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TAYLOR SERIES OF FUNCTIONS WITH VALUES IN DUAL QUATERNION

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Abstract. We define an ε-regular function in dual quaternions. From the properties of ε-regular functions, we represent the Taylor series of ε-regular functions with values in dual quaternions.

1. Introduction

Fueter [2] and Naser [8] have studied properties of quaternionic differential equations as a generalization of the extended Cauchy-Riemann equations in the complex holomorphic function theory and Nôno [9, 10, 11] has given a definition of regular functions over the quaternion field $\mathcal{T}$ identified with $\mathbb{R}^4$. In 1979, Sudbery [15] developed quaternionic regular function theories. By using a generalization of the Cauchy-Riemann equation, Ryan [12, 13] has developed regular function theories on complex Clifford algebra of quaternion valued functions.

In 1873, Clifford [1] originally conceived the algebra of dual numbers. Dual algebra has been often used for closed form solutions in the field of displacement analysis. Kotelnikov [6] and Study [14] developed dual vectors and dual quaternions for use in the application of mechanics and realized that this associative algebra was ideal for describing the group of motions of three-dimensional spaces.

In 2011, Koriyama, Mae and Nôno [5] investigated hyperholomorphic functions and holomorphic functions in quaternion analysis. In 2012, Gotô and Nôno [3] researched regular functions with values in a commutative subalgebra of matrix algebra in four real dimension and we [7] obtained regularities of functions with values in subalgebra of matrix algebras in complex n-dimensional.

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In this paper, we introduce the dual quaternion numbers and give some properties of ε-regular functions in dual quaternions by using the associated Pauli matrices. We give the notation of the derivative for functions with values in dual quaternions and obtain the representation of the Taylor series of ε-regular functions.

2. Preliminary

A dual quaternion is an ordered pair of quaternions and is constructed from eight base elements $e_0, e_1, e_2, e_3, \varepsilon, e_1\varepsilon, e_2\varepsilon$ and $e_3\varepsilon$. We consider the associated Pauli matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. And, we let the dual quaternion identity

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is a nonzero and satisfy $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$, $\varepsilon^2 = 0$ and

$$\mathcal{DH} := \{z = \zeta + \zeta^*\varepsilon \mid \zeta,\ \zeta^* \in \mathcal{T}\} \cong \mathcal{T} \times \mathcal{T},$$

where $\zeta = \sum_{j=0}^{3} e_j x_j$, $\zeta^* = \sum_{j=0}^{3} e_j x_j^*$ and x_j^* is a dual quaternion component of x_j $(x_j, x_j^* \in \mathbb{R})$. The element e_0 is the identity, the element ε is the dual identity of $\mathcal{DH}$ and the element e_1 identifies the imaginary unit $i = \sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. We can identify $\mathcal{DH}$ with $\mathbb{C}^4$.

The dual quaternionic conjugation z^* of z, the absolute value $|z|$ of z and an inverse z^{-1} of z in $\mathcal{DH}$ are defined, respectively, by

$$\begin{aligned} z^* &= \overline{\zeta} + \overline{\zeta^*}\varepsilon, \\ |z|^2 &= zz^* = \sum_{j=0}^{3}(x_j + \varepsilon x_j^*)^2, \\ z^{-1} &= \frac{z^*}{|z|^2}, \end{aligned}$$

where $\overline{\zeta} = \sum_{j=0}^{3} \overline{e_j} x_j$, $\overline{\zeta^*} = \sum_{j=0}^{3} \overline{e_j} x_j^*$ and $\overline{e_0} = e_0$, $\overline{e_j} = -e_j$ $(j = 1, 2, 3)$.

Let Ω be an open subset of $\mathbb{C}^2 \times \mathbb{C}^2$ and the dual quaternion function

$$f : \Omega \longrightarrow \mathcal{DH}$$

satisfy

$$z \in \Omega \longmapsto f(z) = \sum_{j=0}^{3} e_j f_j(\zeta, \zeta^*) \in \mathcal{DH},$$

where $f_j(\zeta, \zeta^*) = u_j(\zeta, \zeta^*) + \varepsilon u_j^*(\zeta, \zeta^*)$ and u_j, u_j^* $(j = 0, 1, 2, 3)$ are real-valued functions.

