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SCALAR CURVATURE DECREASE FROM A HYPERBOLIC METRIC

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Abstract. We find an explicit C^∞-continuous path of Riemannian metrics g_t on the 4-d hyperbolic space $\mathbb{H}^4$, for $0 \leq t \leq \varepsilon$ for some number $\varepsilon > 0$ with the following property: g_0 is the hyperbolic metric on $\mathbb{H}^4$, the scalar curvatures of g_t are strictly decreasing in t in an open ball and g_t is isometric to the hyperbolic metric in the complement of the ball.

1. Introduction

For any Riamannian manifold (M^k, g_0), $k \geq 3$ and a ball $B \subset M$, is there a C^∞-continuous path of Riemannian metrics g_t, $0 \leq t \leq \varepsilon$ on M such that the scalar curvatures of g_t are strictly decreasing in t on B and that $g_t \equiv g_0$ on $M \backslash B$? This family, if exists, may be called a *scalar curvature melting* of g_0 in B. This question is actually a small step toward Lohkamp's conjecture on ricci curvature version [6, Section 10].

If there is a scalar curvature melting g_t, then the scalar curvatures satisfy

$$\frac{ds(g_t)}{dt}|_{t=0} \leq 0$$

on B. As g_t is deforming only inside a ball, it is more relevant to the linearization L_g of the scalar curvature functional on the space of Riemannian metrics restricted to a domain. According to Corvino [3, Theorem 4], a scalar curvature melting of g seems to exist when the formal adjoint L_g^* (as defined on the space of functions which are square integrable on each compact subset of B) is injective. Although this

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injectivity condition holds for generic metrics by Theorem 6.1 and Theorem 7.4 in [1], it is not easy to check which metrics satisfy this.

In the previous works we have studied explicit scalar curvature meltings of Euclidean metrics and one positive Einstein metric [4, 5]. In this article we study the hyperbolic metric g_h, i.e. the metric with constant curvature -1. The derivative of the scalar curvature functional ds_{g_h} (defined on a whole manifold M) is surjective, but we do not know whether the above (locally defined) L_g^* is injective or not. In any case, a merit of our construction is that it is explicit and provides a large scale melting.

We shall first construct a family of Riemannian metrics on the 4-dimensional hyperbolic space $\mathbb{H}^4$ whose scalar curvatures decrease on a precompact open subset and are hyperbolic away from it. Then by conformal change of the metrics, we spread the negativity inside the subset over to a larger ball. In the process, we find a natural choice of parameter t to get g_t. In this way we get a scalar curvature melting;

Theorem 1.1. *There exists a C^∞-continuous path of Riemannian metrics g_t on $\mathbb{H}^4$, for $0 \leq t \leq \varepsilon$ for some number $\varepsilon > 0$ with the following property: g_0 is the hyperbolic metric on $\mathbb{H}^4$, the scalar curvatures of g_t are strictly decreasing in t in an open ball and g_t is isometric to g_0 in the complement of the ball.*

2. Metrics on the 4-d Hyperbolic Space

We start with a metric on $\mathbb{R}^4$ of the form

$$g_0 = f^2 dr^2 + \frac{r^2}{f^2} d\theta^2 + h^2 d\rho^2 + \frac{\rho^2}{h^2} d\sigma^2,$$

where $(r,\theta), (\rho,\sigma)$ are the polar coordinates for each summand of $\mathbb{R}^4 := \mathbb{R}^2 \times \mathbb{R}^2$ respectively, and f, h are smooth positive functions on $\mathbb{R}^4$, which are functions of r and ρ only. Then by a straightforward computation one gets the scalar curvature:

$$\begin{aligned} s_{g_0} &= 2(R_{2112} + R_{3113} + R_{4114} + R_{3223} + R_{4224} + R_{4334}) \\ &= 2\Big(\frac{f_{rr}}{f^3} + \frac{3f_r}{rf^3} - \frac{3f_r^2}{f^4} - \frac{f_\rho^2}{h^2 f^2} + \frac{h_{\rho\rho}}{h^3} + \frac{3h_\rho}{\rho h^3} - \frac{3h_\rho^2}{h^4} - \frac{h_r^2}{h^2 f^2}\Big), \end{aligned}$$

where $f_r = \frac{\partial f}{\partial r}, f_{rr} = \frac{\partial^2 f}{\partial r \partial r}$, etc..

