J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2014.21.1.1
Volume 21, Number 1 (February 2014), Pages 1–10

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

# TWO CHARACTERIZATION THEOREMS FOR HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KENMOTSU MANIFOLD

Dae Ho Jin

Abstract. In this paper, we study the curvature of locally symmetric or semi-symmetric half lightlike submanifolds $M$ of an indefinite Kenmotsu manifold $\bar{M}$, whose structure vector field is tangent to $M$. After that, we study the existence of the totally geodesic screen distribution of half lightlike submanifolds of indefinite Kenmotsu manifolds with parallel co-screen distribution subject to the conditions: (1) $M$ is locally symmetric, or (2) the lightlike transversal connection is flat.

## 1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the electromagnetic field theory. The study of such notion was initiated by Duggal and Bejancu [2] and later studied by many authors (see up-to date results in two books [4, 5]). The class of lightlike submanifolds of codimension 2 is compose of two classes by virtue of the rank of its radical distribution, which are called the *half lightlike* and *coisotropic submanifolds* [3]. Half lightlike submanifold is a special case of $r$-lightlike submanifold such that $r = 1$ and its geometry is more general form than that of coisotrophic submanifold. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to general $r$-lightlike submanifolds of arbitrary codimension $n$ and arbitrary rank $r$.

In the theory of Sasakian manifolds, the following result is well-known [9]: *If a Sasakian manifold is locally symmetric, then it is of constant positive curvature* 1. In 1971, K. Kenmotsu proved the following result [8]: *If a Kenmotsu manifold is locally symmetric, then it is of constant negative curvature* $-1$.

Received by the editors February 22, 2013. Revised January 13, 2014. Accepted Jan. 15, 2014
2010 *Mathematics Subject Classification.* 53C25, 53C40, 53C50.
*Key words and phrases.* locally symmetric, semi-symmetric manifold, half lightlike submanifolds, indefinite Kenmotsu manifold.

In this paper, we study the curvature of locally symmetric or semi-symmetric half lightlike submanifolds of an indefinite Kenmotsu manifold $\bar{M}$, whose structure vector field is tangent to $M$. After that, we study the existence of the totally geodesic screen distribution of half lightlike submanifolds of indefinite Kenmotsu manifolds with parallel co-screen distribution subject such that either $M$ is locally symmetric or the lightlike transversal connection is flat. We prove the following results:

**Theorem 1.1.** *Let $M$ be a half lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$, whose structure vector field is tangent to $M$. If $M$ is locally symmetric or semi-symmetric, then $M$ is a space of constant negative curvature $-1$. In this case, the induced connection on $M$ is a torsion-free metric connection and the lightlike transversal connection is flat.*

**Theorem 1.2.** *Let $M$ be a half lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with parallel co-screen distribution. If either $M$ is locally symmetric or the lightlike transversal connection is flat, then the screen distribution $S(TM)$ of $M$ is never totally geodesic in $M$.*

## 2. Half Lightlike Submanifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite Kenmotsu manifold* [7, 8, 10] if there exist a structure set $(J, \zeta, \theta, \bar{g})$, where $J$ is a $(1,1)$-type tensor field, $\zeta$ is a vector field and $\theta$ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta, \quad J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\zeta) = 1, \tag{2.1}$$

$$\theta(X) = \bar{g}(\zeta, X), \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \theta(X)\theta(Y),$$

$$\bar{\nabla}_X \zeta = -X + \theta(X)\zeta, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = -\bar{g}(JX, Y)\zeta + \theta(Y)JX, \tag{2.3}$$

for any vector fields $X$, $Y$ on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$.

