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STIELTJES DERIVATIVE METHOD FOR INTEGRAL INEQUALITIES WITH IMPULSES

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Abstract. The purpose of this paper is to obtain some integral inequalities with impulses by using the method of Stieltjes derivatives, and we use our results in the study of Lyapunov stability of solutions of a certain nonlinear impulsive integro-differential equation.

1. Introduction

In this paper, we discuss various integral inequalities with impulses.

Differential equations with impulses arise in various real world phenomena in mathematical physics, mechanics, engineering, biology and so on. We refer to the monograph of Samoilenko and Perestyuk [6]. Also integral inequalities are very useful tools in global existence, uniqueness, stability and other properties of the solutions of various nonlinear differential equations, see,e.g., [5].

To obtain our results in the paper we need some preliminaries. Now we state them.

Assume that $[a,b],[c,d]\subset \mathbf{R}$ are bounded intervals, where $\mathbf{R}$ is the set of all real numbers.

A function $f:[a,b]\longrightarrow \mathbf{R}$ is called *regulated* on $[a,b]$ if both

$$f(s+)=\lim_{\eta\to 0+} f(s+\eta),\ \text{and}\ f(s-)=\lim_{\eta\to 0+} f(s-\eta)$$

exist for every point $s\in[a,b]$. As a convention we define $f(a-)=f(a)$ and $f(b+)=f(b)$. Let $G[a,b]$ be the set of all regulated functions on $[a,b]$. If we let for $f\in G[a,b]$, $\|f\|=\sup_{s\in[a,b]}|f(s)|$, then $(G[a,b],\|\cdot\|)$ becomes a Banach space. For regulated functions, see [1, 2].

For a closed interval $I=[c,d]$, we define $f(I)=f(d)-f(c)$. A function $f:[a,b]\longrightarrow \mathbf{R}$ is *of bounded variation* on $[a,b]$ if

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$$V_a^b(f) \equiv \sup\{\sum_{i=1}^{n} |f([t_{i-1}, t_i])|\} < \infty, \tag{1.1}$$

where the supremum is taken over all partitions

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Let $BV[a,b]$ be the set of all functions of bounded variation on $[a,b]$. We use the following notations for the convenience:

$$\Delta^+ f(s) = f(s+) - f(s), \Delta^- f(s) = f(s) - f(s-) \text{ and } \Delta f(s) = f(s+) - f(s-). \tag{1.2}$$

A *tagged* interval $(\tau, [c,d])$ in $[a,b]$ consists of an interval $[c,d] \subset [a,b]$ and a point $\tau \in [c,d]$. Let $I_i = [c_i, d_i] \subset [a,b]$. A finite collection $\{(\tau_i, [c_i, d_i]) : i = 1, 2, ..., m\}$ of pairwise non-overlapping tagged intervals is called a *tagged partition of* $[a,b]$ if $\cup_{i=1}^{n} I_i = [a,b]$. A positive function δ on $[a,b]$ is called a *gauge* on $[a,b]$.

Definition 1.1 ([4, 7])**.** Let δ be a gauge on $[a,b]$. A tagged partition P=$\{(\tau_i$,$[t_{i-1}$,$t_i])$ $: i = 1, 2, ..., m\}$ of $[a,b]$ is said to be $\delta-$*fine* if for every $i = 1, ..., m$ we have

$$\tau_i \in [t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

If moreover a $\delta-$fine partition P satisfies the implications

$$\tau_i = t_{i-1} \Rightarrow i = 1, \quad \tau_i = t_i \Rightarrow i = m,$$

then it is called a δ^*-*fine partition* of $[a,b]$.

The following lemma implies that for a gauge δ on $[a,b]$ there exists a δ^*-fine partition of $[a,b]$. This also implies the existence of a $\delta-$fine partition of $[a,b]$.

Lemma 1.2 ([4])**.** *Let δ be a gauge on $[a,b]$ and a dense subset $\Omega \subset (a,b)$ be given. Then there exists a δ^*-fine partition $P = \{(\tau_i, [t_{i-1}, t_i]) : i = 1, 2, ..., m\}$ of $[a,b]$ such that $t_i \in \Omega$ for $i = 1, ..., m-1$.*

We are now ready to give a formal definition of both types of the Kurzweil integral.

