BOUNDEDNESS IN THE PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. Alexseev's formula generalizes the variation of constants formula and permits the study of a nonlinear perturbation of a system with certain stability properties. In recent years M. Pinto introduced the notion of h-stability. S.K. Choi et al. investigated h-stability for the nonlinear differential systems using the notion of t_∞-similarity. Applying these two notions, we study bounds for solutions of the perturbed differential systems.

1. Introduction

Integral inequalities play a vital role in the study of boundedness and other qualitative properties of solutions of differential equations. The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : the use of integral inequalities, the method of variation of constants formula, and Lyapunov's second method.

Pinto [11, 12] introduced the notion of h-stability (hS) which is an important extension of exponential asymptotic stability. He introduced hS with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptyotic stability to a variety of reasonable systems called h-systems.

The aim of this paper is to obtain some results on boundedness of the perturbed differential systems under suitable conditions on perturbed term. To do this, we need some integral inequalities.

2. Preliminaries

We are interested in the relations of the unperturbed system

(2.1) $$x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

and the solutions of the perturbed systems

(2.2) $$x'(t) = f(t, x) + g(t, x), \quad x(t_0) = x_0,$$

and

(2.3) $$y' = f(t, y) + \int_{t_0}^t g(s, y(s))ds, \ y(t_0) = y_0,$$

Here x, y, f and g are elements of $\mathbb{R}^n$, an n-dimensional real Euclidean space.

We assume that $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, and that f is continuously differentiable with respect to the components of x on $\mathbb{R}^+ \times \mathbb{R}^n$, $f(t, 0) = 0$ for all $t \in \mathbb{R}^+$. The symbol $|\cdot|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$

Let $x(t, t_0, x_0)$ denote the unique solutions of (2.1) and (2.2), satisfying the initial conditions $x(t_0, t_0, x_0) = x_0$, and $y(t_0, t_0, y_0) = y_0$, existing on $[t_0, \infty)$, respectively. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

(2.4) $$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0$$

and

(2.5) $$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0.$$

Here, $f_x(t, x)$ is the matrix whose element in the ith row, jth column is the partial derivative of the ith component of f with respect to the jth component of x. The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.5) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.4).

We recall some notions of h-stability [11].

Definition 2.1. The system (2.1)(the zero solution $x = 0$ of (2.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough(here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called *h-stable(hS)* if there exists $\delta > 0$ such that (2.1) is an *h*-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$(2.6) \qquad \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

We give some related properties that we need in the sequal.

Lemma 2.4 ([12])**.** *The linear system*

$$(2.7) \qquad x' = A(t)x, \ x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system(respectively h-stable) if and only if there exist $c \geq 1$ and a positive continuous(repectively bounded) function h defined on $\mathbb{R}^+$ such that

$$(2.8) \qquad |\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.7).

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.5. *Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.6 ([3])**.** *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is hS.*

Theorem 2.7 ([4]). *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.2) is hS, then the solution $z = 0$ of (2.3) is hS.*

Lemma 2.8 ([9]). *(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t \lambda(s) ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \dfrac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s) ds \in \mathrm{dom} W^{-1}\Big\}.$$

3. MAIN RESULTS

In this section, we investigate bounds for the nonlinear differential systems.
We need the lemma to prove the following theorem.

Lemma 3.1. *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) w(u(s)) ds + \int_{t_0}^t \lambda_2(s)\Big(\int_{t_0}^s \lambda_3(\tau) u(\tau) d\tau\Big) ds, \ \ 0 \leq t_0 \leq t.$$

Then

$$(3.1) \qquad u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) d\tau) ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \dfrac{ds}{w(s)}$, $u > 0$, $u_0 > 0$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)) ds \in \mathrm{dom} W^{-1}\Big\}.$$