A submanifold $(M, g)$ of a semi-Riemannian manifold $\bar{M}$ of codimension 2 is called a *half lightlike submanifold* if the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of $M$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^{\perp}$ of rank 1. Then there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in $TM$ and $TM^{\perp}$ respectively, which are called the *screen* and *co-screen distributions* on $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \pm 1$. In this paper we may assume that $\bar{g}(L, L) = 1$, without loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. For any null section $\xi$ of $Rad(TM)$, certainly $\xi$ and $L$ belong to $\Gamma(S(TM)^{\perp})$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section $\xi$ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \ \forall\, X \in \Gamma(S(TM)). \tag{2.5}$$

We call $N$, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of $M$ with respect to $S(TM)$ respectively. Therefore $T\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.6}$$

Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then the local Gauss and Weingarten formulas of $M$ and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.8}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N\,; \tag{2.9}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.10}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \ \forall\, X,\, Y \in \Gamma(TM). \tag{2.11}$$

where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, $B$ and $D$ are called the *local second fundamental forms* of $M$, $C$ is called the *local second fundamental form* on $S(TM)$. $A_N$, $A^*_\xi$ and $A_L$ are linear operators on $TM$ and $\tau$, $\rho$ and $\phi$ are 1-forms on $TM$.

Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free and both $B$ and $D$ are symmetric. From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we know that $B$ and $D$ are independent of the choice of $S(TM)$ and satisfy

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \quad \forall\, X \in \Gamma(TM). \tag{2.12}$$

The induced connection $\nabla$ of $M$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{2.13}$$

for all $X$, $Y$, $Z \in \Gamma(TM)$, where $\eta$ is a 1-form on $TM$ such that

$$\eta(X) = \bar{g}(X,N),\ \forall\, X \in \Gamma(TM). \tag{2.14}$$

But the connection $\nabla^*$ on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.15}$$

$$C(X,PY) = g(A_{_N} X, PY), \qquad \bar{g}(A_{_N} X, N) = 0, \tag{2.16}$$

$$D(X,PY) = g(A_{_L} X, PY), \qquad \bar{g}(A_{_L} X, N) = \rho(X), \tag{2.17}$$

$$D(X,Y) = g(A_{_L} X, Y) - \phi(X)\eta(Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{2.18}$$

In case $C = 0$ on any coordinate neighborhood $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in $M$. From (2.10), we show that $S(TM)$ is totally geodesic in $M$ if and only if $S(TM)$ is a parallel distribution on $M$, i.e.,

$$\nabla_X Y \in \Gamma(S(TM)), \quad \forall\, X \in \Gamma(TM) \text{ and } Y \in \Gamma(S(TM)).$$

In the sequel, we let $X$, $Y$, $Z$, $U$, $\cdots$ be the vector fields of $M$, unless otherwise specified. Denote by $\bar{R}$ and $R$ the curvature tensors of $\bar{\nabla}$ and $\nabla$ respectively. Using (2.7)$\sim$(2.11), we have the Gauss-Codazzi equations for $M$ and $S(TM)$:

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + B(X,Z)A_{_N}Y - B(Y,Z)A_{_N}X + D(X,Z)A_{_L}Y - D(Y,Z)A_{_L}X \\ &\quad + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\qquad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N, \\ &\quad + \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \bar{R}(X,Y)N &= -\nabla_X(A_{_N}Y) + \nabla_Y(A_{_N}X) + A_{_N}[X,Y] \\ &\quad + \tau(X)A_{_N}Y - \tau(Y)A_{_N}X + \rho(X)A_{_L}Y - \rho(Y)A_{_L}X \\ &\quad + \{B(Y,A_{_N}X) - B(X,A_{_N}Y) + 2d\tau(X,Y) + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N \\ &\quad + \{D(Y,A_{_N}X) - D(X,A_{_N}Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \bar{R}(X,Y)L &= -\nabla_X(A_{_L}Y) + \nabla_Y(A_{_L}X) + A_{_L}[X,Y] \\ &\quad + \phi(X)A_{_N}Y - \phi(Y)A_{_N}X \\ &\quad + \{B(Y,A_{_L}X) - B(X,A_{_L}Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X)\}N, \end{aligned} \tag{2.21}$$

$$R(X,Y)\xi = -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y)\}\xi. \tag{2.22}$$

A half lightlike submanifold $M = (M, g, \nabla)$ equipped with a degenerate metric $g$ and a linear connection $\nabla$ is said to be of *constant curvature* $c$ if there exists a constant $c$ such that the curvature tensor $R$ of $\nabla$ satisfies