Definition 1.3 ([4, 7])**.** Assume that $f, g : [a,b] \longrightarrow \mathbf{R}$ are given. We say that fdg is *Kurzweil integrable* (or shortly, *K-integrable*) on $[a,b]$ and $v \in \mathbf{R}$ is its integral if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for

$$S(fdg, P) \equiv \sum_{i=1}^{n} f(\tau_i) g(I_i),$$

we have

$$|S(fdg, P) - v| \leq \varepsilon,$$

provided $P=\{(\tau_i,I_i):i=1,...,n\}$ is a $\delta-$fine tagged partition of $[a,b]$. In this case we denote $v=\int_a^b f(s)dg(s)$ (or, shortly,$v=\int_a^b fdg$).

If, in the above definition, $\delta-$fine is replaced by δ^*-fine, then we say that fdg is Kurzweil* integrable(or, shortly, K*-integrable) on $[a,b]$ and we denote $v=(K^*)\int_a^b fdg$.

Remark 1.4. By the above definition it is obvious that K-integrability implies K*-integrability.

The integrals have the following properties. For the proofs, see, e.g., [7, 8].

Theorem 1.5. *Assume that $f,f_1,f_2,g:[a,b]\longrightarrow\boldsymbol{R}$ and that $f_1\mathrm{d}g$ and $f_2\mathrm{d}g$ are integrable in the sense of Kurzweil or Kurzweil* on $[a,b]$. Let $k_1,k_2\in\boldsymbol{R}$. Then we have*

$$\int_a^b(k_1f_1+k_2f_2)\mathrm{d}g=k_1\int_a^b f_1\mathrm{d}g+k_2\int_a^b f_2\mathrm{d}g.$$

If for $c\in[a,b]$, integrals $\int_a^c f\,\mathrm{d}g,\int_c^b f\,\mathrm{d}g$ exist, then $\int_a^b f\,\mathrm{d}g$ exists also and we have

$$\int_a^b f\mathrm{d}g=\int_a^c f\mathrm{d}g+\int_c^b f\mathrm{d}g.$$

For the integrability we have the following fundamental result.

Theorem 1.6. *Assume that $f\in G[a,b]$ and $g\in BV[a,b]$. Then fdg is K-integrable on $[a,b]$.*

Theorem 1.7. *Assume that $f,g:[a,b]\longrightarrow\boldsymbol{R}$ and that fdg is K-integrable. If g is a regulated function on $[a,b]$, then we have*

$$\lim_{\eta\to0+}\int_a^{s\pm\eta}fdg=\int_a^s fdg+f(s)(g(s\pm)-g(s)).$$

2. The Stieltjes Derivatives

In this section we state the results in [3] that are essential to verify our main results.

Throughout this section, we assume that $f\in G[a,b]$ and g is a nondecreasing function on $[a,b]$.

A *neighborhood* of $t\in[a,b]$ is an open interval containing t. We say that the function g is *not locally constant* at $t\in(a,b)$ if there exists $\eta>0$ such that g is not constant on $(t-\varepsilon,t+\varepsilon)$ for every $\varepsilon<\eta$. We also say that the function g is *not locally constant* at a and b, respectively if there exists $\eta>0$ such that g is not constant on $[a,a+\varepsilon),(b-\varepsilon,b]$, respectively for every $\varepsilon<\eta$.

Definition 2.1. If g is not locally constant at $t \in (a,b)$, we define

$$\frac{df(t)}{dg(t)} = \lim_{\eta,\delta\to 0+} \frac{f(t+\eta)-f(t-\delta)}{g(t+\eta)-g(t-\delta)},$$

provided that the limit exists. If g is not locally constant at $t = a$ and $t = b$ respectively, we define

$$\frac{df(a)}{dg(a)} = \lim_{\eta\to 0+} \frac{f(a+\eta)-f(a)}{g(a+\eta)-g(a)}, \quad \frac{df(b)}{dg(b)} = \lim_{\delta\to 0+} \frac{f(b)-f(b-\delta)}{g(b)-g(b-\delta)},$$

respectively. Sometimes we use $f'_g(t)$ instead of $\frac{df(t)}{dg(t)}$.

If both f and g are constant on some neighborhood of t, we define $\frac{df(t)}{dg(t)} = 0$.