Proof. Define a function $v(t)$ by the right member of the above inequality . Then

$$v'(t) = \lambda_1(t) w(u(t)) + \lambda_2(t) \int_{t_0}^t \lambda_3(s) u(s) ds, v(t_0) = c, u(t) \leq v(t),$$

which implies

$$v'(t) \leq \lambda_1(t)w(v(t)) + \lambda_2(t)\int_{t_0}^t \lambda_3(s)u(s)ds$$

(3.2)
$$\leq \lambda_1(t)w(v(t)) + \lambda_2(t)\int_{t_0}^t \lambda_3(s)dsw(v(t))$$

$$\leq [\lambda_1(t) + \lambda_2(t)\int_{t_0}^t \lambda_3(s)ds]w(v(t)),$$

since v and w are nondecreasing . Now, by integrating the above inequality on $[t_0, t]$, we have

(3.3)
$$v(t) \leq v(t_0) + \int_{t_0}^t [\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau]w(v(s))ds.$$

It follows from Lemma 2.8 that (3.2) yields the estimate (3.1). □

Theorem 3.2. *Let* $a, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u , $u \leq w(u)$ *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$. *Suppose that the solution* $x = 0$ *of (2.1) is hS with a nondecreasing function h and the perturbed term g in (2.2) satisfies*

$$|\Phi(t, s, y(\tau))g(t, y(\tau))| \leq a(s)(w(|y(\tau)|) + \int_{t_0}^s k(\tau)|y(\tau)|d\tau), \ t \geq t_0 \geq 0,$$

where $\int_{t_0}^\infty a(s)ds < \infty$ *and* $\int_{t_0}^\infty k(s)ds < \infty$. *Then any solution* $y(t) = y(t, t_0, y_0)$ *of (2.2) is bounded on $[t_0, \infty)$ and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + \int_{t_0}^t a(s)(1 + \int_{t_0}^s k(\tau)d\tau)ds\Big], \ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t a(s)(1 + \int_{t_0}^s k(\tau)d\tau)ds \in \text{dom}W^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Applying Lemma 2.5 and the increasing property of the function h, we obtain

$$
\begin{aligned}
|y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))g(s, y(s))|ds \\
&\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t a(s)\Big[w(|y(s)|) + \int_{t_0}^s k(\tau)|y(\tau)|d\tau\Big]ds
\end{aligned}
$$

$$\leq\ c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t a(s)h(t)w(\frac{|y(s)|}{h(s)})ds$$

$$+ \int_{t_0}^t a(s) \int_{t_0}^s h(t)k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds.$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 3.1, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + \int_{t_0}^t a(s)(1 + \int_{t_0}^s k(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation implies the boundedness of $y(t)$, and the proof is complete. □

Remark 3.3. Letting $k(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.1 in [8].

Also, we examine the bounded property for the perturbed system

$$(3.4) \qquad\qquad y' = f(t,y) + \int_{t_0}^t g(s,y(s))ds,\ y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t,0) = 0$.

Theorem 3.4. *Let* $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u *,$u \leq w(u)$ and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$. *Suppose that* $f_x(t,0)$ *is* t_∞-*similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$, *the solution* $x = 0$ *of (2.1) is* hS *with the increasing function* h, *and* g *in (3.4) satisfies*

$$\Big|\int_{t_0}^s g(\tau, y(\tau))d\tau\Big| \leq a(s)w(|y(s)|) + b(s)\int_{t_0}^s k(\tau)|y(\tau)|d\tau,$$

where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, *and* $\int_{t_0}^\infty k(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (3.4) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^t (a(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big],$$

where W, W^{-1} *are the same functions as in Lemma 2.8 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2\int_{t_0}^t (a(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds \in \mathrm{dom}W^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.4), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is hS.

Using two Lemma 2.4 and 2.5, we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds$$

$$\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)a(s)w(\frac{|y(s)|}{h(s)})ds$$

$$+ \int_{t_0}^{t} c_2 h(t)b(s) \int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds.$$

since h is increasing. Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 3.1 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and the theorem is proved. □

Remark 3.5. Letting $k(t) = 0$ in Theorem 3.2, we obtain the same result as that of Theorem 3.2 in [8].

We need the lemma to prove the following theorem.

