J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2013.20.4.243
Volume 20, Number 4 (November 2013), Pages 243–249

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

THE RANGE INCLUSION RESULTS FOR ALGEBRAIC NIL DERIVATIONS ON COMMUTATIVE AND NONCOMMUTATIVE ALGEBRAS

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Abstract. Let A be an algebra and D a derivation of A. Then D is called *algebraic nil* if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)}(P(x)) = 0$, for all $P \in \mathbb{C}[X]$ (by convention $D^{n(x)}(\alpha) = 0$, for all $\alpha \in \mathbb{C}$). In this paper, we show that any algebraic nil derivation (possibly unbounded) on a commutative complex algebra A maps into $N(A)$, where $N(A)$ denotes the set of all nilpotent elements of A. As an application, we deduce that any nilpotent derivation on a commutative complex algebra A maps into $N(A)$.

Finally, we deduce two noncommutative versions of algebraic nil derivations inclusion range.

1. Introduction

Let A be a complex algebra. A linear map D from A to A is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$. A derivation of D on A is called *nil* if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)} = 0$ (see [6]). Here, if the number n can be taken independently of x, D is called *nilpotent*. A derivation D of A is called *algebraic nil* if for any $x \in A$ there is a positive integer $n = n(x)$ such that $D^{n(x)}(P(x)) = 0$, for all $P \in \mathbb{C}[X]$ (by convention $D^{n(x)}(\alpha) = 0$, for all $\alpha \in \mathbb{C}$).

We will denote by $Q(A)$ the set of all quasinilpotent elements in a Banach algebra A. In 1955, Singer and Wermer [12] proved that a continuous derivation on a commutative Banach algebra maps into the (Jacobson) radical, and they conjectured that this result holds even if the derivation is discontinuous. In 1988, Thomas [13] solved the long standing problem by showing that the conjecture is true.

In 1991, Kim and Jun [10] proved that if D is a derivation on a noncommutative Banach algebra A satisfying the condition $[[A, A], A] = 0$ then $D(A) \subset Q(A)$. In

Received by the editors April 2, 2013. Revised August 26, 2013. Accepted August 29, 2013.
2010 *Mathematics Subject Classification.* Primary 13N15, 46E25; Secondary 16N40, 06F25.
Key words and phrases. algebraic nil derivation, d-algebra, nilpotent derivation.

1992, Vukman [15] proved that if D is a linear Jordan derivation on a noncommutative Banach algebra A such that the map $F(x) = [[Dx, x], x]$ is commuting on A then $D = 0$. In 1992, Mathieu and Runde [11] proved that if D is a centralizing derivation on a Banach algebra A; then $D(A) \subset rad(A)$. In 1994, Bresar [5] showed that if D is a bounded derivation of a Banach algebra such that $[D(x), x] \in Q(A)$ for every $x \in A$; then $D(A) \subset rad(A)$ where $rad(A)$ denotes the Jacobson radical of A.

To the best of our knowledge, there is no inclusion versions for derivations on arbitrary algebra, except the paper of Colville, Davis, and Keimel [9] in which they began studying positive derivations on f-rings (i.e., $D(a) \geq 0$, for all $a \geq 0$) and the papers of Boulabiar [4], A. Toumi et al [14] and Ben Amor [2], in which the authors studied exclusively positive and order bounded derivations on Archimedean almost f-algebras.

It is well-known that the notion of nil derivations is a generalization of the notion of nilpotent derivations. The latter, because of its close relation with automorphisms and the existence of a Jordan decomposition into semisimple and nilpotent parts for a large family of derivations (it is a generalization of that of algebraic derivations), has received considerable attention (see [6,7,8]). In this paper we shall be concerned principally with the range of algebraic nil derivations D on commutative algebra, on noncommutative archimedean d-algebra and on noncommutative algebra A satisfying the following condition; $[[A, A], A] = 0$.

2. The Main Results

To prove our first theorem, we shall need the following algebraic result.

Proposition 2.1. *Let A be a commutative complex algebra, n be a positive integer, D be a derivation on A and $x \in A$ such that*

$$D^n(x), D^n(x^2), D^n(x^n) \in N(A),$$

where $N(A)$ denotes the set of all nilpotent elements of A. Then $D(x) \in N(A)$.