Remark 2.2. It is obvious that if g is not continuous at t then $f'_g(t)$ exists. Thus if $f'_g(t)$ does not exist then g is continuous at t. $f'_g(t)$ is called the Stieltjes derivative.

K*-integrals recover Stieltjes derivatives.

Theorem 2.3. *Assume that if g is constant on some neighborhood of t then f is also constant there. Suppose that $f'_g(t)$ exists at every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*)\int_a^b f'_g(s)dg(s) = f(b) - f(a).$$

3. MAIN RESULTS

In this section we will state and prove our results.

Let

$$0 < t_1 < t_2 < \cdots < t_m < 1,$$

and let $0 < a < 1$. Two sorts of Heaviside functions $H_a, H_a^* : [0,1] \longrightarrow \{0,1\}$ are defined respectively by

$$H_a(t) = \begin{cases} 0, & \text{if } t \le a \\ 1, & \text{if } t > a, \end{cases} \quad H_a^*(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \ge a. \end{cases}$$

Using the Heaviside functions H_a, H_a^* we define functions $\phi, \psi : [0,1] \longrightarrow [0,\infty)$ by

$$\phi(t) = t + \sum_{k=1}^{m} H_{t_k}(t), \ \psi(t) = t + \sum_{k=1}^{m} H_{t_k}^*(t), \quad t \in [0,1],$$

respectively.

It is obvious that the functions ϕ, ψ are strictly increasing and of bounded variation on $[0,1]$.

From now on, we assume that $c \geq 0$ and that all the functions $u, f, g, g_i, i = 1, ..., n$ are nonnegative functions defined on $[0, 1]$ that are regulated on $[0, 1]$ and continuous at every $t \neq t_k, k = \overline{1, m}$, where $\overline{1, m} = 1, ..., m$.

Lemma 3.1. *Assume that $f'(t)$ exists for $t \neq t_k, k = \overline{1, m}$. Then we have*

(a)

$$f'_\phi(t) = f'_\psi(t) = f'(t), \quad f'_\phi(t_k) = f'_\psi(t_k) = f(t_k+) - f(t_k-).$$

Let $f'_\psi(t_k-) = \lim_{\eta \to 0+} \frac{f(t_k) - f(t_k - \eta)}{\psi(t_k) - \psi(t_k - \eta)}$. Then we have

$$f'_\psi(t_k-) = f(t_k) - f(t_k-).$$

(b) *If a left-continuous function f is positive, nondecreasing, and differentiable at $t \neq t_k, k = \overline{1, m}$, then*

$$\frac{d[\log f(t)]}{d\phi(t)} = \frac{f'(t)}{f(t)}, \quad \frac{d[\log f(t_k)]}{d\phi(t_k)} \leq \frac{f'_\phi(t_k)}{f(t_k)}.$$

(c)

$$\int_0^t f(s) d\phi(s) = \int_0^t f(s) ds + \sum_{0 < t_k < t} f(t_k),$$

and

$$\int_0^t f(s) d\psi(s) = \int_0^t f(s) ds + \sum_{0 < t_k \leq t} f(t_k).$$

Proof. (a) By definition, for $t_k < t < t_{k+1}$ and for sufficiently small δ and η we have

$$\begin{aligned} \phi(t + \delta) - \phi(t - \eta) &= \left[t + \delta + \sum_{i=1}^{m} H_{t_i}(t + \delta) \right] - \left[t - \eta + \sum_{i=1}^{m} H_{t_i}(t - \eta) \right] \\ &= \left[t + \delta + \sum_{i=1}^{k} 1 \right] - \left[t - \eta + \sum_{i=1}^{k} 1 \right] = \delta + \eta, \end{aligned}$$

so we have

$$f'_\phi(t) = \lim_{\delta, \eta \to 0+} \frac{f(t + \delta) - f(t - \eta)}{\phi(t + \delta) - \phi(t - \eta)} = \lim_{\delta, \eta \to 0+} \frac{f(t + \delta) - f(t - \eta)}{\delta + \eta} = f'(t).$$