We use the following two dual quaternion differential operators which are defined as

$$D := \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \zeta^*}\varepsilon \text{ and } D^* = \frac{\partial}{\partial \overline{\zeta}} + \frac{\partial}{\partial \overline{\zeta^*}}\varepsilon,$$

where

$$\frac{\partial}{\partial \zeta} = \sum_{j=0}^{3} \overline{e_j}\frac{\partial}{\partial x_j}, \ \frac{\partial}{\partial \zeta^*} = \sum_{j=0}^{3} \overline{e_j}\frac{\partial}{\partial x_j^*}, \ \frac{\partial}{\partial \overline{\zeta}} = \sum_{j=0}^{3} e_j\frac{\partial}{\partial x_j}, \ \frac{\partial}{\partial \overline{\zeta^*}} = \sum_{j=0}^{3} e_j\frac{\partial}{\partial x_j^*}.$$

Then we have

$$\begin{aligned} D^* f &= \left(\frac{\partial}{\partial \overline{\zeta}} + \frac{\partial}{\partial \overline{\zeta^*}}\varepsilon\right) f \\ &= \begin{pmatrix} \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_0^*} + i\frac{\partial}{\partial x_1^*} \\ -\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2^*} + i\frac{\partial}{\partial x_3^*} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} u_0 + iu_1 & u_2 + iu_3 + u_0^* + iu_1^* \\ -u_2 + iu_3 & u_0 - iu_1 - u_2^* + iu_3^* \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} a_1 =& \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} - \frac{\partial u_2}{\partial x_0^*} - \frac{\partial u_3}{\partial x_1^*} \\ &+ i\left(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_0^*} - \frac{\partial u_2}{\partial x_1^*}\right), \\ a_2 =& \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0^*}{\partial x_0} - \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_0}{\partial x_2} - \frac{\partial u_2^*}{\partial x_2} + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3^*}{\partial x_3} + \frac{\partial u_0}{\partial x_0^*} - \frac{\partial u_2^*}{\partial x_0^*} \\ &+ \frac{\partial u_1}{\partial x_1^*} - \frac{\partial u_3^*}{\partial x_1^*} + i\Bigg(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_1^*}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_0^*}{\partial x_1} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2} + \frac{\partial u_0}{\partial x_3} - \frac{\partial u_2^*}{\partial x_3} \\ &- \frac{\partial u_1}{\partial x_0^*} + \frac{\partial u_3^*}{\partial x_0^*} + \frac{\partial u_0}{\partial x_1^*} - \frac{\partial u_2^*}{\partial x_1^*}\Bigg), \end{aligned}$$

$$\begin{aligned}a_3 = &-\frac{\partial u_0}{\partial x_2}-\frac{\partial u_1}{\partial x_3}-\frac{\partial u_2}{\partial x_0}+\frac{\partial u_3}{\partial x_1}+\frac{\partial u_2}{\partial x_2^*}-\frac{\partial u_3}{\partial x_3^*}\\ &+i\left(-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_0}{\partial x_3}+\frac{\partial u_3}{\partial x_0}+\frac{\partial u_2}{\partial x_1}-\frac{\partial u_3}{\partial x_2^*}-\frac{\partial u_2}{\partial x_3^*}\right),\\ a_4 = &-\frac{\partial u_2}{\partial x_2}-\frac{\partial u_0^*}{\partial x_2}-\frac{\partial u_3}{\partial x_3}-\frac{\partial u_1^*}{\partial x_3}+\frac{\partial u_0}{\partial x_0}-\frac{\partial u_2^*}{\partial x_0}-\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3^*}{\partial x_1}-\frac{\partial u_0}{\partial x_2^*}+\frac{\partial u_2^*}{\partial x_2^*}\\ &+\frac{\partial u_1}{\partial x_3^*}-\frac{\partial u_3^*}{\partial x_3^*}-i\Big(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1^*}{\partial x_2}-\frac{\partial u_2}{\partial x_3}-\frac{\partial u_0^*}{\partial x_3}+\frac{\partial u_1}{\partial x_0}-\frac{\partial u_3^*}{\partial x_0}+\frac{\partial u_0}{\partial x_1}-\frac{\partial u_2^*}{\partial x_1}\\ &-\frac{\partial u_1}{\partial x_2^*}+\frac{\partial u_3^*}{\partial x_2^*}\frac{\partial u_0}{\partial x_3^*}+\frac{\partial u_2^*}{\partial x_3^*}\Big).\end{aligned}$$

Definition 2.1. Let Ω be an open set in $\mathbb{C}^2\times\mathbb{C}^2$. A function $f(z)$ is said to be ε-regular in Ω if the following two conditions are satisfied:
(a) f_j $(j=0,1,2,3)$ are continuously differential functions in Ω, and
(b) $D^*f(z)=0$ in Ω.