Proof. Let $x \in A$ with $D^n(x), D^n(x^2), D^n(x^n) \in N(A)$.
It follows that

$$D^n(x^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x) \in N(A). \tag{1}$$

Since $D^n(x) \in N(A)$, we have

$$\sum_{k=1}^{n-1} \binom{n}{k} D^k(x) D^{n-k}(x) \in N(A). \tag{2}$$

Moreover, letting $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n$, this one has

$$D^n(x^n) = \sum_{k=0}^{n} \binom{n}{k} D^k(x^{n_1}) D^{n-k}(x^{n_2}) \in N(A). \tag{3}$$

By using the Leibnitz rule for $D^k(x^{n_1})$ and $D^k(x^{n_2})$ in Equality (3) and by using the relation (2), we deduce that

$$(D(x))^n \in N(A)$$

and then $D(x) \in N(A)$. □

From the above result, we deduce the following:

Proposition 2.2. *Let A be a commutative complex algebra, n be a positive integer, D be a derivation on A and $x \in A$ such that*

$$D^n(x) = D^n(x^2) = D^n(x^n) = 0.$$

Then $(D(x))^n = 0$.

The below theorem is an immediate consequence of Proposition 2.2.

Theorem 2.3. *Let A be a commutative complex algebra and let D be an algebraic nil derivation on A. Then $D(A)$ is contained in $N(A)$.*

Since any nilpotent derivation is algebraic nil, we have the following:

Corollary 2.4. *Let A be a commutative complex algebra and let D be a nilpotent derivation on A. Then $D(A)$ is contained in $N(A)$.*

In what follows, we shall deal with the range of algebraic nil derivation on non-commutative algebras. In order to hit this mark, we will need the following lemma.

Lemma 2.5 ([10, Lemma 3.1])**.** *Let A be a complex algebra satisfying the condition $[[A, A], A] = 0$. Let $A \oplus A$ be the vector space direct sum. Define a multiplication in $A \oplus A$ by setting*

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + a_2a_1, b_1b_2 + b_2b_1)$$

for all $(a_1, b_1), (a_2, b_2)$ in $A \oplus A$. Then $A \oplus A$ is a commutative algebra.

Using the previous lemma, we deduce the following result. Its proof is inspired from [10, Theorem 3.2].

Theorem 2.6. *Let A be a complex algebra satisfying the condition $[[A, A], A] = 0$ and let D be an algebraic nil derivation on A. Then $D(A)$ is contained in $N(A)$.*

Proof. By the previous lemma, $A \oplus A$ is a commutative algebra. Now we define the linear mapping $\overline{D} : A \oplus A \to A \oplus A$ by

$$\overline{D}(a, b) = (D(a), D(b)).$$

Since D is an algebraic nil derivation on A, it is not hard to prove that $\overline{D}$ is an algebraic nil derivation on $A \oplus A$. By Theorem 1, we have $\overline{D}(A \oplus A) \subset N(A \oplus A) = N(A) \oplus N(A)$. Therefore $D(A) \subset N(A)$. □

Corollary 2.7. *Let A be a complex algebra satisfying the condition $[[A, A], A] = 0$ and let D be a nilpotent derivation on A. Then $D(A)$ is contained in $N(A)$.*

Next, we will be interested with the range of derivations on noncommutative algebra A satisfying the following condition;

$$(\chi) \qquad a[A, A]b = 0$$

for all $a, b \in A$.

Theorem 2.8. *Let A be a complex algebra satisfying the condition (χ) and let D be an algebraic nil derivation on A. Then $D(A)$ is contained in $N(A)$.*

Proof. Let $x \in A$. Then there exists $n =: n(x) \in \mathbb{N}$ such that $D^n(x) = D^n(x^2) = D^n(x^n) = 0$. Let $a, b \in A$. It follows that

$$(4) \qquad aD^n(x^2)b = a\left(\sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x)\right) b = 0.$$

Moreover, let $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n(x)$, then

$$(5) \qquad aD^n(x^n)b = a\left(\sum_{k=0}^{n} \binom{n}{k} D^k(x^{n_1}) D^{n-k}(x^{n_2})\right) b = 0$$

By using the Leibnitz rule for $aD^k(x^{n_1})b$ and $aD^k(x^{n_2})b$ in Equality (5), by using Equality (4) and taking into account that $D^n(x) = 0$, we deduce that

$$a(D(x))^n b = 0$$

for all $a, b \in A$. Consequently $(D(x))^{n+2} = 0$. Therefore $D(A) \subset N(A)$. □

Corollary 2.9. *Let A be a complex algebra satisfying the condition (χ) and let D be a nilpotent derivation on A, then $D(A)$ is contained in $N(A)$.*

In the following lines, we recall definitions and some basic facts about lattice-ordered algebras. For more information about this field, one can refer to [1,3]. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, that is $a, b \in A^+$ implies $ab \in A^+$ is called *lattice-ordered algebra*(briefly *ℓ-algebra*). The ℓ-algebra A is said to be a d-algebra whenever $a \wedge b = 0$ in A implies $ac \wedge bc = ca \wedge cb = 0$, for all $0 \leq c \in A$. In general, d-algebras are not commutative, see [3].