And

$$\begin{aligned} \phi(t_k + \delta) - \phi(t_k - \eta) &= \left[t_k + \delta + \sum_{i=1}^{m} H_{t_i}(t_k + \delta) \right] - \left[t_k - \eta + \sum_{i=1}^{m} H_{t_i}(t_k - \eta) \right] \\ &= \left[t_k + \delta + \sum_{i=1}^{k} 1 \right] - \left[t_k - \eta + \sum_{i=1}^{k-1} 1 \right] = 1 + \delta + \eta, \end{aligned}$$

This implies

$$f'_\phi(t_k) = \lim_{\delta,\eta\to 0+} \frac{f(t_k+\delta) - f(t_k-\eta)}{\phi(t_k+\delta) - \phi(t_k-\eta)} = \lim_{\delta,\eta\to 0+} \frac{f(t_k+\delta) - f(t_k-\eta)}{1+\delta+\eta} = f(t_k+) - f(t_k-).$$

Similarly we can verify

$$f'_\psi(t) = f'_\phi(t), f'_\psi(t_k) = f'_\phi(t_k).$$

And

$$f'_\psi(t_k-) = \lim_{\eta\to 0+} \frac{f(t_k) - f(t_k-\eta)}{\psi(t_k) - \psi(t_k-\eta)} = \lim_{\eta\to 0+} \frac{f(t_k) - f(t_k-\eta)}{1+\eta} = f(t_k) - f(t_k-).$$

This completes the proof for (a).

(b) By (a) if $t \neq t_k$ then it is obvious that

$$\frac{d[\log f(t)]}{d\phi(t)} = \frac{d[\log f(t)]}{dt} = \frac{f'(t)}{f(t)},$$

and by the Mean Value Theorem and since f is nondecreasing and left-continuous we have

$$\frac{d[\log f(t_k)]}{d\phi(t_k)} = \log f(t_k+) - \log f(t_k-)$$
$$\leq \frac{f(t_k+) - f(t_k)}{f(t_k)} = \frac{f'_\phi(t_k)}{f(t_k)}.$$

(c) By Theorem 1.7, we have for $t_k < t$

$$\int_0^t f(s) dH_{t_k}(s) = \lim_{\delta,\eta\to 0+} \int_{t_k-\eta}^{t_k+\delta} f(s) dH_{t_k}(s) = f(t_k)[H_{t_k}(t_k+) - H_{t_k}(t_k-)] = f(t_k).$$

Through the same process, we can obtain that $\int_0^t f(s) dH^*_{t_k}(s) = f(t_k)$. By the same method we can easily verify that

$$\int_0^{t_k} f(s) dH_{t_k}(s) = 0, \int_0^{t_k} f(s) dH^*_{t_k}(s) = f(t_k).$$

Since for $t_k > t$, $H_{t_k}(s) = 0 = H^*_{t_k}(s)$ for every $s \in [0, t]$ we have

$$\int_0^t f(s) dH_{t_k}(s) = 0 = \int_0^t f(s) dH^*_{t_k}(s).$$

Using the above results and the definition and properties of K-integral we get

$$\int_0^t f(s) d\phi(s) = \int_0^t f(s) \ \mathrm{d}\left[s + \sum_{k=1}^m H_{t_k}(s)\right]$$
$$= \int_0^t f(s) ds + \sum_{k=1}^m \int_0^t f(s) dH_{t_k}(s) = \int_0^t f(s) ds + \sum_{0<t_k<t} f(t_k).$$

Considering

$$\sum_{k=1}^{m}\int_0^t f(s)dH^*_{t_k}(s)=\sum_{0<t_k\le t}\int_0^t f(s)dH^*_{t_k}(s)=\sum_{0<t_k\le t} f(t_k),$$

we get

$$\int_0^t f(s)d\psi(s)=\int_0^t f(s)ds+\sum_{0<t_k\le t} f(t_k).$$

The proof is complete. □

Now we define functions $A, B_i : [0,1] \longrightarrow [0,\infty)$ as follows:

$$A(t)=\begin{cases}0, & t\in\{t_1,...,t_m\}\\ 1, & \text{otherwise,}\end{cases}\quad B_i(t)=\begin{cases}0, & t\ne t_i,\\ 1, & t=t_i,\ i=\overline{1,m}.\end{cases}$$

The following theorem is a Gronwall-Bellman type integral inequality with impulses.

