4(\mathcal{C}l_F)) = (3,3), (4,3)$ or $(4,4)$ if $s_F \ge 2$.

3.1. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 4$ with $s_F = 0$ and $D_F = \gamma P_1P_2P_3P_4$. In this case, by (2.1), R_F is a zero matrix, so $(\frac{P_1}{P_i}) = (\frac{P_2}{P_i}) = 1$ for $i = 3, 4$. Hence F has an infinite Hilbert 2-class field tower by Lemma 2.2.

3.2. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 3$ with $s_F \ge 2$ and $D_F = \gamma P_1P_2P_3P_4$. In this case, we have $\operatorname{rank} R_F = 1$ by (2.1). Since $\vec{R}_5 \ne \vec{0}$, we have $\vec{R}_i \in \{\vec{0}, \vec{R}_5\}$ for $1 \le i \le 4$. But, since $\sum_{j=1}^4 e_{1j} = 1$, $\vec{R}_1 \ne \vec{0}$ and $\vec{R}_1 \ne \vec{R}_5$, which is a contradiction. Hence this case can not occur.

3.3. Case $r_2(\mathcal{C}l_F) = 4, r_4(\mathcal{C}l_F) = 3$ with $s_F \ge 2$ and $D_F = \gamma P_1P_2P_3P_4P_5$. In this case, we have $\operatorname{rank} R_F = 2$ by (2.1). Note that $\vec{R}_6 \ne \vec{0}$. Since $\sum_{j=1}^5 e_{1j} = 1$, we have $\vec{R}_1 \notin \{\vec{0}, \vec{R}_6\}$. Then $\{\vec{R}_1, \vec{R}_6\}$ is a basis of the row space of R_F. Hence $\vec{R}_i \in \{\vec{0}, \vec{R}_1, \vec{R}_6, \vec{R}_1 + \vec{R}_6\}$ for all $1 \le i \le 5$. Assume that $s_F = 2$. Then $\vec{R}_6 = (1\ 1\ 0\ 0\ 0)$. If $\vec{R}_i = \vec{0}$ for some $3 \le i \le 5$, say $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so F has infinite 2-class field tower by Lemma 2.1. We may assume $\vec{R}_i \ne \vec{0}$ for all $3 \le i \le 5$. Since $\sum_{j=1}^5 e_{ij} = 0$ for $3 \le i \le 5$, we have $\vec{R}_3 = \vec{R}_4 = \vec{R}_5 = \vec{R}_6$. Then we have $(\frac{P_3P_4}{P_i}) = (\frac{P_3P_5}{P_i}) = 1$ for $i = 1, 2$, so F has infinite Hilbert 2-class field tower by Lemma 2.2.

Assume that $s_F = 4$. Then $\vec{R}_6 = (1\ 1\ 1\ 1\ 0)$. If $\vec{R}_5 = \vec{0}$, then $(\frac{P_5}{P_i}) = 1$ for $1 \le i \le 4$, so F has infinite 2-class field tower by Lemma 2.1. We may assume $\vec{R}_5 \ne \vec{0}$. Since $\sum_{j=1}^5 e_{5j} = 0$, we have $\vec{R}_5 = \vec{R}_6$. Since $\sum_{j=1}^5 e_{ij} = 1$ for $1 \le i \le 4$, we have $\vec{R}_i \in \{\vec{R}_1, \vec{R}_1 + \vec{R}_6\}$. If three of $\vec{R}_1, \vec{R}_2, \vec{R}_3, \vec{R}_4$ are equal, say $\vec{R}_1 = \vec{R}_2 = \vec{R}_3$, then $(\frac{P_1P_2}{P_i}) = (\frac{P_1P_3}{P_i}) = 1$ for $i = 4, 5$, so F has infinite Hilbert 2-class field tower by Lemma 2.2. We may assume that $\vec{R}_1 = \vec{R}_2$ and $\vec{R}_3 = \vec{R}_4 = \vec{R}_1 + \vec{R}_6$. Then, by the quadratic reciprocity law ([5, Theorem 3.3]), we have $e_{11} + e_{12} = 1, e_{13} = e_{14} = e_{11}$ and $e_{15} = 1$, so $\sum_{j=1}^5 e_{1j} = 0$, which is a contradiction.