Consider the unit ball centered at the origin in $\mathbb{R}^4$. Then the hyperbolic metric corresponds to $g_h = \frac{4}{(1-r^2-\rho^2)^2}(dr^2 + r^2 d\theta^2 + d\rho^2 + \rho^2 d\sigma^2)$ in the unit ball

$\{(r,\theta,\rho,\sigma)|\ r^2+\rho^2<1\}$. Note that $g_h=\frac{4}{(1-|x|^2)^2}(dx_1{}^2+dx_2{}^2+dx_3{}^2+dx_4{}^2)$ in the rectangular coordinates. If we consider the deformation

$$\tilde{g}=\frac{4}{(1-r^2-\rho^2)^2}\big(f^2dr^2+\frac{r^2}{f^2}d\theta^2+h^2d\rho^2+\frac{\rho^2}{h^2}d\sigma^2\big)=\psi^2g_0,$$

where $\psi=\frac{2}{(1-r^2-\rho^2)}$, the scalar curvature is given [2, p.59] by

$$s(\tilde{g})=\psi^{-3}\{6\triangle_{g_0}\psi+s(g_0)\psi\}.$$

Substituting $\triangle_{g_0}\psi=-\frac{\psi_r}{rf^2}+\frac{2f_r\psi_r}{f^3}-\frac{\psi_{rr}}{f^2}-\frac{\psi_\rho}{\rho h^2}+\frac{2h_\rho\psi_\rho}{h^3}-\frac{\psi_{\rho\rho}}{h^2}$, $\psi_r=\frac{4r}{(1-r^2-\rho^2)^2}$, $\psi_{rr}=\frac{12r^2-4\rho^2+4}{(1-r^2-\rho^2)^3}$, $\psi_\rho=\frac{4\rho}{(1-r^2-\rho^2)^2}$ and $\psi_{\rho\rho}=\frac{12\rho^2-4r^2+4}{(1-r^2-\rho^2)^3}$, we get;

$$\begin{aligned}s(\tilde{g})=&3(1-r^2-\rho^2)^2\big\{\frac{(-2-2r^2+2\rho^2)}{f^2(1-r^2-\rho^2)^2}+\frac{(-2+2r^2-2\rho^2)}{h^2(1-r^2-\rho^2)^2}\\&+\frac{f_r}{f^3}\cdot\frac{2r}{1-r^2-\rho^2}+\frac{h_\rho}{h^3}\cdot\frac{2\rho}{1-r^2-\rho^2}\\&+\frac{1}{6}\big(\frac{f_{rr}}{f^3}+\frac{3f_r}{rf^3}-\frac{3f_r^2}{f^4}-\frac{f_\rho^2}{f^2h^2}+\frac{h_{\rho\rho}}{h^3}+\frac{3h_\rho}{\rho h^3}-\frac{3h_\rho^2}{h^4}-\frac{h_r^2}{f^2h^2}\big)\big\}.\end{aligned}$$

Put $F+1=\frac{1}{f^2}$ and $H+1=\frac{1}{h^2}$. Then

$$\begin{aligned}\frac{s(\tilde{g})+12}{6(1-r^2-\rho^2)^2}=&-\frac{1}{24}\big\{F_{rr}+(\frac{3}{r}+\frac{12r}{1-r^2-\rho^2})F_r-\frac{24(-1-r^2+\rho^2)}{(1-r^2-\rho^2)^2}F\\&+H_{\rho\rho}+(\frac{3}{\rho}+\frac{12\rho}{1-r^2-\rho^2})H_\rho-\frac{24(-1+r^2-\rho^2)}{(1-r^2-\rho^2)^2}H\big\}-\frac{f_\rho^2+h_r^2}{12f^2h^2}.\end{aligned}$$

