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HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

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Abstract. We study half lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature subject to the condition that the 1-form θ and the vector field ζ, defined by (1.1), are identical with the 1-form θ and the vector field ζ of the indefinite trans-Sasakian structure $\{J,\, \theta,\, \zeta\}$ of $\bar{M}$.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. The study of such notion was initiated by Duggal-Bejancu [3] and later studied by many authors (see two books [5, 6]). Half lightlike submanifold M is a lightlike submanifold of codimension 2 such that $rank\{Rad(TM)\} = 1$, where $Rad(TM)$ is the radical distribution of M. It is a special case of an r-lightlike submanifold [3] such that $r = 1$. Its geometry is more general than that of lightlike hypersurfaces or coisotropic submanifolds which are lightlike submanifolds M of codimension 2 such that $rank\{Rad(TM)\} = 2$. Much of its theory will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds. For this reason, we study half lightlike submanifolds.

B.Y. Chen and K. Yano [2] introduced the notion of a *Riemannian manifold of quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with the curvature tensor $\bar{R}$ satisfying the following form:

$$(1.1)\ \bar{R}(X,Y)Z = \ell\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \hbar\{\bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y\},$$

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for any vector fields X, Y and Z of $\bar{M}$, where ℓ and $\hbar$ are smooth functions, ζ is a smooth vector field and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. If $\hbar = 0$, then $\bar{M}$ is a space of constant curvature ℓ.

J.A. Oubina [10] introduced the notion of a trans-Sasakian manifold of type (α, β). We say that a trans-Sasakian manifold $\bar{M}$ of type (α, β) is an *indefinite trans-Sasakian manifold* if $\bar{M}$ is a semi-Riemannian manifold. Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of trans-Sasakian manifold such that $\alpha = 1,\ \beta = 0$, and $\alpha = 0,\ \beta = 1$, and $\alpha = \beta = 0$, respectively.

In this paper, we study half lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature subject to the condition that the 1-form θ and the vector field ζ, defined by (1.1), are identical with the 1-form θ and the vector field ζ of the indefinite trans-Sasakian structure $\{J,\ \zeta,\ \theta\}$ of $\bar{M}$. The paper contains several new results which are related to the induced structure on M.

2. Half Lightlike Submanifold

Let (M, g) be a half lightlike submanifold, with the radical distribution $Rad(TM)$, and screen and coscreen distributions $S(TM)$ and $S(TM^{\perp})$ respectively, of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. We follow Duggal and Jin [4] for notations and structure equations used in this article. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M and by $(*\,.\,*)_i$ the i-th equation of $(*\,.\,*)$. We use the same notations for any others. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.1}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.2}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.4}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.5}$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM. From now and in the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified.

Since the connection $\bar\nabla$ on $\bar M$ is torsion-free, the induced connection ∇ on M is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar g(A^*_\xi X, N) = 0, \tag{2.6}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar g(A_N X, N) = 0, \tag{2.7}$$

$$D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \qquad \bar g(A_L X, N) = \rho(X), \tag{2.8}$$

where η is a 1-form on TM such that $\eta(X) = \bar g(X, N)$ for any $X \in \Gamma(TM)$. From (2.6), (2.7) and (2.8), we see that B and D satisfy

$$B(X,\,\xi) = 0, \qquad D(X,\,\xi) = -\phi(X), \tag{2.9}$$

$$\bar\nabla_X \xi = -A^*_\xi X - \tau(X)\xi - \phi(X)L. \tag{2.10}$$

A^*_ξ and A_N are $S(TM)$-valued, and A^*_ξ is self-adjoint on TM such that $A^*_\xi \xi = 0$.

Denote by $\bar R$, R and R^* the curvature tensors of the connections $\bar\nabla$, ∇ and ∇^* respectively. Using the local Gauss-Weingarten formulas for M and $S(TM)$, we have the Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar R(X,Y)Z = {} & R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ & + D(X,Z)A_L Y - D(Y,Z)A_L X \\ & + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ & + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N \\ & + \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) \\ & \quad - \rho(Y)B(X,Z)\}L,\end{aligned} \tag{2.11}$$

$$\begin{aligned}R(X,Y)PZ = {} & R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A_\xi X \\ & + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\}\xi\end{aligned} \tag{2.12}$$

In the case $R = 0$, we say that M is *flat*.