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.23}$$

For any $X \in \Gamma(TM)$, let $\nabla^\ell_X N = Q(\bar{\nabla}_X N)$, where $Q$ is the projection morphism of $\Gamma(T\bar{M})$ on $\Gamma(ltr(TM))$ with respect to (2.6). Then $\nabla^\ell$ is a linear connection on the lightlike transversal vector bundle $ltr(TM)$ of $M$. We say that $\nabla^\ell$ is the *lightlike transversal connection* of $M$. We define the curvature tensor $R^\ell$ on $ltr(TM)$ by

$$R^\ell(X,Y)N = \nabla^\ell_X\nabla^\ell_Y N - \nabla^\ell_Y\nabla^\ell_X N - \nabla^\ell_{[X,Y]}N. \tag{2.24}$$

If $R^\ell$ vanishes identically, then the transversal connection is said to be *flat*.

From (2.8) and the definition of $\nabla^\ell$, we get $\nabla^\ell_X N = \tau(X)N$ for all $X \in \Gamma(TM)$. Substituting this equation into the right side of (2.24), we get

$$R^\ell(X,Y)N = 2d\tau(X,Y)N.$$

From this result we deduce the following theorem:

**Theorem 2.1** ([6]). *Let $M$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the lightlike transversal connection of $M$ is flat, if and only if the 1-form $\tau$ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

**Note 1.** We know that $d\tau$ is independent of the choice of the section $\xi$ on $Rad(TM)$, where $\tau$ is given by $\tau(X) = \bar{g}(\bar{\nabla}_X N, \xi)$. In fact, if we take $\tilde{\xi} = \gamma\xi$ and $\tilde{\tau}(X) = \bar{g}(\bar{\nabla}_X \tilde{N}, \tilde{\xi})$, it follows that $\tau(X) = \tilde{\tau}(X) + X(\text{In}\,\gamma)$. If we take the exterior derivative $d$ on the last equation, then we have $d\tau = d\tilde{\tau}$.

## 3. Proof of Theorem 1.1

Assume that $\zeta$ is tangent to $M$. It is well known [1] that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$. Replacing $Y$ by $\zeta$ to (2.7) and using (2.2), we have

$$\nabla_X\zeta = -X + \theta(X)\zeta, \quad B(X,\zeta) = D(X,\zeta) = 0. \tag{3.1}$$

Substituting $(3.1)_1$ into $R(X,Y)\zeta = \nabla_X\nabla_Y\zeta - \nabla_Y\nabla_X\zeta - \nabla_{[X,Y]}\zeta$ and using (2.19), (3.1) and the fact that $\nabla$ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = R(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta.$$

Taking the scalar product with $\zeta$ to this and using the fact $g(\bar{R}(X,Y)\zeta,\zeta)=0$ and (2.1), we show that $\theta$ is closed, i.e., $d\theta=0$ on $TM$. Thus we obtain

$$R(X,Y)\zeta=\theta(X)Y-\theta(Y)X. \tag{3.2}$$

Applying $\bar{\nabla}_X$ to $\theta(Y)=g(Y,\zeta)$ and using (2.2), (2.5) and $\bar{g}(\zeta,N)=0$, we have

$$(\nabla_X\theta)(Y)=-g(X,Y)+\theta(X)\theta(Y). \tag{3.3}$$

Case 1. Assume that $M$ is locally symmetric, i.e., $\nabla R=0$. Applying $\nabla_Z$ to (3.2) and using the first equation of (3.1)[denote by $(3.1)_1$], (3.2) and (3.3), we have

$$R(X,Y)Z=g(X,Z)Y-g(Y,Z)X. \tag{3.4}$$

Thus $M$ is a space of constant curvature $-1$. Applying $\nabla_U$ to (3.4), we have

$$(\nabla_U g)(X,Z)Y=(\nabla_U g)(Y,Z)X.$$

Taking $Z=Y=\xi$ to this and using $(2.12)_1$ and (2.13), we get $B=0$. Thus $\nabla$ is a torsion-free metric connection on $M$ by (2.13). As $B=0$, we have $A^*_\xi=0$ by (2.15). From (2.22), we get $R(X,Y)\xi=-2d\tau(X,Y)\xi$. On the other hand, replacing $Z$ by $\xi$ to (3.4), we have $R(X,Y)\xi=0$. These two results imply $d\tau=0$. Thus the lightlike transversal connection $\nabla^\ell$ is flat.