We shall find F and H which satisfy

$$F_{rr}+(\frac{3}{r}+\frac{12r}{1-r^2-\rho^2})F_r-\frac{24(-1-r^2+\rho^2)}{(1-r^2-\rho^2)^2}F=\alpha(r,\rho)$$

and

$$H_{\rho\rho}+(\frac{3}{\rho}+\frac{12\rho}{1-r^2-\rho^2})H_\rho-\frac{24(-1+r^2-\rho^2)}{(1-r^2-\rho^2)^2}H=-\alpha(r,\rho)$$

for some function $\alpha(r,\rho)$. For convenience we denote $F_r=F'$, $F_{rr}=F''$, $C=(\frac{3}{r}+\frac{12r}{1-r^2-\rho^2})$ and $D=-\frac{24(-1-r^2+\rho^2)}{(1-r^2-\rho^2)^2}$, hence the equation is $F''+CF'+DF=\alpha$. If we assume the solution is of the form $F(r,\rho)=u(r,\rho)v(r,\rho)$, the equation becomes

$$v''+(\frac{2}{u}u'+C)v'+(\frac{1}{u}u''+\frac{C}{u}u'+D)v=\frac{\alpha}{u}. \tag{2.1}$$

Choose u so that $\frac{2}{u}u'+C=0$, i.e.,

$$u=e^{-\frac{1}{2}\int Cdr}=e^{-\frac{1}{2}\int(\frac{3}{r}+\frac{12r}{1-r^2-\rho^2})dr}=r^{-\frac{3}{2}}(1-r^2-\rho^2)^3\tilde{c}(\rho).$$

Then $\frac{1}{u}u'' + \frac{C}{u}u' + D = -\frac{3}{4}\frac{1}{r^2}$. Therefore the equation (2.1) becomes

$$v'' - \frac{3}{4r^2}v = \frac{r^{\frac{3}{2}}}{(1-r^2-\rho^2)^3}\alpha,$$

which is a well-known Euler-Cauchy equation. The general solution of this equation is

$$v = c_1(\rho)r^{\frac{3}{2}} + c_2(\rho)r^{-\frac{1}{2}} + \frac{1}{2}r^{\frac{3}{2}}\int \frac{r}{(1-r^2-\rho^2)^3}\alpha dr - \frac{1}{2}r^{-\frac{1}{2}}\int \frac{r^3}{(1-r^2-\rho^2)^3}\alpha dr.$$

Hence we have the solution

$$\begin{aligned} F = u(r,\rho)v(r,\rho) =& c_1(\rho)\tilde{c}(\rho)(1-r^2-\rho^2)^3 + c_2(\rho)\tilde{c}(\rho)r^{-2}(1-r^2-\rho^2)^3 \\ &+ \frac{1}{2}\tilde{c}(\rho)(1-r^2-\rho^2)^3\int \frac{r}{(1-r^2-\rho^2)^3}\alpha dr \\ &- \frac{1}{2}\tilde{c}(\rho)r^{-2}(1-r^2-\rho^2)^3\int \frac{r^3}{(1-r^2-\rho^2)^3}\alpha dr. \end{aligned}$$

Choosing $c_1(\rho) = c_2(\rho) = 0$ and $\tilde{c}(\rho) = 1$ we have a solution

$$F = \frac{1}{2}(1-r^2-\rho^2)^3\Big\{\int_0^r \frac{t}{(1-t^2-\rho^2)^3}\alpha(t,\rho)dt - \frac{1}{r^2}\int_0^r \frac{t^3}{(1-t^2-\rho^2)^3}\alpha(t,\rho)dt\Big\}.$$

Similarly we have

$$H = -\frac{1}{2}(1-r^2-\rho^2)^3\Big\{\int_0^\rho \frac{s}{(1-r^2-s^2)^3}\alpha(r,s)ds - \frac{1}{\rho^2}\int_0^\rho \frac{s^3}{(1-r^2-s^2)^3}\alpha(r,s)ds\Big\}.$$