3. Indefinite Trans-Sasakian Manifolds

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite trans-Sasakian manifold* [10] if there exists a structure set $\{J, \zeta, \theta, \bar{g}\}$, where J is a tensor field of type (1, 1), ζ is a vector field which is called the *structure vector field* of $\bar{M}$ and θ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta,\ \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\ \theta(\zeta) = 1, \tag{3.1}$$

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\}, \tag{3.2}$$

for any vector fields X and Y on $\bar{M}$, where $\epsilon = 1$ or -1 according as the vector field ζ is spacelike or timelike respectively. In this case, the set $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this paper, we may assume that ζ is unit spacelike, *i.e.*, $\epsilon = 1$, without loss generality. From (3.1) and (3.2), we get

$$\bar{\nabla}_X\zeta = -\alpha JX + \beta(X - \theta(X)\zeta), \qquad d\theta(X, Y) = \bar{g}(X,\, JY). \tag{3.3}$$

Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ such that the structure vector field ζ of $\bar{M}$ is tangent to M. Călin [1] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which assume in this paper. It is known [8] that, for any half lightlike submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$, of rank 1. Thus there exists a non-degenerate almost complex distribution H_o with respect to J, *i.e.*, $J(H_o) = H_o$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o.$$

Denote by H the almost complex distribution with respect to J such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o,$$
$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})).$$

Consider two local null vector fields U and V, a local unit spacelike vector field W on $S(TM)$, and their 1-forms u, v and w defined by

$$U = -JN, \qquad V = -J\xi, \qquad W = -JL, \tag{3.4}$$

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W). \tag{3.5}$$

Let S be the projection morphism of TM on H and F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + u(X)N + w(X)L. \tag{3.6}$$

Applying J to (3.6) and using (3.1) and (3.4), we have

$$F^2 X = -\,X \,+\, u(X)U \,+\, w(X)W + \theta(X)\zeta. \tag{3.7}$$

Applying $\bar\nabla_X$ to (3.4) ∼ (3.6) by turns and using (2.1), (2.2), (2.3), (2.6) ∼ (2.8), (2.10) and (3.4) ∼ (3.6), we have

$$B(X,U) = C(X,V),\ B(X,W) = D(X,V),\ C(X,W) = D(X,U), \tag{3.8}$$

$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.9}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \phi(X)W - \beta u(X)\zeta, \tag{3.10}$$

$$\nabla_X W = F(A_L X) + \phi(X)U - \beta w(X)\zeta, \tag{3.11}$$

$$\begin{aligned}(\nabla_X F)(Y) = {} & u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar g(JX,Y)\zeta - \theta(Y)FX\},\end{aligned} \tag{3.12}$$

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\phi(X) - \beta\theta(Y)u(X) - B(X,FY), \tag{3.13}$$

$$\begin{aligned}(\nabla_X v)(Y) = {} & v(Y)\tau(X) + w(Y)\rho(X) - \theta(Y)\{\alpha\eta(X) + \beta v(X)\} \\ & - g(A_N X, FY),\end{aligned} \tag{3.14}$$

$$(\nabla_X w)(Y) = -u(Y)\rho(X) - \beta\theta(Y)w(X) - D(X,FY). \tag{3.15}$$

Substituting (3.6) into (3.3) and using (2.1), we see that

$$\nabla_X \zeta = -\alpha FX + \beta(X - \theta(X)\zeta), \tag{3.16}$$

$$B(X,\,\zeta) = -\alpha u(X), \qquad D(X,\,\zeta) = -\alpha w(X). \tag{3.17}$$

Applying $\bar\nabla_X$ to $\bar g(\zeta, N) = 0$ and using (3.1) and (3.3), we have

$$C(X,\,\zeta) = -\,\alpha v(X) + \beta\eta(X). \tag{3.18}$$

4. Manifold of Quasi-constant Curvature

Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar M$ of quasi-constant curvature. Comparing the tangential, lightlike transversal and co-screen components of the two equations (2.11) and (4.1), we get

