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COMPLETE MONOTONICITY OF A DIFFERENCE BETWEEN THE EXPONENTIAL AND TRIGAMMA FUNCTIONS

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Abstract. In the paper, by directly verifying an inequality which gives a lower bound for the first order modified Bessel function of the first kind, the authors supply a new proof for the complete monotonicity of a difference between the exponential function $e^{1/t}$ and the trigamma function $\psi'(t)$ on $(0,\infty)$.

1. Introduction

In [3, Lemma 2], the inequality

$$\psi'(t) < e^{1/t} - 1 \tag{1.1}$$

on $(0,\infty)$ was discovered and employed, where $\psi(t)$ denotes the digamma function

$$\psi(t) = [\ln\Gamma(t)]' = \frac{\Gamma'(t)}{\Gamma(t)}$$

and Γ is the classical Euler gamma function which may be defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t.$$

The functions $\psi'(z)$ and $\psi''(z)$ are respectively called the trigamma function and the tetragamma function. As a whole, the derivatives $\psi^{(k)}(z)$ for $k \in \{0\}\cup\mathbb{N}$ are called polygamma functions.

An infinitely differentiable function f defined on an interval I is said to be a completely monotonic function on I if it satisfies

$$(-1)^k f^{(k)}(x) \ge 0 \tag{1.2}$$

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for all $k \in \{0\} \cup \mathbb{N}$ on I. Some properties of the completely monotonic functions can be found in, for example, [2, 8].

In [5, Theorem 3.1] and [6, Theorem 1.1], the following theorem was proved by three methods totally.

Theorem 1.1. *The function*

$$h(t) = e^{1/t} - \psi'(t) \tag{1.3}$$

is completely monotonic on $(0, \infty)$ *and*

$$\lim_{t\to\infty} h(t) = 1. \tag{1.4}$$

The second main result of the paper [6] is [6, Theorem 1.2] which has been referenced in [4, Section 1.2] and [5, Lemma 2.1] as follows.

Theorem 1.2. *For* $k \in \{0\} \cup \mathbb{N}$ *and* $z \neq 0$, *let*

$$H_k(z) = e^{1/z} - \sum_{m=0}^{k} \frac{1}{m!} \frac{1}{z^m}. \tag{1.5}$$

For $\Re(z) > 0$, *the function* $H_k(z)$ *has the integral representations*

$$H_k(z) = \frac{1}{k!(k+1)!} \int_0^\infty {}_1F_2(1; k+1, k+2; t) t^k e^{-zt} \,\mathrm{d}\,t \tag{1.6}$$

and

$$H_k(z) = \frac{1}{z^{k+1}} \left[\frac{1}{(k+1)!} + \int_0^\infty \frac{I_{k+2}(2\sqrt{t})}{t^{(k+2)/2}} e^{-zt} \,\mathrm{d}\,t \right], \tag{1.7}$$

where the hypergeometric series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} \tag{1.8}$$

for $b_i \notin \{0, -1, -2, \dots\}$, *the shifted factorial* $(a)_0 = 1$ *and*

$$(a)_n = a(a+1) \cdots (a+n-1) \tag{1.9}$$

for $n > 0$ *and any real or complex number* a, *and the modified Bessel function of the first kind*

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k+\nu} \tag{1.10}$$

for $\nu \in \mathbb{R}$ *and* $z \in \mathbb{C}$.

When $k = 0$, the integral representations (1.6) and (1.7) may be written as

$$e^{1/z} = 1 + \int_0^\infty \frac{I_1(2\sqrt{t})}{\sqrt{t}} e^{-zt} \,\mathrm{d}\,t \tag{1.11}$$

and

$$e^{1/z} = 1 + \frac{1}{z}\left[1 + \int_0^\infty \frac{I_2(2\sqrt{t})}{t} e^{-zt} \,\mathrm{d}\, t\right] \tag{1.12}$$

for $\Re(z) > 0$. Hence, by the well known formula

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1 - e^{-u}} e^{-zu} \,\mathrm{d}\, u \tag{1.13}$$

for $\Re(z) > 0$ and $n \in \mathbb{N}$, see [1, p. 260, 6.4.1], the function $h(t)$ defined by (1.3) has the following integral representation

$$h(t) = 1 + \int_0^\infty \left[\frac{I_1(2\sqrt{u})}{\sqrt{u}} - \frac{u}{1 - e^{-u}}\right] e^{-tu} \,\mathrm{d}\, u. \tag{1.14}$$

Proposition 1.3 (Hausdorff-Bernstein-Widder Theorem [8, p. 161, Theorem 12b]). *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that*

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\, \alpha(t), \tag{1.15}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

Combining the complete monotonicity in Theorem 1.1 and the integral representation (1.14) with the necessary and sufficient condition in Proposition 1.3, it was revealed in [6] that

$$\frac{I_1(2\sqrt{u})}{\sqrt{u}} \ge \frac{u}{1 - e^{-u}}, \quad u > 0. \tag{1.16}$$

Replacing $2\sqrt{u}$ by t in (1.16) yields [6, Theorem 1.3] below.