Hence

$$\frac{s(\tilde{g})+12}{6(1-r^2-\rho^2)^2} = -\frac{f_\rho^2+h_r^2}{12f^2h^2} = -\frac{1}{48}\Big\{\frac{H+1}{(F+1)^2}F_\rho^2 + \frac{F+1}{(H+1)^2}H_r^2\Big\}. \tag{2.2}$$

We choose $\alpha(r,\rho) = a(r)b(\rho)(1-r^2-\rho^2)^3$ where $a(r)$ and $b(\rho)$ are smooth functions satisfying

$$1)\ a(r) = 0,\ r \le 0,\ r \ge \frac{1}{2}$$

$$2)\ \int_0^{\frac{1}{2}} (t-4t^3)a(t)dt = 0$$

$$3)\ b(\rho) = 0,\ \rho \le 0,\ \rho \ge \frac{1}{2}$$

$$4)\ \int_0^{\frac{1}{2}} (s-4s^3)b(s)ds = 0.$$

Note that this will make $F(r,\rho) = 0$ and $H(r,\rho) = 0$ when $r \ge \frac{1}{2}$ or $\rho \ge \frac{1}{2}$.

A graph of a typical such function a (or b) is given in the picture below:

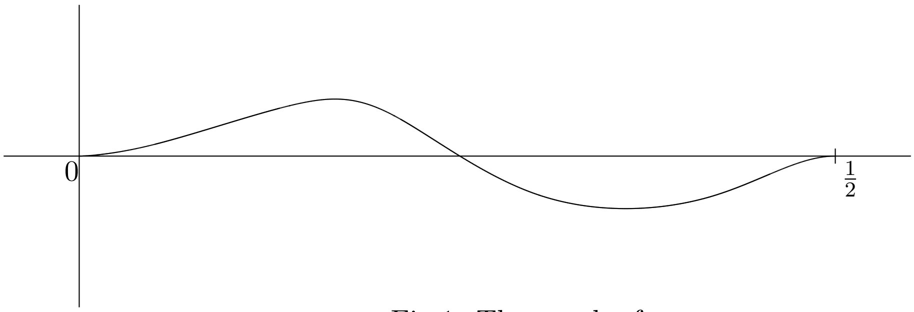

Fig.1. The graph of a.

Then

$$F(r,\rho) = \frac{1}{2}(1-r^2-\rho^2)^3 b(\rho)\{\int_0^r ta(t)dt - \frac{1}{r^2}\int_0^r t^3 a(t)dt\}, \tag{2.3}$$

$$H(r,\rho) = -\frac{1}{2}(1-r^2-\rho^2)^3 a(r)\{\int_0^\rho sb(s)ds - \frac{1}{\rho^2}\int_0^\rho s^3 b(s)ds\}$$

and

$$F_\rho = \frac{1}{2}(1-r^2-\rho^2)^2\{-6\rho b(\rho) + (1-r^2-\rho^2)b^{'}(\rho)\}(\int_0^r ta(t)dt - \frac{1}{r^2}\int_0^r t^3 a(t)dt),$$

$$H_r = -\frac{1}{2}(1-r^2-\rho^2)^2\{-6ra(r) + (1-r^2-\rho^2)a^{'}(r)\}(\int_0^\rho sb(s)ds - \frac{1}{\rho^2}\int_0^\rho s^3 b(s)ds).$$

We set $\mathcal{D} = \{(r,\theta,\rho,\phi)|\quad 0 \le r,\rho < \frac{1}{2},\ 0 \le \theta,\phi < 2\pi\}$. Due to the conditions $1)-4)$ on a and b, the support of F and H lie in $\mathcal{D}$. So, $\tilde{g} = g_h$ away from $\mathcal{D}$ and from (2.2) its scalar curvature $s_{\tilde{g}} < s(g_h)$ inside $\mathcal{D}$ except the subset $\mathfrak{T} := \{(r,\theta,\rho,\phi) \in \mathcal{D} \mid\ F_\rho = 0,\ \ H_r = 0\}$. By choosing a and b properly, $\mathfrak{T}$ becomes a thin subset in $\mathcal{D}$.