3.4. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 4$ with $s_F \ge 2$ and $D_F = \gamma P_1P_2P_3P_4P_5$. In this case, we have $\operatorname{rank} R_F = 1$ by (2.1). Since $\vec{R}_6 \ne \vec{0}$, we have $\vec{R}_i \in \{\vec{0}, \vec{R}_6\}$ for $1 \le i \le 5$. But, since $\sum_{j=1}^5 e_{1j} = 1$, $\vec{R}_1 \ne \vec{0}$ and $\vec{R}_1 \ne \vec{R}_6$, which is a contradiction. Hence this case can not occur.

4. Proof of Theorem 1.2

For any positive even integer n and positive integer t, let $\mathcal{R}(n,t) = \mathcal{P}(n,t) \setminus \mathcal{P}_2(n,t)$ and $\mathcal{R}'(n,t) = \mathcal{P}'(n,t) \setminus \mathcal{P}'_2(n,t)$.

Lemma 4.1. *As $n \to \infty$, we have $|\mathcal{R}(n,t+1) \cup \mathcal{P}_2(n,t)| \sim |\mathcal{R}'(n,t+1)|$.*

Proof. By (2.4), (2.5) and (2.7), as $n \to \infty$, we have

$$|\mathcal{P}_2(n,t)| = O\Big(\frac{q^n(\log n)^{t-1}}{n}\Big), \qquad |\mathcal{P}_2(n,t) \setminus \mathcal{P}'_2(n,t)| = o\Big(\frac{q^n(\log n)^{t-1}}{n}\Big),$$

$$|\mathcal{R}'(n,t+1)| = O\Big(\frac{q^n(\log n)^{t}}{n}\Big), \quad |\mathcal{R}(n,t+1) \setminus \mathcal{R}'(n,t+1)| = o\Big(\frac{q^n(\log n)^{t}}{n}\Big).$$

Then $|\mathcal{P}_2(n,t) \setminus \mathcal{P}'_2(n,t)| = o(|\mathcal{P}'_2(n,t)|)$ and $|\mathcal{R}(n,t+1) \setminus \mathcal{R}'(n,t+1)| = o(|\mathcal{R}'(n,t+1)|)$. Also by (2.7), $|\mathcal{P}'_2(n,t)| = o(|\mathcal{R}'(n,t+1)|)$. Hence we get the result. ☐

Recall that for an inert imaginary quadratic function field F, $r_2(\mathcal{C}l_F)$ is equal to $\omega(D_F)$ or $\omega(D_F) - 1$ according as D_F is special or non-special. Let n be a positive even integer and r, s be integers with $0 \le s \le r$. Then we have

$$Z_{r;n} = \big\{k(\sqrt{\gamma N}) : N \in \mathcal{R}(n, r+1) \cup \mathcal{P}_2(n,r)\big\}.$$

Let $\bar{Z}_{r;n}$ be the subset of $Z_{r;n}$ consisting of $k(\sqrt{\gamma N})$ with $N \in \mathcal{R}'(n, r+1)$ and $\bar{Z}^*_{r,s;n} = \bar{Z}_{r;n} \cap Z^*_{r,s;n}$. By Lemma 4.1, we have

$$\varrho^*_{r,s} = \liminf_{n\to\infty} \frac{|\bar{Z}^*_{r,s;n}|}{|\bar{Z}_{r;n}|}. \tag{4.1}$$

By (2.7), we have

$$|\bar{Z}_{r;n}| = (1 - 2^{-r})\frac{q^n(\log n)^r}{r!n} + O\Big(\frac{q^n(\log n)^{r-1}}{n}\Big) \tag{4.2}$$

as $n \to \infty$. For any $N \in \mathcal{R}'(n, r+1)$, let $\mathcal{S}(N)$ be the set of inert imaginary quadratic function fields $k(\sqrt{\gamma M})$ with $M \in \mathcal{N}(N)$. Then $\mathcal{S}(N)$ is a subset of $\bar{Z}_{r;n}$ and by (2.6), we have

$$|\mathcal{S}(N)| = 2^{-\frac{r(r+3)}{2}} \cdot \frac{q^n(\log n)^r}{r!n} + O\Big(\frac{q^n(\log n)^{r-1}}{n}\Big) \tag{4.3}$$

as $n \to \infty$. Thus, from (4.2) and (4.3), we get

$$\lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{r;n}|} = 2^{-\frac{r(r+1)}{2}}(1 - 2^{-r})^{-1}. \tag{4.4}$$

4.1. $\varrho^*_{3,s}$ Consider an inert imaginary quadratic function field $F = k(\sqrt{\gamma N})$ with $N = P_1P_2P_3P_4 \in \mathcal{R}'(n, 4)$ such that $2 \nmid \deg P_i$, $2 | \deg P_j$ and $(\frac{P_i}{P_j}) = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Then $r_2(\mathcal{C}l_F) = 3$ and F has infinite Hilbert 2-class field tower by Lemma 2.2. Moreover, every fields in $\mathcal{S}(N)$ also has infinite Hilbert 2-class field tower.