Since any Archimedean d-algebra satisfies the condition (χ), see [3, Corollary 5.7], we deduce the following result:

Corollary 2.10. *Let A be an Archimedean d-algebra and let D be an algebraic nil derivation on A. Then $D(A)$ is contained in $N(A)$.*

Definition 2.11. Let A be an algebra. For a fixed $a \in A$, define $D : A \to A$ by $D(x) = [x, a] = xa - ax$, for all $x \in A$. Then D is called *inner derivation* of A associated with a and is generally denoted by D_a.

Theorem 2.12. *Let A be an Archimedean d-algebra with the condition $Z(A) = \{0\}$, where $Z(A)$ denotes the center of A and let D be an inner derivation on A. Then the following assertions are equivalent:*

- i) *D is nilpotent;*
- ii) *$D^3 = 0$;*
- iii) *D is induced by a nilpotent element.*

Proof. i) $\Rightarrow$ ii) Let $a \in A$ such that $D = D_a$. Since any Archimedean d-algebra satisfies the condition (χ), then for all $k \in \mathbb{N}$, we have

$$D_a^{2k+1}(x) = xa^{2k+1} - a^{2k+1}x$$

for all $x \in A$. Since D_a is nilpotent, there exists $n \in \mathbb{N}$ such that $D_a^n = 0$. Therefore

$$D_a^{2n+1}(x) = xa^{2n+1} - a^{2n+1}x = 0$$

for all $x \in A$. Consequently $a^{2n+1} \in Z(A) = \{0\}$. Hence $a^{2n+1} = 0$. By [3, Theorem 5.5], we deduce that $a^3 = 0$. It follows that $D_a^3 = 0$.
ii) $\Rightarrow$ iii) $D^3 = D_a^3 = 0$ means that $a^3 = 0$. Therefore $a \in N(A)$.
iii) $\Rightarrow$ i) This path is obvious. □

Remark 2.13. It is obvious that algebraic nil derivations are nil derivations. The simple-minded attempt to extend Theorem 1,2 and 3 to nil derivations obviously fails. This is illustrated in the following example.

Example 2.14. *Let* $A = \mathbb{C}[X]$ *and* $D : A \to A$ *defined by*

$$D\left(\sum_{i=1}^{n} a_i X^i\right) = a_1 + 2a_2 X + ... + na_n X^{n-1}.$$

It is not hard to prove that D *is a nil derivation but not an algebraic nil derivation, whereas* $D(A) = A \neq N(A)$.

Acknowledgment

The referees have reviewed the paper very carefully. The author expresses his deep thanks for the comments.

References

1. C.D. Aliprantis & O. Burkinshaw: *Positive Operators.* Academic Press. Orlando, 1985.
2. F. Ben Amor: On orthosymmetric bilinear maps. *Positivity* **14** (2010), no. 1, 123-134.
3. S.J. Bernau & C.B. Huijsmans: Almost f-algebras and d-algebras. *Math. Proc. Cam. Phil. Soc.* **107** (1990), 287-308.
4. K. Boulabiar: Positive Derivations on Almost f-rings. *Order* **19** (2002), 385-395.
5. M. Bresar: Derivations of noncommutative Banach algebras II. *Arch. Math.* **63** (1994), 56-59.
6. L.O. Chung: Nil derivations. *J. Algebra* **95** (1985), 20-30.
7. L.O. Chung & J. Luh: Nilpotency of derivations on an ideal. *Proc. Amer. Math. Soc.* **90** (1984), no. 2, 211-214.
8. L.O. Chung & Y. Kobayashi: Nil derivations and chain conditions in prime rings. *Proc. Amer. Math. Soc.* **94** (1985), no. 2, 201-205.
9. P. Colville, G. Davis & K. Keimel: Positive derivations on f-rings. *J. Austral. Math. Soc.* **23** (1977), 371-375.
10. B.D. Kim & K.-W. Jun: The range of derivations on noncommutative Banach algebras. *Bull. Korean Math. Soc.* **28** (1991), 65-68.
11. M. Mathieu & V. Runde: Derivations mapping into the radical II. *Bull. London Math. Soc.* **24** (1992), 485-487.
12. I.M. Singer & J. Wermer: Derivations on commutative normed algebras. *Math. Ann.* **129** (1955), 260-264.
13. M. Thomas: The image of a derivations contained in the radical. *Ann. of Math.* **128** (1988), 435-460.

14. A. Toumi & M.A. Toumi: Order bounded derivations on Archimedean almost f-algebras. *Positivity* **14** (2010), no. 2, 239-245.
15. J. Vukman: On derivations in prime rings and Banach algebras. *Proc. Amer. Math. Soc.* **116** (1992), no. 4, 877-884.

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