One can check that the region $\mathcal{D}$ lies within the g_h-distance 4 from the origin $(0,0,0,0) \in \mathbb{H}^4$.

Proposition 1. *There exist Riemannian metrics on* $\mathbb{H}^4$ *such that their scalar curvatures are less than that of the hyperbolic metric on the subset* $\mathcal{D}\backslash\mathfrak{T}$ *and they are hyperbolic away from* $\mathcal{D}$.

3. A Scalar-curvature-decreasing Family

We are going to show that there is a C^∞-continuous path $\tilde{g}_t$ among the metrics in the previous section such that its scalar curvature $s(\tilde{g}_t)$ is decreasing in $\mathcal{D}\backslash\mathfrak{T}$ and $\tilde{g}_t$ is hyperbolic in the complement of $\mathcal{D}$.

We define a path of metrics:

$$\tilde{g}_t = \frac{4}{(1-r^2-\rho^2)^2}(f_t^2 dr^2 + \frac{r^2}{f_t^2}dr^2 + h_t^2 d\rho^2 + \frac{\rho^2}{h_t^2}d\sigma^2), \tag{3.1}$$

where $\frac{1}{f_t^2} = tF + 1$ and $\frac{1}{h_t^2} = tH + 1$ for the functions F and H as in (2.3). Then $\tilde{g}_0 = g_h$.

From (2.2) the scalar curvature is as follows;

$$\frac{s(\tilde{g}_t) + 12}{6(1 - r^2 - \rho^2)^2} = -\frac{1}{48}\{\frac{tH+1}{(tF+1)^2}t^2F_\rho^2 + \frac{tF+1}{(tH+1)^2}t^2H_r^2\}.$$

One can easily check $\frac{d(s(\tilde{g}_t))}{dt}|_{t=0} = 0$ and

$$\frac{d^2(s(\tilde{g}_t))}{dt^2}|_{t=0} = -\frac{1}{4}(1 - r^2 - \rho^2)^2(F_\rho^2 + H_r^2) \leq 0. \tag{3.2}$$

Note that inside $\mathcal{D}$ the set of points with $\frac{d^2}{dt^2}(s(\tilde{g}_t))|_{t=0} = 0$ is identical to the set $\mathfrak{T}$. We see that $s(\tilde{g}_t)$ is strictly decreasing only on $\mathcal{D}\backslash\mathfrak{T}$. In order to have the right decreasing property, we need to diffuse the negativity (of scalar curvature) onto a ball containing $\mathcal{D}\backslash\mathfrak{T}$.

4. Diffusion of Negative Scalar Curvature onto a Ball

Our argument in this section follows those in [4, Section 4] and [5, Section 4] with just a few differences in estimation.

We use the following functions; $F_{t,m}^M(\rho) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ for $m, M > 0$, $t \geq 0$ defined by $F_{t,m}^M(\rho) = m \cdot t^2 \cdot \exp(-\frac{M}{\rho})$ on $\mathbb{R}^{>0}$ and $F_{t,m}^M = 0$ on $\mathbb{R}^{\leq 0}$. Also choose an $H \in C^\infty(\mathbb{R}, [0, 1])$ with $H = 0$ on $\mathbb{R}^{\geq 1}$, $H = 1$ on $\mathbb{R}^{\leq 0}$ and $H_\epsilon^b(\rho) = H(\frac{1}{\epsilon}(\rho - b))$, for $b > 0$, $\epsilon > 0$.

Let $B_r(x)$ be the open ball of radius r with respect to $\tilde{g}_0 = g_h$ centered at x. We may choose a point p and a number $\epsilon_1 < 0.1$ so that $B_{2\epsilon_1}(p) \subset \mathcal{D}\backslash\mathfrak{T}$ as $\mathfrak{T}$ is a thin subset. Then $s(\tilde{g}_t) < 0$ on $B_{\epsilon_1}(p)$ when $0 < t < c$ for some number c.