$$
\begin{aligned}
(4.1)\qquad R(X,Y)Z \;&= \ell\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\
&\quad + \hbar\{\bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta + \theta(Y)\theta(Z)X \\
&\quad - \theta(X)\theta(Z)Y\} \\
&\quad + B(Y,Z)A_{_N}X - B(X,Z)A_{_N}Y \\
&\quad + D(Y,Z)A_{_L}X - D(X,Z)A_{_L}Y,
\end{aligned}
$$

$$
\begin{aligned}
(4.2)\qquad &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
&\quad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z) = 0,
\end{aligned}
$$

$$
(4.3)\qquad (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z) = 0.
$$

Taking the scalar product with N to (2.12), we have

$$
\begin{aligned}
g(R(X,Y)PZ,\, N) \;&=\; (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\
&\quad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ).
\end{aligned}
$$

Substituting (4.1) into the last equation, we see that

$$
\begin{aligned}
(4.4)\qquad &(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\
&\qquad - \rho(X)D(Y,PZ) + \rho(Y)D(X,PZ) \\
&= \ell\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\
&\qquad + \hbar\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ).
\end{aligned}
$$

Theorem 4.1. *Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. Then α is a constant, and*

$$
\beta = 0, \qquad \ell = \alpha^2, \qquad \hbar = 0.
$$

Proof. Applying ∇_Y to (3.16), we obtain

$$
\begin{aligned}
\nabla_X\nabla_Y\zeta = &-(X\alpha)FY - \alpha(\nabla_X F)Y - \alpha F(\nabla_X Y) \\
&+ (X\beta)Y + \beta\nabla_X Y + \alpha\beta\theta(Y)FX - \beta^2\theta(Y)X \\
&- \{(X\beta)\theta(Y) + \beta X(\theta(Y)) - \beta^2\theta(X)\theta(Y)\}\zeta.
\end{aligned}
$$

Using this equation, $(2.3)_2$, (3.12) and (3.16), we have

$$
\begin{aligned}
(4.5)\qquad R(X,Y)\zeta = &-(X\alpha)FY + (Y\alpha)FX + (X\beta)Y - (Y\beta)X \\
&+ \alpha\{u(X)A_{_N}Y - u(Y)A_{_N}X + w(X)A_{_L}Y - w(Y)A_{_L}X\} \\
&+ (\alpha^2 - \beta^2)\{\theta(Y)X - \theta(X)Y\}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\alpha\beta\{\theta(Y)FX - \theta(X)FY\} \\
&- \{(X\beta)\theta(Y) - (Y\beta)\theta(X) + 2\beta(1-\alpha)d\theta(X,Y)\}\zeta.
\end{aligned}
$$

Replacing Z by ζ to (2.11) and then, taking the scalar product with ζ and using (3.17) and the fact that $\bar{g}(\bar{R}(X,Y)\zeta,\zeta) = 0$, we have

$$g(R(X,Y)\zeta,\zeta) = \alpha\{u(X)g(A_N Y,\zeta) - u(Y)g(A_N X,\zeta)\}.$$

Taking the scalar product with ζ to (4.5) and using (3.17), we have

$$\beta(\alpha-1)\bar{g}(X,JY) = 0.$$

Taking $X = U$ and $Y = \xi$ to this equation, we obtain $\beta(\alpha-1) = 0$.

Applying ∇_X to $(3.8)_1 : B(Y,U) = C(Y,V)$, we have

$$(\nabla_X B)(Y,U) = (\nabla_X C)(Y,V) + g(A_N Y,\nabla_X V) - g(A^*_\xi Y,\nabla_X U).$$

Using (3.8), (3.9), (3.10), (3.17) and (3.18), the last equation is reduced to

$$
\begin{aligned}
&(\nabla_X B)(Y,U) \\
&= (\nabla_X C)(Y,V) - 2\tau(X)C(Y,V) - \phi(X)D(Y,U) - \rho(X)D(Y,V) \\
&\quad -\alpha^2 u(Y)\eta(X) - \beta^2 u(X)\eta(Y) + \alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\
&\quad - g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).
\end{aligned}
$$