Lemma 3.6. *Let $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u and $u \le w(u)$. Suppose that for some $c \ge 0$,*

$$(3.5)\quad u(t) \le c + \int_{t_0}^{t}\Big(p(s)\int_{t_0}^{s}\Big(q(\tau)w(u(\tau)) + v(\tau)\int_{t_0}^{\tau}r(a)u(a)da\Big)d\tau\Big)ds, \ t \ge t_0.$$

Then
(3.6)
$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(p(s)\int_{t_0}^{s}\Big(q(\tau) + v(\tau)\int_{t_0}^{\tau}r(a)da\Big)d\tau\Big)ds\Big], \ t_0 \le t < b_1,$$

where $W(u) = \displaystyle\int_{u_0}^{u}\frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Big(p(s)\int_{t_0}^{s}\Big(q(\tau) + v(\tau)\int_{t_0}^{\tau}r(a)da\Big)d\tau\Big)ds \in \mathrm{dom}W^{-1}\Big\}.$$

Proof. Setting $z(t) = c + \displaystyle\int_{t_0}^{t}(p(s)\int_{t_0}^{s}(q(\tau)w(u(\tau)) + v(\tau)\int_{t_0}^{\tau}r(a)u(a)da)d\tau)ds$, we have $z(t_0) = c$ and

$$z'(t) = p(t) \int_{t_0}^{t} \Big(q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a)u(a)da \Big) d\tau$$

(3.7)
$$\leq p(t) \int_{t_0}^{t} \Big(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da \Big) w(u(\tau)) d\tau$$

$$\leq \Big[p(t) \int_{t_0}^{t} \Big(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da \Big) d\tau \Big] w(z(t)), \ t \geq t_0,$$

since $z(t)$ and $w(u)$ are nondecreasing and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

(3.8) $$z(t) \leq c + \int_{t_0}^{t} \Big(p(s) \int_{t_0}^{s} \Big(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da \Big) d\tau w(z(s)) \Big) ds.$$

It follows from Lemma 2.8 that (3.8) yields the estimate (3.6). □

Theorem 3.7. *Let $w \in C(\mathbb{R}^{+})$, $w(u)$ be nondeacreasing in u, $u \leq w(u)$, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (2.1)is an h-system with a positive continuous function h and g in (3.4) satisfies*

(3.9) $$|g(t, y)| \leq a(t)(w(|y(t)|) + \int_{t_0}^{t} k(s)|y(s)|ds), \ t \geq t_0, \ y \in \mathbb{R}^n$$

where $a : \mathbb{R}^{+} \to \mathbb{R}^{+}$ is continuous with

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} \Big(a(\tau) \Big(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \Big) d\tau ds < \infty,$$

for all $t_0 \geq 0$, then any solution $y(t) = y(t, t_0, y_0)$ of (3.4) satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} a(\tau) \Big(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \Big) d\tau ds \Big],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup \Big\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} a(\tau) \Big(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \Big) d\tau ds \in \text{dom}W^{-1} \Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.4), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is a h-system, the solution $v = 0$ of (2.2) is a h-system. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is a h-system. Applying Lemma 2.5 and (3.9) , we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))| \int_{t_0}^s |g(\tau,y(\tau))| d\tau ds$$

$$\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 \frac{h(t)}{h(s)} \Big(\int_{t_0}^s h(\tau) a(\tau) w(\frac{|y(\tau)|}{h(\tau)})$$

$$+ \int_{t_0}^s a(\tau) \int_{t_0}^\tau h(r) k(r) \frac{|y(r)|}{h(r)} dr \Big) d\tau ds.$$

Using Lemma 3.6 with $u(t) = |y(t)| h(t)^{-1}$, we obtain

$$|y(t)| \leq h(t) W^{-1} \Big[W(c) + \int_{t_0}^t \frac{c_2}{h(s)} \int_{t_0}^s a(\tau) \Big(h(\tau) + \int_{t_0}^\tau h(r)k(r)dr \Big) d\tau ds \Big],$$

$t_0 \leq t < b_1$, where $c = c_1 |y_0| h(t_0)^{-1}$. Hence, the proof is complete. $\qquad \square$

Remark 3.8. Letting $k(s) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.5 in [8].

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