Theorem 3.2 ([6])**.** *Let* $a_k \ge 0,\ k=\overline{1,m}$. *If*

$$u(t)\le c+\int_0^t f(s)u(s)ds+\sum_{0<t_k<t} a_k u(t_k), \tag{3.1}$$

then we have

$$u(t)\le c\cdot\exp\left(\int_0^t f+\sum_{0<t_k<t} a_k\right).$$

Proof. Define a function $z(t)$ by the right side of (3.1); then we observe that $z(0)= c, u(t)\le z(t)$ and for $t\ne t_k, k=\overline{1,m}$, we have by Lemma 3.1

$$z'_\phi(t)=f(t)u(t)\le f(t)z(t),\quad z'_\phi(t_k)=a_k u(t_k)\le a_k z(t_k).$$

So, we have

$$z'_\phi(t)\le A(t)f(t)z(t)+\sum_{i=1}^{m} B_i(t)a_i z(t_i)=\left[A(t)f(t)+\sum_{i=1}^{m} B_i(t)a_i\right]z(t).$$

By Lemma 3.1 this implies

$$\frac{d\log z(t)}{d\phi(t)}\le\frac{z'_\phi(t)}{z(t)}\le\left[A(t)f(t)+\sum_{i=1}^{m} B_i(t)a_i\right]. \tag{3.2}$$

By setting $t = s$ in (3.2) and integrating it with respect to ϕ from 0 to t then by Theorem 2.3 and Lemma 3.1 we get

$$\log\frac{z(t)}{z(0)}\le\int_0^t\left[A(s)f(s)+\sum_{i=1}^{m} B_i(s)a_i\right]d\phi(s)=\int_0^t f+\sum_{0<t_k<t} a_k.$$

Since $z(0) = c$ we get

$$u(t) \leq z(t) \leq c \cdot \exp\left(\int_0^t f + \sum_{0<t_k<t} a_k\right).$$

This completes the proof. $\square$

A generalization of Theorem 3.2 is the following result.

Theorem 3.3. *Let* $0 < m_1 < m_2 < \cdots < m_n$ *and let* $a_k, a_{ik} \geq 0, i = \overline{1,n}, k = \overline{1,m}$. *If*

$$u(t) \leq c + \int_0^t f(s)u(s)ds + \int_0^t \sum_{i=1}^n g_i(s)[u(s)]^{m_i+1}ds + \sum_{0<t_k<t}\left[a_k u(t_k) + \sum_{i=1}^n a_{ik}[u(t_k)]^{m_i+1}\right], \tag{3.3}$$

then

$$u(t) \leq c \cdot \exp\left(\int_0^t f + \sum_{0<t_k<t} a_k\right)[1 - M(t) - N(t)]^{-\frac{1}{m_n}},$$

where

$$M(t) = m_n \int_0^t \sum_{i=1}^n g_i(s)c^{m_i} \cdot \exp\left(m_n \int_0^s f + m_n \sum_{0<t_j<s} a_j\right) ds,$$

$$N(t) = m_n \sum_{0<t_k<t} \sum_{i=1}^n a_{ik}c^{m_i} \cdot \exp\left(m_n \int_0^{t_k} f + m_n \sum_{0<t_j<t_k} a_j\right),$$

provided that $M(t) + N(t) < 1$.

Proof. Inequality (3.3) is written as:

$$u(t) \leq c + \int_0^t \left(f(s) + \sum_{i=1}^n g_i(s)[u(s)]^{m_i}\right)u(s)ds + \sum_{0<t_k<t}\left[a_k + \sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right]u(t_k). \tag{3.4}$$

By applying Theorem 3.2, we get

$$u(t) \leq c \cdot \exp\left(\int_0^t f + \sum_{0<t_k<t} a_k + \int_0^t \sum_{i=1}^n g_i u^{m_i} + \sum_{0<t_k<t} \sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right). \tag{3.5}$$

Then for every $m_j,\ j=\overline{1,n}$, we have

$$[u(t)]^{m_j} \le c^{m_j}\cdot\exp\Bigg(m_j\int_0^t f + m_j\sum_{0<t_k<t} a_k + m_j\int_0^t\sum_{i=1}^n g_i u^{m_i} + m_j\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\Bigg)$$
$$\le c^{m_j}\cdot\exp\Bigg(m_n\int_0^t f + m_n\sum_{0<t_k<t} a_k + m_n\int_0^t\sum_{i=1}^n g_i u^{m_i} + m_n\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\Bigg).$$