Case 2. Assume that $M$ is semi-symmetric, i.e., $R(X,Y)R=0$. Applying $\nabla_Z$ to (3.2) and using $(3.1)_1$, (3.2) and (3.3), we have

$$(\nabla_Z R)(X,Y)\zeta=R(X,Y)Z-g(X,Z)Y+g(Y,Z)X. \tag{3.5}$$

Substituting (3.5) into $(R(U,Z)R)(X,Y)\zeta=0$ and using $(3.1)_1$, we have

$$\begin{aligned}0=&\ \theta(Z)(\nabla_U R)(X,Y)\zeta-\theta(U)(\nabla_Z R)(X,Y)\zeta\\&+\{B(U,Y)\eta(Z)-B(Z,Y)\eta(U)\}X-\{B(U,X)\eta(Z)-B(Z,X)\eta(U)\}Y.\end{aligned} \tag{3.6}$$

Replacing $U$ by $\zeta$ to (3.6) and using $(\nabla_\zeta R)(X,Y)\zeta=0$ due to (3.2) and (3.5), we have $(\nabla_Z R)(X,Y)\zeta=0$. From this and (3.5), we show that

$$R(X,Y)Z=g(X,Z)Y-g(Y,Z)X. \tag{3.7}$$

Thus $M$ is a space of constant negative curvature $-1$. Replacing $U$ by $\xi$ to (3.6) and using $(2.12)_1$, (3.7) and $(\nabla_Z R)(X,Y)\zeta=0$, we have

$$B(Y,Z)X=B(X,Z)Y.$$

Replacing $Y$ by $\xi$ to this and using $(2.12)_1$, we get $B=0$. Thus, by (2.13), $\nabla$ is a torsion-free metric connection on $M$. Using (2.22), (3.7) and the method of Case 1, we see that the lightlike transversal connection is flat. $\square$

## 4. Proof of Theorem 1.2

From the decomposition (2.6) of $T\bar{M}$, the vector field $\zeta$ is decomposed as

$$\zeta = W + mN + nL, \tag{4.1}$$

where $W$ is a smooth vector field on $M$ and $m = \theta(\xi)$ and $n = \theta(L)$ are smooth functions. Substituting (4.1) in (2.2) and using (2.8) and (2.9), we have

$$\nabla_X W = -X + \theta(X)W + mA_N X + nA_L X, \tag{4.2}$$

$$Xm + m\tau(X) + n\phi(X) + B(X,\, W) = m\theta(X), \tag{4.3}$$

$$Xn + m\rho(X) + D(X,\, W) = n\theta(X). \tag{4.4}$$

Substituting (4.3) and (4.4) into the following two equations

$$[X,Y]m = X(Ym) - Y(Xm), \quad [X,Y]n = X(Yn) - Y(Xn),$$

and using (2.19), (2.20), (2.21), (4.1), (4.3), (4.4), we have respectively

$$2m\, d\theta(X,Y) = \bar{g}(\bar{R}(X,Y)\zeta,\, \xi),\ 2n\, d\theta(X,Y) = \bar{g}(\bar{R}(X,Y)\zeta,\, L). \tag{4.5}$$

Substituting (4.2) into $R(X,Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W$ and using (2.19)∼(2.21), (4.2)∼(4.5) and the fact $\nabla$ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta. \tag{4.6}$$

Taking the scalar product with $\zeta$ to (4.6) and using (2.1), we show that the structure 1-form $\theta$ is closed, i.e., $d\theta = 0$ on $TM$.