Let $f_{t,m}^M \in C^\infty(\mathbb{H}^4, \mathbb{R}^{\geq 0})$ be $f_{t,m}^M(q) = F_{t,m}^M(\varrho(q))$, where $\varrho(q)$ is the $\tilde{g}_0$-distance from p to $q \in \mathbb{H}^4$ and let $h_\epsilon^b \in C^\infty(\mathbb{H}^4, \mathbb{R}^{\geq 0})$ be $h_\epsilon^b(q) = H_\epsilon^b(\varrho(q))$. We choose $b = 9$ and $\epsilon = \epsilon_1$. We consider the Riemannian metric $e^{2\phi_t}\tilde{g}_t$, where

$$\phi_t(\varrho) = f_{t,m}^M(9 + \epsilon_1 - \varrho) \cdot h_{\epsilon_1}^9(9 + \epsilon_1 - \varrho) = mt^2e^{-\frac{M}{9+\epsilon_1-\varrho}}h_{\epsilon_1}^9(9 + \epsilon_1 - \varrho).$$

Here m and M will be determined below. The scalar curvature is as follows;

$$s(e^{2\phi_t}\tilde{g}_t) = e^{-2\phi_t}(s_{\tilde{g}_t} + 6\Delta_{\tilde{g}_t}\phi_t - 6|\nabla_{\tilde{g}_t}\phi_t|^2).$$

Setting $B = s_{\tilde{g}_t} + 6\Delta_{\tilde{g}_t}\phi_t - 6|\nabla_{\tilde{g}_t}\phi_t|^2$, we have

$$\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt} = -2\frac{d\phi_t}{dt}e^{-2\phi_t}B + e^{-2\phi_t}(\frac{ds_{\tilde{g}_t}}{dt} + 6\frac{d\Delta_{\tilde{g}_t}\phi_t}{dt} - 6\frac{d|\nabla_{\tilde{g}_t}\phi_t|^2}{dt})$$

and

$$\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2} = 4(\frac{d\phi_t}{dt})^2 e^{-2\phi_t}B - 2\frac{d^2\phi_t}{dt^2}e^{-2\phi_t}B - 4\frac{d\phi}{dt}e^{-2\phi_t}(\frac{ds_{\tilde{g}_t}}{dt} + 6\frac{d\Delta_{\tilde{g}_t}\phi_t}{dt} - 6\frac{d|\nabla_{\tilde{g}_t}\phi_t|^2}{dt}) + e^{-2\phi_t}(\frac{d^2 s_{\tilde{g}_t}}{dt^2} + 6\frac{d^2\Delta_{\tilde{g}_t}\phi_t}{dt^2} - 6\frac{d^2|\nabla_{\tilde{g}_t}\phi_t|^2}{dt^2}).$$

As ϕ_t is of second degree in t and $B|_{t=0} = -12$, we readily get

$$\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt}|_{t=0} = 0 \qquad \text{and}$$

$$\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2}|_{t=0} = \quad 48me^{-\frac{M}{9+\epsilon_1-\varrho}}h^9_{\epsilon_1}(9+\epsilon_1-\varrho) \quad + \quad \frac{d^2 s_{\tilde{g}_t}}{dt^2}|_{t=0} +12m\Delta_{\tilde{g}_0}e^{-\frac{M}{9+\epsilon_1-\varrho}}h^9_{\epsilon_1}(9+\epsilon_1-\varrho)\ .$$

On $B_{9+\epsilon_1}(p) - B_{\epsilon_1}(p)$, since $h^9_{\epsilon_1}(9+\epsilon_1-\varrho) = 1$ and $\frac{d^2 s_{\tilde{g}_t}}{dt^2}|_{t=0} \leq 0$,

$$\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2}|_{t=0} \leq 48me^{-\frac{M}{9+\epsilon_1-\varrho}} + 12m\Delta_{\tilde{g}_0}e^{-\frac{M}{9+\epsilon_1-\varrho}}. \tag{4.1}$$

As $\Delta_{\tilde{g}_0}f = -f'' - \frac{3}{\varrho}f'$ for a function $f := f(\varrho)$, we compute

$$\Delta_{\tilde{g}_0}e^{-\frac{M}{9+\epsilon_1-\varrho}} = -e^{-\frac{M}{9+\epsilon_1-\varrho}}\frac{M}{(9+\epsilon_1-\varrho)^4}\{M - 2(9+\epsilon_1-\varrho) - \frac{3}{\varrho}(9+\epsilon_1-\varrho)^2\}.$$

Then we can readily see in (4.1) that $\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2}|_{t=0} < 0$ for some large $M > 0$.

