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LIPSCHITZ AND ASYMPTOTIC STABILITY FOR PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. The present paper is concerned with the notions of Lipschitz and asymptotic stability for perturbed nonlinear differential system knowing the corresponding stability of nonlinear differential system. We investigate Lipschitz and asymtotic stability for perturbed nonlinear differential systems. The main tool used is integral inequalities of the Bihari-type, in special some consequences of an extension of Bihari's result to Pinto and Pachpatte, and all that sort of things.

1. Introduction

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [8] . For linear systems, the notions of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. In fact, uniformly Lipschitz stability lies somewhere between uniformly stability on one side and the notions of asmptotic stability in variation of Brauer[4] and uniformly stability in variation of Brauer and Strauss[3] on the other side. Gonzalez and Pinto[9] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems.

In this paper, we investigate Lipschitz and asymptotic stability for solutions of the nonlinear differential systems. To do this we need some integral inequalities. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

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2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed differential system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds, \; y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel[8].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(S) *stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,
(US) *uniformly stable* if the δ in (S) is independent of the time t_0,
(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$,

and $\delta > 0$ such that

$$|x(t)| \le K\,|x_0|e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \le K\, e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \infty$.

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ \ y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.2. *Let x and y be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.3 ([7])**.** *Let $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. If , for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)\Big\{ \int_{t_0}^{s} \lambda_2(\tau)w(u(\tau))d\tau \Big\}ds,\ \ t \ge t_0 \ge 0,$$

then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_2(s)ds\Big] \exp\Big(\int_{t_0}^{t} \lambda_1(s)ds\Big),\ \ t_0 \le t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}, u > 0,\ u_0 > 0,\ W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s)ds \in \mathrm{domW}^{-1} \Big\}.$$

Lemma 2.4 ([10]). *Let u, p, q, w, and $r \in C(\mathbb{R}^+)$ and suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau)u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)u(a)da]d\tau ds,\ t \geq t_0. \tag{2.6}$$

Then

$$u(t) \leq c\exp(\int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)da]d\tau ds),\ t \geq t_0. \tag{2.7}$$

Lemma 2.5 ([15]). *Let $u(t)$, $f(t)$,and $g(t)$ be real-valued nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(\tau)u(\tau)d\tau)ds,\ t \in \mathbb{R}^+,$$

holds, where u_0 is a nonnegative constant. Then,

$$u(t) \leq u_0(1 + \int_0^t f(s)\exp(\int_0^s (f(\tau) + g(\tau))d\tau)ds),\ t \in \mathbb{R}^+.$$

Lemma 2.6 ([12]). *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)(\int_{t_0}^{s} \lambda_3(\tau)u(\tau)d\tau)ds,\ \ 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds\Big],\ t_0 \leq t < b_1, \tag{2.8}$$

where W, W^{-1} are the same functions as in Lemma 2.3 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.7 ([13]). *Let $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a)w(u(a))da)d\tau)ds,\ t \geq t_0. \tag{2.9}$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds\Big],\ t_0 \leq t < b_1, \tag{2.10}$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.8 ([14]). *Let the following condition hold for functions $u(t), v(t) \in C[[t_0, \infty), \mathbb{R}^+)$ and $k(t, u) \in C[[t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^+)$:*

$$u(t) - \int_{t_0}^{t} k(s, u(s))ds \leq v(t) - \int_{t_0}^{t} k(s, v(s))ds,$$

$t \geq t_0$ and $k(s, u)$ is strictly increasing in u for each fixed $s \geq 0$. If $u(t_0) < v(t_0)$, then $u(t) < v(t)$, $t \geq t_0 \geq 0$.

Lemma 2.9 ([5]). *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)(\int_{t_0}^{s} \lambda_3(\tau) w(u(\tau))d\tau)ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}, u > 0$, $u_0 > 0$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds \in \mathrm{domW}^{-1}\Big\}.$$

3. Main Results

In this section, we investigate Lipschitz and asymptotic stability for solutions of the nonlinear perturbed differential systems.