- *Case* $(\frac{P_3}{P_4}) = -1$. In this case, the matrix R_F is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 3, so $r_4(\mathcal{C}l_F) = 1$. Hence $F \in \bar{Z}^*_{3,1;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{3,1;n}$. By (4.4), we have

$$\varrho^*_{3,1} \geq \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{3;n}|} = 2^{-3}(2^3 - 1)^{-1}.$$

- *Case* $(\frac{P_3}{P_4}) = 1$. In this case, the matrix R_F is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 2, so $r_4(\mathcal{C}l_F) = 2$. Hence $F \in \bar{Z}^*_{3,2;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{3,2;n}$. By (4.4), we have

$$\varrho^*_{3,2} \geq \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{3;n}|} = 2^{-3}(2^3 - 1)^{-1}.$$

4.2. $\varrho^*_{4,s}$ Consider an inert imaginary quadratic function field $F = k(\sqrt{\gamma N})$ with $N = P_1P_2P_3P_4P_5 \in \mathcal{R}'(n, 5)$ such that $2 \nmid \deg P_i$ for $i \in \{1, 2\}$, $2 | \deg P_j$ for $j \in \{3, 4, 5\}$ and $(\frac{P_i}{P_j}) = 1$ for $i \in \{1, 2\}, j \in \{3, 4\}$. Then $r_2(\mathcal{C}l_F) = 4$ and F has infinite Hilbert 2-class field tower by Lemma 2.2. Moreover, every fields in $\mathcal{S}(N)$ also has infinite Hilbert 2-class field tower.

- *Case* $(\frac{P_i}{P_5}) = -1$ *for* $i \in \{1, 3\}$, $(\frac{P_i}{P_5}) = 1$ *for* $i \in \{2, 4\}$ *and* $(\frac{P_3}{P_4}) = -1$. In this case, the matrix R_F is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 5, so $r_4(\mathcal{C}l_F) = 0$. Hence $F \in \bar{Z}^*_{4,0;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{4,0;n}$. By (4.4), we have

$$\varrho^*_{4,0} \geq \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{4;n}|} = 2^{-6}(2^4 - 1)^{-1}.$$

• *Case* $(\frac{P_1}{P_5}) = -1$, $(\frac{P_i}{P_5}) = 1$ *for* $2 \le i \le 4$ *and* $(\frac{P_3}{P_4}) = -1$. In this case, the matrix R_F is

$$\begin{pmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 1&0&0&0&1\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 1&1&0&0&1\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 1&0&0&0&1\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 4, so $r_4(\mathcal{C}l_F) = 1$. Hence $F \in \bar{Z}^*_{4,1;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{4,1;n}$. By (4.4), we have

$$\varrho^*_{4,1} \ge \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{4;n}|} = 2^{-6}(2^4-1)^{-1}.$$

• *Case* $(\frac{P_i}{P_5}) = 1$ *for* $1 \le i \le 4$ *and* $(\frac{P_3}{P_4}) = -1$. In this case, the matrix R_F is

$$\begin{pmatrix} 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 0&0&0&0&0\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 0&1&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 0&0&0&0&0\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 3, so $r_4(\mathcal{C}l_F) = 2$. Hence $F \in \bar{Z}^*_{4,2;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{4,2;n}$. By (4.4), we have

$$\varrho^*_{4,2} \ge \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{4;n}|} = 2^{-6}(2^4-1)^{-1}.$$

• *Case* $(\frac{P_i}{P_5}) = 1$ *for* $1 \le i \le 4$ *and* $(\frac{P_3}{P_4}) = 1$. In this case, the matrix R_F is

$$\begin{pmatrix} 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = 1, \qquad \begin{pmatrix} 0&1&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&1&0&0&0 \end{pmatrix} \text{ if } (P_1/P_2) = -1$$

whose rank is 2, so $r_4(\mathcal{C}l_F) = 3$. Hence $F \in \bar{Z}^*_{4,3;n}$ and $\mathcal{S}(N) \subset \bar{Z}^*_{4,3;n}$. By (4.4), we have

$$\varrho^*_{4,3} \ge \lim_{n\to\infty} \frac{|\mathcal{S}(N)|}{|\bar{Z}_{4;n}|} = 2^{-6}(2^4-1)^{-1}.$$

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