Multiplying the last inequality by a negative term $-m_n g_j(t)$, we have

$$-m_n g_j(t)[u(t)]^{m_j}\exp\left(-m_n\int_0^t\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right)$$
$$\ge -m_n g_j(t)c^{m_j}\cdot\exp\left(m_n\int_0^t f + m_n\sum_{0<t_k<t} a_k\right).$$

By summing the inequality for $j=\overline{1,n}$, we obtain

$$-m_n\sum_{i=1}^n g_i(t)[u(t)]^{m_i}\exp\left(-m_n\int_0^t\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right)$$
$$\ge -m_n\sum_{i=1}^n g_i(t)c^{m_i}\cdot\exp\left(m_n\int_0^t f + m_n\sum_{0<t_k<t} a_k\right).$$

This implies that for $t\neq t_k, k=\overline{1,m}$

$$\frac{d}{d\phi(t)}\exp\left(-m_n\int_0^t\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right)$$
$$\ge -m_n\sum_{i=1}^n g_i(t)c^{m_i}\cdot\exp\left(m_n\int_0^t f + m_n\sum_{0<t_k<t} a_k\right). \tag{3.6}$$

And by Lemma 3.1 we have

$$\begin{aligned}
&\frac{d}{d\phi(t_k)}\exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_j<t_k}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right)\\
&= \exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{j=1}^{k}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right)\\
&\quad - \exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{j=1}^{k-1}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right)\\
&= \left[\exp\left(-m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right) - 1\right]\\
&\quad \times \exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{j=1}^{k-1}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right).
\end{aligned}$$

By the Mean Value Theorem we get for some $\omega \in [-m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}, 0]$,

$$\begin{aligned}
&\left[\exp\left(-m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right) - 1\right]\\
&= -m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\exp(\omega) \geq -m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}.
\end{aligned}$$

So we conclude that

$$\begin{aligned}
(3.7)\quad &\frac{d}{d\phi(t_k)}\exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_j<t_k}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right)\\
&\geq -m_n\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\cdot\exp\left(-m_n\int_0^{t_k}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_j<t_k}\sum_{i=1}^n a_{ij}[u(t_j)]^{m_i}\right)\\
&\geq -m_n\sum_{i=1}^n a_{ik}c^{m_i}\cdot\exp\left(m_n\int_0^{t_k} f + m_n\sum_{0<t_j<t_k} a_j\right).
\end{aligned}$$

By (3.6) and (3.7) we obtain

$$\frac{d}{d\phi(t)}\exp\left(-m_n\int_0^{t}\sum_{i=1}^n g_i u^{m_i} - m_n\sum_{0<t_k<t}\sum_{i=1}^n a_{ik}[u(t_k)]^{m_i}\right)$$

$$\geq -m_n A(t)\sum_{i=1}^{n} g_i(t)c^{m_i}\cdot\exp\left(m_n\int_0^t f + m_n\sum_{0<t_k<t} a_k\right)$$

$$-m_n\sum_{p=1}^{m} B_p(t)\sum_{i=1}^{n} a_{ip}c^{m_i}\cdot\exp\left(m_n\int_0^{t_p} f + m_n\sum_{0<t_j<t_p} a_j\right).$$

Integrating from 0 to t with respect to ϕ we get by Theorem 2.3

$$\exp\left(-m_n\int_0^t\sum_{i=1}^{n} g_i u^{m_i} - m_n\sum_{0<t_j<t}\sum_{i=1}^{n} a_{ij}[u(t_j)]^{m_i}\right)$$

$$\geq 1 - m_n\int_0^t\sum_{i=1}^{n} g_i(s)c^{m_i}\cdot\exp\left(m_n\int_0^s f + m_n\sum_{0<t_j<s} a_j\right) ds$$

$$-m_n\sum_{0<t_k<t}\sum_{i=1}^{n} a_{ik}c^{m_i}\cdot\exp\left(m_n\int_0^{t_k} f + m_n\sum_{0<t_j<t_k} a_j\right).$$

This implies that

$$\exp\left(\int_0^t\sum_{i=1}^{n} g_i u^{m_i} + \sum_{0<t_j<t}\sum_{i=1}^{n} a_{ij}u^{m_i}(t_j)\right) \leq [1 - M(t) - N(t)]^{-\frac{1}{m_n}}.$$