Theorem 3.1. *Assume that $x = 0$ of (2.1) is ULS. Let the following condition hold for (2.2):*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq W(t, |y|), 0 \leq t_0 \leq t,$$

where $W(t, u) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is monotone nondecreasing in u with $W(t, 0) = 0$. Suppose that $u(t)$ is any solution of the scalar differential equation

$$u'(t) = MW(t, u), u(t_0) = u_0 > 0, M \geq 1, \tag{3.1}$$

existing on $\mathbb{R}^+$ such that $m(t_0) < u(t_0)$. If $u = 0$ of (3.1) is ULS, then $y = 0$ of (2.2) is also ULS whenever $M|y_0| < u_0$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using the variation of constants formula, we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds,$$

where $\Phi(t, t_0, y_0)$ is the fundemental matrix of (2.4). Since $x = 0$ of (2.1) is ULS, it is ULSV by Corollary 3.6[5]. Thus there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, y_0)| \le M$ for $t \ge t_0 \ge 0$. Therefore, by the assmption, we have

$$|y(t)| - M \int_{t_0}^{t} W(s, |y(s)|) ds \le M|y_0| < u_0 = u(t) - M \int_{t_0}^{t} W(s, u(s)) ds.$$

Hence $|y(t)| < u(t)$ by Lemma 2.8. Since $u = 0$ of (3.1) is ULS, it easily follows that $y = 0$ of (2.2) is ULS. □

Corollary 3.2. *Assume that $x = 0$ of (2.1) is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t,u) = Ka(t)[u + \int_{t_0}^{t} k(s)u(s)ds], \tag{3.2}$$

where $u_0 \ge 1, K \ge 1$ and $a, k \in C(\mathbb{R}^+)$ satisfy the conditions
(a) $\int_{t_0}^{t} |g(s, y(s))| ds \le W(t, |y|)$, where $\int_{t_0}^{t} g(s, y(s)) ds$ is in (2.2),
(b) $M(t_0) = (1 + K \int_{t_0}^{\infty} a(s) exp(\int_{t_0}^{s} (Ka(\tau) + k(\tau)) d\tau) ds) < \infty$ and $b_1 = \infty$.
Then $y = 0$ of (2.2) is ULS.

Proof. Let $u(t) = u(t, t_0, x_0)$ be any solution of (3.2). Then, by Lemma 2.5 , we have

$$|u(t)| \le u_0(1 + K \int_{t_0}^{t} a(s) exp(\int_{t_0}^{s} (Ka(\tau) + k(\tau)) d\tau) ds) \le M(t_0)|u_0|,$$

Hence $u = 0$ of (3.2) is ULS. This implies that the solution $y = 0$ of (2.2) is ULS by Theorem 3.1. □

Remark 3.3. In Corollary 3.2, it is needed that $b_1 = \infty$. The condition $W(\infty) = \infty$ is too strong and it represents situations which are not stable. For example, if $w(u) = u^{\alpha}$, then only $\alpha \le 1$ satisfies $W(\infty) = \infty$ and $\alpha < 1$ is not stable. See [18].

Corollary 3.4. *Assume that $x = 0$ of (2.1) is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t,u) = Ka(t)[u + \int_{t_0}^{t} k(s)w(u(s))ds], \tag{3.3}$$

where $u_0 \ge 1, K \ge 1$, $u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v} w(u) \le w(\frac{u}{v})$ for some $v > 0$, and $a, k \in C(\mathbb{R}^+)$ satisfy the conditions
(a) $\int_{t_0}^{t} |g(s, y(s))| ds \le W(t, |y|)$, where $\int_{t_0}^{t} g(s, y(s)) ds$ is in (2.2),

(b) $M(t_0) = W^{-1}[W(u_0) + \int_{t_0}^{\infty} k(s)ds] \cdot exp(\int_{t_0}^{\infty} Ka(s)ds) < \infty$, $b_1 = \infty$, *and* $a, k \in L_1(\mathbb{R}^+)$. *Then* $y = 0$ *of (2.2) is ULS.*

Proof. Let $u(t) = u(t, t_0, x_0)$ be any solution of (3.3). Then, by Lemma 2.3, we have

$$|u(t)| \leq W^{-1}[W(u_0) + \int_{t_0}^{t} k(s)ds] \cdot exp(\int_{t_0}^{t} Ka(s)ds) \leq M(t_0) \leq M(t_0)|u_0|.$$

Hence $u = 0$ of (3.3) is ULS. By Theorem 3.1, the solution $y = 0$ of (2.2) is ULS. □