3. Taylor Series of Dual Quaternion Functions

We define the derivative $f'(z)$ of $f(z)$ by the following:

$$f'(z):=Df(z).$$

Lemma 3.1. *Let Ω be a domain in $\mathbb{C}^2\times\mathbb{C}^2$ and $f(z)$ be a holomorphic mapping and ε-regular defined in Ω. Then,*

$$\begin{aligned}f'(z) &= \Big(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\overline{\zeta}}+\varepsilon(\frac{\partial}{\partial\zeta^*}+\frac{\partial}{\partial\overline{\zeta^*}})\Big)f\\ &= 2\Big(\frac{\partial}{\partial x_0}+\varepsilon\frac{\partial}{\partial x_0^*}\Big)f\\ &= -2\sum_{j=1}^{3}e_j\left(\frac{\partial}{\partial x_j}+\varepsilon\frac{\partial}{\partial x_j^*}\right)f.\end{aligned}$$

Proof. Since $f(z)$ is an ε-regular function in Ω, we have

$$\begin{aligned}f'(z) &= \begin{pmatrix}\frac{\partial}{\partial x_0}-i\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2}-i\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_0^*}-i\frac{\partial}{\partial x_1^*}\\ \frac{\partial}{\partial x_2}-i\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2^*}-i\frac{\partial}{\partial x_3^*}\end{pmatrix}\\ &\cdot\begin{pmatrix}u_0+iu_1 & u_2+iu_3+u_0^*+iu_1^*\\ -u_2+iu_3 & u_0-iu_1-u_2^*+iu_3^*\end{pmatrix}\end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \\ &= 2\Big(\frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_0^*}\Big) f \\ &= -2\sum_{j=1}^{3} e_j \left(\frac{\partial}{\partial x_j} + \varepsilon \frac{\partial}{\partial x_j^*}\right) f, \end{aligned}$$

where

$$\begin{aligned} d_1 &= 2\Big((\frac{\partial u_0}{\partial x_0} - \frac{\partial u_2}{\partial x_0^*}) + i(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_3}{\partial x_0^*})\Big), \\ d_2 &= 2\Big((\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0^*}{\partial x_0} + \frac{\partial u_0}{\partial x_0^*} - \frac{\partial u_2^*}{\partial x_0^*}) + i(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_1^*}{\partial x_0} - \frac{\partial u_1}{\partial x_0^*} + \frac{\partial u_3^*}{\partial x_0^*})\Big), \\ d_3 &= -\frac{\partial u_2}{\partial x_0} + i\frac{\partial u_3}{\partial x_0}, \\ d_4 &= 2\Big((\frac{\partial u_0}{\partial x_0} - \frac{\partial u_2^*}{\partial x_0}) + i(-\frac{\partial u_1}{\partial x_0} + \frac{\partial u_3^*}{\partial x_0})\Big). \end{aligned}$$

From

$$\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}} = 2\frac{\partial}{\partial x_0} \text{ and } \frac{\partial}{\partial \zeta^*} + \frac{\partial}{\partial \overline{\zeta^*}} = 2\frac{\partial}{\partial x_0^*},$$

we can get

$$f'(z) = \Big(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}} + \varepsilon(\frac{\partial}{\partial \zeta^*} + \frac{\partial}{\partial \overline{\zeta^*}})\Big) f.$$

□

Theorem 3.2. *Let $f(z)$ be a homogeneous polynomial of degree m with respect to the variables ζ and ζ^*. If $f(z)$ is a holomorphic and ε-regular function in $\mathbb{C}^2 \times \mathbb{C}^2$, then we have*

$$f(z) = \frac{1}{m!} f^{(m)}(z) z^m.$$

Proof. Since $f(z)$ is a homogeneous polynomial, we have

$$\begin{aligned} f(z) &= \sum_{k=0}^{m} \binom{m}{k} \zeta^{m-k} (\varepsilon \zeta^*)^k \\ &= \zeta^m + \varepsilon m \zeta^{m-1} \zeta^*. \end{aligned}$$