So inequality (3.5) becomes

$$u(t) \leq c\cdot\exp\left(\int_0^t f + \sum_{0<t_k<t} a_k\right)[1 - M(t) - N(t)]^{-\frac{1}{m_n}}.$$

The proof is complete. □

From now on a function $\tilde{f} : [0,1] \longrightarrow [0,\infty)$ is defined by

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if} \quad t \neq t_k \\ 1, & \text{if} \quad t = t_k,\ k = \overline{1,m}. \end{cases}$$

Theorem 3.4. *Let* $1 < p$ *and let* $a_k, b_k \geq 0,\ k = \overline{1,m}$. *If*

(3.8)
$$u(t) \leq c + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u^p(\sigma)d\sigma\right) ds + \sum_{0<t_k<t}[a_k u(t_k) + b_k u^p(t_k)],$$

and $1 - M(t) - N(t) > 0$, *where*

$$M(t) = c^{p-1}(p-1)\int_0^t g(s)\cdot\exp\left((p-1)\int_0^s f + (p-1)\sum_{0<t_j<s} a_j\right) ds,$$

$$N(t) = c^{p-1}(p-1)\sum_{0<t_k<t} b_k \cdot \exp\left((p-1)\int_0^{t_k} f + (p-1)\sum_{0<t_j<t_k} a_j\right),$$

then we have

$$u(t) \le c\left[1 + \int_0^t f(s)W(s)ds + \sum_{0<t_k<t}[a_k W(t_k) + c^{p-1}b_k W^p(t_k)]\right],$$

where

$$W(t) = \exp\left(\int_0^t f + \sum_{0<t_k<t} a_k\right)[1 - M(t) - N(t)]^{-\frac{1}{p-1}}.$$

Proof. Define a function $z(t)$ by the right side of (3.8); then we observe that $z(0) = c, u(t) \le z(t)$ and for $t \ne t_k, k = \overline{1,m}$ we have

$$z'_\phi(t) = f(t)\left(z(t) + \int_0^t g(\sigma)z^p(\sigma)d\sigma\right)$$

and

$$z'_\phi(t_k) = a_k u(t_k) + b_k u^p(t_k) \le a_k z(t_k) + b_k z^p(t_k).$$

This implies that

$$\begin{aligned} z'_\phi(t) &\le A(t)f(t)\left(z(t) + \int_0^t g(\sigma)z^p(\sigma)d\sigma\right) + \sum_{i=1}^m B_i(t)[a_i z(t_i) + b_i z^p(t_i)] \\ &= \tilde{f}(t)\left[A(t)\left(z(t) + \int_0^t g(\sigma)z^p(\sigma)d\sigma\right) + \sum_{i=1}^m B_i(t)[a_i z(t_i) + b_i z^p(t_i)]\right]. \end{aligned} \tag{3.9}$$

Define a function $v(t)$ by

$$\begin{aligned} v(t) &= A(t)\left(z(t) + \int_0^t g(\sigma)z^p(\sigma)d\sigma\right) \\ &\quad + \sum_{i=1}^m B_i(t)\left[a_i z(t_i) + b_i z^p(t_i)\right], \end{aligned}$$

and then for $t \ne t_k, k = \overline{1,m}$, by Lemma 3.1 and (3.9)

$$v'_\psi(t) = z'_\psi(t) + g(t)z^p(t) \le f(t)v(t) + g(t)v^p(t),$$

and

$$v'_\psi(t_k) \le a_k z(t_k) + b_k z^p(t_k) \le v(t_k),$$

and

$$v'_\psi(t_k-) = v(t_k) - v(t_k-) \le v(t_k).$$

Thus we have

$$v'_\psi(t) \le A(t)[f(t)v(t) + g(t)v^p(t)] + \sum_{i=1}^{m} B_i(t)v(t_i).$$

This implies that

$$v(t) \le c + \int_0^t [f(s)v(s) + g(s)v^p(s)]ds + \sum_{0<t_k\le t} v(t_k).$$

Since $z(t)$ is left-continuous we have

$$v(t_k) = a_k z(t_k) + b_k z^p(t_k) \le a_k z(t_k-) + b_k z^p(t_k-) \le a_k v(t_k-) + b_k v^p(t_k-).$$