On $B_{\epsilon_1}(p)$, $\frac{d^2 s_{\tilde{g}_t}}{dt^2}|_{t=0} < 0$, so choose $m > 0$ small so that $48me^{-\frac{M}{9+\epsilon_1-\varrho}}h^9_{\epsilon_1}(9+\epsilon_1-\varrho) + \frac{d^2 s_{\tilde{g}_t}}{dt^2}|_{t=0} + 12m\Delta_{\tilde{g}_0}e^{-\frac{M}{9+\epsilon_1-\varrho}}h^9_{\epsilon_1}(9+\epsilon_1-\varrho) < 0$.

In sum, we have $\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt}|_{t=0} = 0$ and $\frac{d^2 s(e^{2\phi_t}\tilde{g}_t)}{dt^2}|_{t=0} < 0$ on $B_{9+\epsilon_1}(p)$ and $e^{2\phi_t}\tilde{g}_t = \tilde{g}_0$ on $\mathbb{H}^4 - B_{9+\epsilon_1}(p)$.

We may have subtlety near the boundary $\partial B_{9+\epsilon_1}(p)$, so we add the following argument.

On $\overline{B_9(p)}$, there exists $\tilde{\varepsilon} > 0$ such that $s(e^{2\phi_t}\tilde{g}_t)$ is strictly decreasing for $0 \leq t \leq \tilde{\varepsilon}$. For a moment we set $\kappa = 9+\epsilon_1-\varrho$, $\tilde{M} = M - 2\kappa - \frac{3}{\varrho}\kappa^2$ and $E = e^{-\frac{M}{\kappa}}$. On $B_{9+\epsilon_1}(p) - \overline{B_9(p)}$, $\tilde{g}_t = g_h$ and $s_{\tilde{g}_t} = -12$, so

$$s(e^{2\phi_t}\tilde{g}_t) = e^{-2\phi_t}(-12 + 6\Delta_{\tilde{g}_0}\phi_t - 6|\nabla_{\tilde{g}_0}\phi_t|^2) = e^{-2\phi_t}\{-12 + t^2\frac{6MmE}{\kappa^4}(-\tilde{M} - Mmt^2E)\}.$$

We have

$$\frac{ds(e^{2\phi_t}\tilde{g}_t)}{dt} = 12te^{-2\phi_t}mE(4 + 2\frac{t^2M\tilde{M}mE}{\kappa^4} + 2\frac{t^4M^2m^2E^2}{\kappa^4} - \frac{M\tilde{M}}{\kappa^4} - 2\frac{M^2mt^2E}{\kappa^4}).$$

As M is large and m small, $4+2\frac{t^2 M\tilde{M}mE}{\kappa^4}+2\frac{t^4M^2m^2E^2}{\kappa^4}-\frac{M\tilde{M}}{\kappa^4}<0$ for $0<t\le t_0$ with some $t_0>0$. Hence $s(e^{2\phi_t}\tilde{g}_t)$ is strictly decreasing for $0\le t\le t_0$ on $B_{9+\epsilon_1}(p)-\overline{B_9(p)}$. Setting $\varepsilon=\min\{\tilde{\varepsilon},t_0\}$, we get a scalar-curvature melting $g_t=e^{2\phi_t}\tilde{g}_t$ on $B_{9+\epsilon_1}(p)$ for $0\le t\le\varepsilon$. Theorem 1.1 is proved.

Remark 1. The argument in this article may be applicable to some other metrics. A more generalization, including spherical metrics, will appear later.

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