Then

$$\begin{aligned} f'(z) &= m\zeta^{m-1} + \varepsilon m(m-1)\zeta^{m-2}\zeta^*, \\ f'(z)z &= m\zeta^m + \varepsilon m^2 \zeta^{m-1}\zeta^*. \end{aligned}$$

Thus, $f(z) = \frac{1}{m} f'(z) z$. And

$$\begin{aligned} f''(z) &= m(m-1)\zeta^{m-2} + \varepsilon m(m-1)(m-2)\zeta^{m-3}\zeta^*, \\ f''(z)z &= m(m-1)\zeta^{m-1} + \varepsilon m(m-1)^2\zeta^{m-2}\zeta^*. \end{aligned}$$

Thus, $f'(z) = \frac{1}{m-1} f''(z) z$. Repeating the above calculation, we have

$$f(z) = \frac{1}{m!} f^{(m)}(z) z^m.$$

□

Theorem 3.3. *Let Ω be a domain in $\mathbb{C}^2 \times \mathbb{C}^2$. Let $f(z)$ be a holomorphic and ε-regular function in Ω and $\alpha \in \Omega$. Then there exists a neighborhood U_α of α such that*

$$f(z) = \sum_{m=0}^{\infty} C_m \zeta^m + \varepsilon \Big\{ C_1 + \sum_{m=2}^{\infty} C_m ((m-1)\zeta^{m-1} + \zeta^{m-2}) \Big\} \zeta^*,$$

where $C_m = \frac{1}{m!} f^{(m)}(\alpha)$.

Proof. From substituting a dual number into Taylor series, we have

$$\begin{aligned} f(z) &= f(\zeta) + \frac{f'(\zeta)}{1!}(\varepsilon\zeta^*) + \frac{f''(\zeta)}{2!}(\varepsilon\zeta^*)^2 + \frac{f^{(3)}(\zeta)}{3!}(\varepsilon\zeta^*)^3 + \cdots \\ &= f(\zeta) + \varepsilon f'(\zeta)\zeta^*. \end{aligned}$$

By Theorem 3.2 and $z^m = \zeta^m + \varepsilon((m-1)\zeta^{m-1}\zeta^* + \zeta^{m-2}\zeta^*)$, we have

$$\begin{aligned} f(z) &= C_0 + C_1 z + \sum_{m=2}^{\infty} C_m \Big\{ \zeta^m + \varepsilon((m-1)\zeta^{m-1}\zeta^* + \zeta^{m-2}\zeta^*) \Big\} \\ &= C_0 + C_1\zeta + \sum_{m=2}^{\infty} C_m \zeta^m + \varepsilon \Big\{ C_1\zeta^* + \sum_{m=2}^{\infty} C_m ((m-1)\zeta^{m-1}\zeta^* + \zeta^{m-2}\zeta^*) \Big\} \\ &= \sum_{m=0}^{\infty} C_m \zeta^m + \varepsilon \Big\{ C_1 + \sum_{m=2}^{\infty} C_m ((m-1)\zeta^{m-1} + \zeta^{m-2}) \Big\} \zeta^*. \end{aligned}$$

□

Remark 3.4. Let Ω be a domain in $\mathbb{C}^2 \times \mathbb{C}^2$. If $g_0(z)$ is a holomorphic function with value in quaternions, then there exists a function $g_1(z)$ with value in quaternions such that

$$f(z) = g_0(z) + \varepsilon g_1(z)$$

is ε-regular in Ω.

By the results of Kenwright [4],

$$f(z) = f(\zeta) + \varepsilon f'(\zeta)\zeta^*.$$

We put $g_0(z) = f(\zeta)$, then $Dg_0(z) = f'(\zeta)$. Hence we put $g_1(z) = Dg_0(z)\zeta^*$, we have

$$f(z) = g_0(z) + \varepsilon g_1(z).$$

Example 3.5. Let Ω be a domain in $\mathbb{C}^2 \times \mathbb{C}^2$. If $g_0 = \sin(n\zeta)$, then $g_0' = n\cos(n\zeta)$, $n \in \mathbb{Z}$. Thus, there exists a function $g_1 = n\cos(n\zeta)\zeta^*$ such that

$$f(z) = g_0(z) + \varepsilon g_1(z).$$

is ε-regular in Ω.

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