So we have

$$v(t) \le c + \int_0^t [f(s)v(s) + g(s)v^p(s)]ds + \sum_{0<t_k\le t} [a_k v(t_k-) + b_k v^p(t_k-)].$$

Thus

$$v(t-) \le c + \int_0^t [f(s)v(s-) + g(s)v^p(s-)]ds + \sum_{0<t_k<t} [a_k v(t_k-) + b_k v^p(t_k-)].$$

By Theorem 3.3 we have

$$v(t-) \le c \cdot \exp\left(\int_0^t f + \sum_{0<t_k<t} a_k\right)[1 - M(t) - N(t)]^{-\frac{1}{p-1}} = cW(t).$$

And

$$v(t_k) \le a_k v(t_k-) + b_k v^p(t_k-) \le c[a_k W(t_k) + b_k c^{p-1} W^p(t_k)].$$

Thus we get

$$z'_\phi(t) \le \tilde{f}(t)v(t) \le cA(t)f(t)W(t) + c\sum_{i=1}^{m} B_i(t)[a_i W(t_i) + b_i c^{p-1} W^p(t_i)].$$

So we have by Theorem 2.3,

$$u(t) \le z(t) \le c\left[1 + \int_0^t f(s)W(s)ds + \sum_{0<t_k<t} [a_k W(t_k) + c^{p-1} b_k W^p(t_k)]\right].$$

This completes the proof. □

4. An Example

There are many applications of the inequalities obtained in Section 3. Here we shall give an example which is sufficient to show the usefulness of our results.

Consider the following impulsive integro-differential equation

$$x'(t) = F\left(t, x(t), \int_0^t G(\sigma, x(\sigma))d\sigma\right), \ t \neq t_k \tag{4.1}$$

$$x(t_k+) - x(t_k) = I_k(x(t_k)), \quad k = \overline{1, m},$$

$$x(0) = x_0,$$

where $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, where a function $F : [0,1] \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ is continuous on $[0,1] \times \mathbf{R}^2$ and satisfies

$$|F(s, x, y)| \leq f(s)(|x| + |y|)$$

for some continuous function $f : [0,1] \longrightarrow [0,\infty)$, and a function $G : [0,1] \times \mathbf{R} \longrightarrow \mathbf{R}$ is continuous on $[0,1] \times \mathbf{R}$ and satisfies

$$|G(s, x)| \leq g(s)|x|^p$$

for some continuous function $g : [0,1] \longrightarrow [0,\infty)$ and $p > 1$. Then we have

$$x(t) = x_0 + \int_0^t F\left(s, x(s), \int_0^s G(\sigma, x(\sigma))d\sigma\right) ds + \sum_{0<t_k<t} I_k(x(t_k)). \tag{4.2}$$

Assume that $F(t, 0, 0) = 0$ and that the function $I_k : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous and $|I_k(x)| \leq a_k|x| + b_k|x|^p, \ a_k, b_k \geq 0, \ k = \overline{1, m}$.

Let $q = p - 1$ and suppose that $M(t) + N(t) < 1$, where

$$M(t) = qc^q \int_0^t g(s) \cdot \exp\left(q \int_0^s f + q \sum_{0<t_k<s} a_k\right) ds,$$

$$N(t) = qc^q \sum_{0<t_k<t} b_k \cdot \exp\left(q \int_0^{t_k} f + q \sum_{0<t_j<t_k} a_j\right).$$

Then we have

$$|x(t)| \leq |x_0| + \int_0^t f(s)\left(|x(s)| + \int_0^s g(\sigma)|x(\sigma)|^p d\sigma\right) ds + \sum_{0<t_k<t} [a_k|x(t_k)| + b_k|x(t_k)|^p].$$

Applying Theorem 3.4 to the above inequality, we get

$$|x(t)| \leq |x_0| \left[1 + \int_0^t f(s)W(s)ds + \sum_{0<t_k<t} [a_k W(t_k) + |x_0|^{p-1} b_k W^p(t_k)]\right] \equiv |x_0| \cdot K(t).$$

Since the function $K(t)$ is bounded on $[0,1]$, the above inequality implies that the zero solution of equation (4.1) is Lyapunov stable. □

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