Substituting this equation into (4.2) such that $Z = U$ and using (3.8), we get

$$
\begin{aligned}
&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V) - \tau(X)C(Y,V) + \tau(Y)C(X,V) \\
&- \rho(X)D(Y,V) + \rho(Y)D(X,V) + 2\alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\
&+ (\alpha^2 - \beta^2)\{u(X)\eta(Y) - u(Y)\eta(X)\} = 0.
\end{aligned}
$$

Comparing this equation with (4.4) such that $PZ = V$, we obtain

$$(\ell - \alpha^2 + \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\} = 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.$$

Taking $X = \xi$ and $Y = U$, and then, $X = V$ and $Y = U$ to this, we have $\ell = \alpha^2 - \beta^2$ and $\alpha\beta = 0$. From the facts that $\alpha\beta = 0$ and $\beta(\alpha-1) = 0$, we obtain $\beta = 0$, *i.e.*,

$$\ell = \alpha^2 - \beta^2, \qquad \beta = 0.$$

Applying ∇_Y to $(3.17)_1$ and using (3.13) and (3.16) $\sim$ (3.18), we have

$$
\begin{aligned}
(\nabla_X B)(Y,\zeta) &= -(X\alpha)u(Y) \\
&\quad + \alpha\{u(Y)\tau(X) + w(Y)\phi(X) + B(X,FY) + B(Y,FX)\}.
\end{aligned}
$$

Substituting this into (4.2) such that $Z = \zeta$, we have

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Replacing Y by U to this equation, we obtain

$$Xα = (Uα)u(X). \tag{4.6}$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y, N)$ and using (2.1) and (2.2) we have

$$(\nabla_X \eta)(Y) = -g(A_N X,\, Y) + \tau(X)\eta(Y).$$

Applying ∇_Y to (3.18) and using (3.14), (3.16) and (3.18), we have

$$\begin{aligned}(\nabla_X C)(Y, \zeta) = &-(X\alpha)v(Y) - \alpha\{\tau(X)v(Y) + \rho(X)w(Y)\} \\ &+ \alpha\{\alpha\theta(Y)\eta(X) + g(A_N X, FY) + g(A_N Y, FX)\}.\end{aligned}$$

Substituting this into (4.4) such that $PZ = \zeta$ and using (4.5), we get

$$\hbar\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\} = (X\alpha)v(Y) - (Y\alpha)v(X).$$

Taking $X = \xi$ and $Y = \zeta$, and then, $X = U$ and $Y = V$ to this, we obtain

$$\hbar = 0, \qquad U\alpha = 0.$$

As $U\alpha = 0$, from (4.6), we see that α is a constant. □

Corollary 1. *Let $\bar{M}$ be an indefinite trans-Sasakian manifold, of type (α, β), of quasi-constant curvature with a half lightlike submanifold. Then $\bar{M}$ is an indefinite α-Sasakian manifold of constant positive curvature α^2.*

Theorem 4.2. *Let M be a half lightlike submanifolds of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If one of the followings;*

(1) *F is parallel with respect to the connection ∇,*
(2) *U is parallel with respect to the connection ∇,*
(3) *V is parallel with respect to the connection ∇, and*
(4) *W is parallel with respect to the connection ∇*

is satisfied, then $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure. In case (1), M is also flat.

Proof. Denote λ, μ, σ and δ by the 1-forms such that

$$\lambda(X) = B(X, U) = C(X, V), \qquad \sigma(X) = D(X, W),$$
$$\mu(X) = B(X, W) = D(X, V), \qquad \delta(X) = B(X, V).$$

(1) If F is parallel, then, as $\beta = 0$, from (3.12) we have

$$\begin{aligned}&u(Y)A_N X + w(Y)A_L X - B(X, Y)U - D(X, Y)W \\ &+ \alpha\{g(X, Y)\zeta - \theta(Y)X\} = 0.\end{aligned} \tag{4.7}$$