Theorem 1.4. *For $t > 0$, we have*

$$I_1(t) > \frac{(t/2)^3}{1 - e^{-(t/2)^2}}. \tag{1.17}$$

We note that the complete monotonicity in Theorem 1.1 is the basis of the inequality (1.17) and some results in the subsequent papers [4, 5].

The aim of this paper is, with the help of the integral representation (1.14) but without using Proposition 1.3, to supply a new proof of Theorems 1.1 and 1.4 in a converse direction with that in [4, 5, 6]. In other words, Theorem 1.4 will be firstly and straightforwardly proved, and then Theorem 1.1 will be done.

2. A New Proof of Theorems 1.1 and 1.4

By the definition of the modified Bessel function $I_\nu(z)$ in (1.10), it is easy to see that

$$\frac{I_1(2\sqrt{u})}{\sqrt{u}} = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+2)} u^k > 1 + \frac{1}{2}u + \frac{1}{12}u^2.$$

Hence, in order to prove (1.16), it suffices to show

$$1 + \frac{1}{2}u + \frac{1}{12}u^2 \ge \frac{u}{1-e^{-u}} \tag{2.1}$$

which is equivalent to

$$\begin{aligned}
&e^u\bigl(12 - 6u + u^2\bigr) - 12 - 6u - u^2 \\
&> \left(1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!}\right)\bigl[3 + (u-3)^2\bigr] - 12 - 6u - u^2 \\
&= \frac{1}{120}u^5\left[\frac{3}{4} + \left(\frac{1}{2} - u\right)^2\right] \\
&\ge 0.
\end{aligned}$$

Consequently, the proof of the inequality (1.16), that is, Theorem 1.4, is thus complete.

Substituting the inequality (1.16) into the integral representation (1.14) leads to $h(t) > 0$ and for $k \in \mathbb{N}$

$$(-1)^k h^{(k)}(t) = \int_0^\infty \left[\frac{I_1(2\sqrt{u})}{\sqrt{u}} - \frac{u}{1-e^{-u}}\right] u^k e^{-tu} \,\mathrm{d}\, u > 0$$

on $(0,\infty)$. As a result, the function $h(t)$ is completely monotonic on $(0,\infty)$.

The limit (1.4) follows immediately from taking $t \to \infty$ on both sides of the integral representation (1.14). Theorem 1.1 is thus proved.

Remark 2.1. The inequality (2.1) is equivalent to

$$Q(u) = e^u\bigl(12 - 6u + u^2\bigr) - 12 - 6u - u^2 > 0.$$

An immediate differentiation yields

$$\begin{aligned}
Q'(u) &= e^u\bigl(u^2 - 4u + 6\bigr) - 2(u+3), \\
Q''(u) &= e^u\bigl(u^2 - 2u + 2\bigr) - 2, \\
Q'''(u) &= u^2 e^u.
\end{aligned}$$

Since $Q'''(u)$ and $Q''(0) = 0$, it follows that $Q''(u) > 0$ on $(0,\infty)$. Owing to $Q'(0) = 0$ and $Q''(u) > 0$, it is derived that $Q'(u) > 0$. Finally, since $Q(0) = 0$, the function $Q(u)$ is positive on $(0,\infty)$. This gives an alternative proof of the inequality (2.1).

Remark 2.2. *This is a slightly modified version of the preprint* [7].

References

1. M. Abramowitz & I.A. Stegun (Eds): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.
2. B.-N. Guo & F. Qi: A completely monotonic function involving the tri-gamma function and with degree one. *Appl. Math. Comput.* **218** (2012), no. 19, 9890–9897; Available online at http://dx.doi.org/10.1016/j.amc.2012.03.075.
3. ______: Refinements of lower bounds for polygamma functions. *Proc. Amer. Math. Soc.* **141** (2013), no. 3, 1007–1015; Available online at http://dx.doi.org/10.1090/S0002-9939-2012-11387-5.
4. F. Qi: Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions. *arXiv preprint*, available online at http://arxiv.org/abs/1302.6731.
5. F. Qi & C. Berg: Complete monotonicity of a difference between the exponential and trigamma functions and properties related to a modified Bessel function. *Mediterr. J. Math.* **10** (2013), no. 4, 1685–1696; Available online at http://dx.doi.org/10.1007/s00009-013-0272-2.
6. F. Qi & S.-H. Wang: Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions. *arXiv preprint*, available online at http://arxiv.org/abs/1210.2012.
7. F. Qi & X.-J. Zhang: Complete monotonicity of a difference between the exponential and trigamma functions. *arXiv preprint*, available online at http://arxiv.org/abs/1303.1582.
8. D.V. Widder: *The Laplace Transform.* Princeton University Press, Princeton, 1946.

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