Corollary 3.5. *Assume that* $x = 0$ *of (2.1) is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t, u) = K[a(t)w(u(t)) + b(s) \int_{t_0}^{t} k(s)u(s)ds], \tag{3.4}$$

where $w \in C((0, \infty)$, $w(u)$ *is nondecreasing on* u *and* $u \leq w(u)$, $u_0 \geq 1, K \geq 1$ *and* $a, b, k \in C(\mathbb{R}^+)$ *satisfy the conditions*
(a) $\int_{t_0}^{t} |g(s, y(s))|ds \leq W(t, |y|)$, *where* $\int_{t_0}^{t} g(s, y(s))ds$ *is in (2.2),*
(b) $M(t_0) = W^{-1}[W(u_0) + K \int_{t_0}^{\infty}(a(s) + b(s) \int_{t_0}^{s} k(s)ds)] < \infty$, $b_1 = \infty$, *and* $a, b, k \in L_1(\mathbb{R}^+)$. *Then* $y = 0$ *of (2.2) is ULS.*

Proof. Let $u(t) = u(t, t_0, x_0)$ be any solution of (3.4). Then, Lemma 2.6, we have

$$|u(t)| \leq W^{-1}[W(u_0) + K \int_{t_0}^{t}(a(s) + b(s) \int_{t_0}^{s} k(s)ds)] \leq M(t_0) \leq M(t_0)|u_0|.$$

Hence $u = 0$ of (3.4) is ULS, and so by Theorem 3.1, the solution $y = 0$ of (2.2) is ULS. □

Theorem 3.6. *For the perturbed (2.2), we asssume that*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)|y(s)|ds,$$

where $a, b, k \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty)$, *and* $w(u)$ *is nondecreasing in* u,$u \leq w(u)$, *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty}(a(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau)ds\Big], \tag{3.5}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS by Theorem 3.3[8]. Applying

Lemma 2.2, we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0| a(s) w(\frac{|y(s)|}{|y_0|})ds \\ &\quad + \int_{t_0}^{t} M|y_0| b(s) \int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{|y_0|} d\tau ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.6 yields

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big].$$

Hence we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Theorem 3.7. *For the perturbed (2.2), we asssume that*

$$|g(t,y)| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)|y(s)|ds,$$

where $a, b, k \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *is nondecreasing in* u,$u \le w(u)$, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau ds\Big], \tag{3.6}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using the nonlinear variation of constants formula and the ULSV condition of $x = 0$ of (2.1), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} [a(\tau) w(\frac{|y(\tau)|}{|y_0|})d\tau ds \\ &\quad + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|} dr] d\tau ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.7 yields

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau ds\Big],$$

Thus we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Theorem 3.8. *Let the solution $x = 0$ of (2.1) be EAS. Suppose that the perturbing term $g(t, y)$ satisfies*

$$|g(t, y(t))| \leq e^{-\alpha t}\Big(a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)|y(s)|ds\Big), \tag{3.7}$$

where $\alpha > 0$, $a, b, k \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If

$$M(t_0) = c\exp(\int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}[a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds) < \infty,\ t \geq t_0, \tag{3.8}$$

where $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EAS, we have $|\Phi(t, t_0, x_0)| \leq Me^{-\alpha(t-t_0)}$ for some $M > 0$ and $c > 0$(Theorem 2[2]). Using Lemma 2.2, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big|ds \\ &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}[a(\tau)e^{-\alpha\tau}|y(\tau)| \\ &\quad + b(\tau)\int_{t_0}^{\tau} k(r)e^{-\alpha r}|y(r)|drd\tau]ds, \end{aligned}$$

since $e^{\alpha t}$ is increasing. Set $u(t) = |y(t)|e^{\alpha t}$. An application of Lemma 2.4 obtains

$$|y(t)| \leq ce^{-\alpha t}\exp(\int_{t_0}^{t} Me^{\alpha s}\int_{t_0}^{s}[a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds) \leq ce^{-\alpha t}M(t_0),\ t \geq t_0.$$

The above estimation yields the desired result. □

Theorem 3.9. *Let the solution $x = 0$ of (2.1) be EAS. Suppose that the perturbing term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq e^{-\alpha t}\Big(a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds\Big), \tag{3.9}$$

where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$ and $w(u)$ is nondecreasing in u. If

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] < \infty, b_1 = \infty, \tag{3.10}$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using Lemma 2.2 and the assmptions, we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}[e^{-\alpha s}a(s)w(|y(s)|) \\ &\quad +Mb(s)e^{-\alpha s}\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau]ds.\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, an application of Lemma 2.9 obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big],$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtains the desired result. □

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