Replacing Y by ξ and using (2.9) and (3.5), we obtain $\phi(X)W = 0$. From this result, we see that $\phi = 0$. Taking the scalar product with U to (4.7), we get

$$u(Y)v(A_{N}X) + w(Y)v(A_{L}X) - \alpha\theta(Y)v(X) = 0.$$

Taking $Y = W$ and $Y = \zeta$ to this equation by turns, we get

$$C(X,W) = D(X,U) = 0, \qquad \alpha v(X) = 0. \tag{4.8}$$

From $(4.8)_2$, we get $\alpha = 0$. By Theorem 4.1, $\ell = 0$ and $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure. Taking $Y = U$ to (4.7), we have

$$A_{N}X = \lambda(X)U, \tag{4.9}$$

due to $(4.8)_1$. Taking the scalar product with N, V and W to (4.7) by turns and using (2.7), (2.8), (3.8) and $(4.8)_1$, we have

$$\begin{aligned} &w(Y)\rho(X) = 0. \\ &g(A^*_{\xi}X,\, Y) = g(\lambda(X)V + \mu(X)W,\, Y), \\ &g(A_{L}X,\, Y) = g(\sigma(X)W,\, Y). \end{aligned}$$

Taking $Y = W$ to the first equation, we obtain $\rho = 0$. As $\rho = 0$, from (2.8) we see that $A_{L}X$ belongs to $S(TM)$. As $A^*_{\xi}X$ and $A_{L}X$ belong to $S(TM)$ and $S(TM)$ is non-degenerate, from the last two equations, we have

$$A^*_{\xi}X = \lambda(X)V + \mu(X)W, \qquad A_{L}X = \sigma(X)W.$$

Taking the scalar product with V to the second equation, we see that

$$\begin{aligned} &\mu(X) = B(X,W) = D(X,V) = 0, \\ &A^*_{\xi}X = \lambda(X)V, \qquad A_{L}X = \sigma(X)W. \end{aligned} \tag{4.10}$$

As $\ell = \hbar = 0$, substituting (4.9) and (4.10) into (4.1), we get

$$\begin{aligned} R(X,Y)Z \;=\; &\{\lambda(Y)\lambda(X) - \lambda(X)\lambda(Y)\}u(Z)U \\ &+ \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}w(Z)W = 0. \end{aligned}$$

Thus M is also flat.

(2) If U is parallel with respect to ∇, then, from (3.6) and (3.9), we have

$$J(A_{N}X) - u(A_{N}X)N - w(A_{N}X)L + \tau(X)U + \rho(X)W - \alpha\eta(X)\zeta = 0.$$

Taking the scalar product with ζ, V and W by turns, we get

$$\alpha\eta(X) = 0, \qquad \tau = 0, \qquad \rho = 0,$$

respectively. Taking $X=\xi$ to the first result, we have $\alpha=0$. As $\alpha=0$, we see that $\ell=0$ and $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure.

(3) If V is parallel with respect to ∇, then, from (3.6) and (3.10), we have

$$J(A^*_\xi X)-u(A^*_\xi X)N-w(A^*_\xi X)L-\tau(X)V-\phi(X)W=0.$$

Taking the scalar product with U and W by turns, we get $\tau=0$ and $\phi=0$, respectively. Applying J to the last equation and using (3.1) and $(3.17)_1$, we have

$$A^*_\xi X=-\alpha u(X)\zeta+\delta(X)U+\mu(X)W.$$

Taking the scalar product with U to this equation, we get

$$B(X,U)=g(A^*_\xi X,U)=0.$$

Replacing X by ζ to this equation and using $(3.17)_1$, we get

$$\alpha=\alpha u(U)=-B(U,\zeta)=0.$$

Thus $\ell=0$ and $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure.