Assume that $S(TM)$ is totally geodesic in $M$. In this case, $\zeta$ is not tangent to $M$ and $l = \theta(N) \neq 0$. In fact, if $\zeta$ is tangent to $M$ or $l = 0$, then $\bar{g}(\zeta, N) = 0$. Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.2) and (2.8), we have $\eta(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction as $\eta(\xi) = 1$. Thus $\zeta$ is not tangent to $M$ and $l \neq 0$. As $\zeta$ is not tangent to $M$, we see that $(m,n) \neq (0,0)$. As $S(TM^{\perp})$ is a parallel distribution, we have $A_L = \phi = 0$ due to (2.9). From (2.17) and (2.18), we also have $D = \rho = 0$.

Substituting (2.19)∼(2.21) into (4.6) and using (4.5), we get

$$R(X,Y)W = \theta(X)Y - \theta(Y)X. \tag{4.7}$$

Applying $\bar{\nabla}_X$ to $\theta(Y) = g(Y, \zeta)$ and using (2.2) and (2.6), we have

$$(\nabla_X\theta)(Y) = lB(X,Y) - g(X,Y) + \theta(X)\theta(Y). \tag{4.8}$$

Case 1. Assume $M$ is locally symmetric. Applying $\nabla_Z$ to (4.7), we have

$$R(X,Y)\nabla_Z W = (\nabla_Z\theta)(X)Y - (\nabla_Z\theta)(Y)X.$$

Substituting (4.2) and (4.8) in this equation and using (4.7), we obtain

$$R(X,Y)Z = \{g(X,Z) - lB(X,Z)\}Y - \{g(Y,Z) - lB(Y,Z)\}X. \tag{4.9}$$

Replacing $Z$ by $\xi$ to (4.9) and using $(2.12)_1$, we have $R(X,Y)\xi = 0$. Comparing the $Rad(TM)$-components of this and (2.22), we have $d\tau = 0$. Thus by Theorem 2.1 the lightlike transversal connection is flat. From (2.19), (2.20) and (4.9), we have

$$\begin{aligned} 0 &= \bar{g}(\bar{R}(X,Y)N, Z) = -\bar{g}(\bar{R}(X,Y)Z, N) = -\bar{g}(R(X,Y)Z, N) \\ &= \{g(Y,Z) - lB(Y,Z)\}\eta(X) - \{g(X,Z) - lB(X,Z)\}\eta(Y), \end{aligned} \tag{4.10}$$

Replacing $Y$ by $\xi$ to (4.10) and using $(2.12)_1$, we get

$$lB(X,Y) = g(X,Y). \tag{4.11}$$

From (4.9) and (4.11), we show that $R = 0$. From this and (4.7), we have

$$\theta(X)Y = \theta(Y)X.$$

Replacing $Y$ by $\xi$ to this equation and using $X = PX + \eta(X)\xi$, we have

$$m\,PX = g(X,W)\xi.$$

As the left term of this equation belongs to $S(TM)$ and the right term belongs to $Rad(TM)$, we have $m\,PX = 0$ and $g(X,W)\xi = 0$ for all $X \in \Gamma(TM)$. Thus $m = 0$ and $g(X,W) = 0$ for all $X \in \Gamma(TM)$. This imply $W = l\xi$ and

$$\zeta = l\xi + nL. \tag{4.12}$$

From this and the fact $\bar{g}(\zeta,\zeta) = 1$, we show that $n^2 = 1$.

It is known [6] that, for any half lightlike submanifold of an indefinite almost contact metric manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$ of rank 1 respectively. Applying $\bar{\nabla}_X$ to $\bar{g}(JN, L) = 0$ and using (2.1), (2.3), (2.8) and (2.9), we have

$$ng(X, JN) = lg(X, JL). \tag{4.13}$$

Replacing $X$ by $J\xi$ to (4.13) and using $(2.1)_6$, we have $n = 0$. It is a contradiction as $n^2 = 1$. Thus $S(TM)$ is not totally geodesic in $M$.