(4) If W is parallel with respect to ∇, then, from (3.6) and (3.11), we get

$$J(A_L X)-u(A_L X)N-w(A_L X)L+\phi(X)U=0.$$

Taking the scalar product with V and U by turns, we have

$$\phi=0, \qquad \rho=0,$$

respectively. Applying J to the last equation and using (3.1), $(3.17)_2$, we have

$$A_L X=-\alpha w(X)\zeta+\mu(X)U+\sigma(X)W.$$

Taking the scalar product with U to this, we have $D(X,U)=0$ and

$$C(X,W)=0.$$

Applying ∇_X to $C(Y,W)=0$ and using (3.10) and $\phi=\beta=0$, we have

$$(\nabla_X C)(Y,W)=-g(A_N Y,F(A_L X)).$$

Taking $PZ=W$ to (4.4) and using the last two equations, we obtain

$$g(A_N X,F(A_L Y))-g(A_N Y,F(A_L X))=\ell\{w(Y)\eta(X)-w(X)\eta(Y)\}$$

as $\rho=0$. Taking $X=\xi$ and $Y=W$ to this and using the facts that $F(A_L W)=0$ and $A_L\xi=0$, we obtain $\ell=0$. As $\ell=0$, we see that $\alpha=0$ and $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure. $\square$

5. Recurrent Half Lightlike Submanifolds

Definition. The structure tensor field F on M is said to be *recurrent* [9] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Theorem 5.1. *Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If F is recurrent, then it is parallel, $\bar{M}$ and M are flat, and the transversal connection of M is flat.*

Proof. As F is recurrent, from (3.12) and the fact that $\beta = 0$, we get

$$\begin{aligned}\varpi(X)FY \;=\;& u(Y)A_{_N}X + w(Y)A_{_L}X - B(X,Y)U - D(X,Y)W \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\}.\end{aligned}$$

Replacing Y by ξ to this and using (2.10), (3.1), (3.4), (3.5) and the fact that $F\xi = -V$, we get $-\varpi(X)V = \phi(X)W$. Taking the scalar product with U, we get $\varpi = 0$. Thus F is parallel with respect to ∇. From Theorem 4.3, we see that $\bar{M}$ and M are flat, and the transversal connection of M is flat. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* [9] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{5.1}$$

where $\mathcal{L}_X F$ denotes the Lie derivative on M of F with respect to X, that is,

$$\begin{aligned}(\mathcal{L}_X F)Y \;&=\; [X, FY] - F[X,Y] \\ &=\; (\nabla_X F)Y - \nabla_{FY}X + F\nabla_Y X.\end{aligned} \tag{5.2}$$

The structure tensor field F is called *Lie parallel* if $\vartheta = 0$.

Theorem 5.2. *Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ of quasi-constant curvature. If F is Lie recurrent, then it is Lie parallel, and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure.*

Proof. As F is Lie recurrent, from (3.12), (5.1) and (5.2) we get

$$\begin{aligned}\vartheta(X)FY \;=\;& u(Y)A_{_N}X + w(Y)A_{_L}X - B(X,Y)U - D(X,Y)W \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} - \nabla_{FY}X + F\nabla_Y X.\end{aligned} \tag{5.3}$$

Replacing Y by ξ to (5.3) and using (2.6), (3.4) and $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \phi(X)W. \tag{5.4}$$

Taking the scalar product with V, W and ζ to this by turns, we get

$$(5.5)\qquad u(\nabla_V X) = 0, \qquad w(\nabla_V X) = -\phi(X), \qquad \theta(\nabla_V X) = 0.$$

On the other hand, taking $Y = V$ to (5.3) and using (3.4), we have

$$\vartheta(X)\xi = -B(X,V)U - D(X,V)W - \nabla_\xi X + F\nabla_V X + \alpha u(X)\zeta.$$

Applying F and using (3.7), (5.5) and $FU = FW = F\zeta = 0$, we get

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \phi(X)W.$$

Comparing this with (5.4), we get $\vartheta = 0$. Therefore F is Lie parallel.

Taking $X = U$ to (5.3) and using (3.7), (3.8), (3.9) and (3.18), we get

$$\begin{aligned} u(Y)A_N U + w(Y)A_L U - F(A_N FY) - \tau(FY)U - \rho(FY)W \\ - A_N Y + \alpha\{v(Y)\zeta - \theta(Y)U\} = 0. \end{aligned}$$

Taking the scalar product with ζ to this equation and using (3.18), we get $\alpha v(Y) = 0$. Taking $Y = V$ to this result, we have $\alpha = 0$. Therefore, $\ell = 0$ and $\bar{M}$ is a flat manifold with indefinite cosymplectic structure. □

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