Case 2. Assume that the transversal connection is flat. We have $d\tau = 0$. Substituting (4.1) into (4.6) with $d\theta = 0$ and using (2.19)∼(2.21) and (4.5), we have

$$\bar{R}(X,Y)W = \theta(X)Y - \theta(Y)X.$$

Taking the scalar product with $W$ to this and using the facts $\theta(X) - m\eta(X) = g(X,W)$ and $\bar{g}(\bar{R}(X,Y)W,W) = 0$, we have

$$\theta(Y)\eta(X) - \theta(X)\eta(Y) = 0.$$

Replacing $Y$ by $\xi$ to this equation, we have $g(X, W) = 0$ for all $X \in \Gamma(TM)$. This implies $W = l\xi$. Thus $\zeta$ is decomposed as

$$\zeta = l\xi + mN + nL. \tag{4.14}$$

From the fact $\bar{g}(\zeta, \zeta) = 1$ and (4.14), we show that $2lm = 1 - n^2$. Applying $\bar{\nabla}_X$ to (4.14) and using (2.2), (2.8), (2.9) and (2.11), we have

$$\begin{aligned} &-lA^*_\xi X + \{X[l] - l\tau(X)\}\xi + \{X[m] + m\tau(X)\}N + X[n]L \\ &= -PX + \{l\theta(X) - \eta(X)\}\xi + m\theta(X)N + n\theta(X)L. \end{aligned}$$

Taking the scalar product with $\xi$, $N$ and $L$ to this result by turns, we get

$$X[l] - l\tau(X) = l\theta(X) - \eta(X), \;\; Xm + m\tau(X) = m\theta(X), \;\; Xn = n\theta(X), \tag{4.15}$$

respectively. From (2.15) and (4.15), we have

$$lA^*_\xi X = PX, \qquad lB(X, Y) = g(X, Y). \tag{4.16}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(JN, L) = 0$ and using (2.1), (2.3), (2.8), (2.9) and the fact $S(TM)$ is non-degenerate, we have

$$nJN = lJL. \tag{4.17}$$

Taking the scalar product with $J\xi$ to this and using $(2.1)_6$, we have $n(1 - ml) = -lmn$. This implies $n = 0$. As $(m, n) \neq (0, 0)$ and $n = 0$, we have $m \neq 0$ and $2lm = 1$. Consequently we get $JL = 0$ by (4.17). It is a contradiction as

$$0 = g(JL, JL) = \bar{g}(L, L) - \theta(L)^2 = 1 - n^2 = 1.$$

Thus $S(TM)$ is not totally geodesic in $M$. □

**Corollary 1.** *Let $M$ be a half lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the structure 1-form $\theta$, given by* (2.1)*, is closed on $TM$.*

## References


1. C. Călin: *Contributions to geometry of CR-submanifold.* Thesis, University of Iasi (Romania), 1998.
2. K.L. Duggal & A. Bejancu: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.* Kluwer Acad. Publishers, Dordrecht, 1996.
3. K.L. Duggal & D.H. Jin: Half-Lightlike Submanifolds of Codimension 2. *Math. J. Toyama Univ.* **22** (1999), 121-161.
4. ______: *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds.* World Scientific, 2007.

5. K.L. Duggal & B. Sahin: *Differential geometry of lightlike submanifolds.* Frontiers in Mathematics, Birkhäuser, 2010.
6. D.H. Jin: Half lightlike submanifolds of an indefinite Sasakian manifold. *J. Korean Soc. Math. Edu. Ser. B: Pure Appl. Math.* **18** (2011), no. 2, 173-183.
7. ______: The curvatures of lightlike hypersurfaces in an indefinite Kenmotsu manifold. *Balkan J. of Geo. and Its Appl.* **17** (2012), no. 1, 49-57.
8. K. Kenmotsu: A class of almost contact Riemannian manifolds. *Tôhoku Math. J.* **21** (1972), 93-103.
9. M. Okumura: Some remarks on space with a certain contact structure. *Tôhoku Math. J.* **14** (1962), no. 2, 135-145.
10. R. Shankar Gupta & A. Sharfuddin: Lightlike submanifolds of indefinite Kenmotsu manifold. *Int. J. Contemp. Math. Sciences* **5** (2010), no. 10, 475-496.


DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, GYEONGJU 780-714, REPUBLIC OF KOREA
*Email address*: jindh@dongguk.ac.kr