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CHARACTERIZATIONS OF BOOLEAN RANK PRESERVERS OVER BOOLEAN MATRICES

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Abstract. The Boolean rank of a nonzero $m \times n$ Boolean matrix A is the least integer k such that there are an $m\times k$ Boolean matrix B and a $k\times n$ Boolean matrix C with $A = BC$. In 1984, Beasley and Pullman showed that a linear operator preserves the Boolean rank of any Boolean matrix if and only if it preserves Boolean ranks 1 and 2. In this paper, we extend this characterization of linear operators that preserve the Boolean ranks of Boolean matrices. We show that a linear operator preserves all Boolean ranks if and only if it preserves two consecutive Boolean ranks if and only if it strongly preserves a Boolean rank k with $1 \le k \le \min\{m, n\}$.

1. Introduction

The *binary Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathbb{M}_{m,n}$ denote the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper, we shall adopt the convention that $3 \le m \le n$ unless otherwise specified.

The *(Boolean) rank*, $b(A)$, of nonzero $A \in \mathbb{M}_{m,n}$ is the least integer k such that there are Boolean matrices $B \in \mathbb{M}_{m,k}$ and $C \in \mathbb{M}_{k,n}$ with $A = BC$. It follows that $1 \le b(A) \le m$ for all nonzero $A \in \mathbb{M}_{m,n}$. The Boolean rank of the zero Boolean matrix is 0.

A mapping $T : \mathbb{M}_{m,n} \to \mathbb{M}_{m,n}$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathbb{M}_{m,n}$ and for all $\alpha, \beta \in \mathbb{B}$.

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A linear operator T on $\mathbb{M}_{m,n}$ is called a (P,Q)-*operator* if there are permutation matrices P and Q of orders m and n, respectively, such that $T(X) = PXQ$ for all X, or $m = n$ and $T(X) = PX^tQ$ for all X, where X^t is the transpose of X.

Let $1 \leq k \leq m$. For a linear operator T on $\mathbb{M}_{m,n}$, we say that

(1) T *preserves Boolean rank* k if $b(T(X)) = k$ whenever $b(X) = k$ for all X;

(2) T *strongly preserves Boolean rank* k if, $b(T(X)) = k$ if and only if $b(X) = k$ for all X;

(3) T *preserves Boolean rank* if it preserves Boolean rank k for all $k \in \{1, 2, \ldots, m\}$.

Beasley and Pullman ([1]) have characterized linear operators on $\mathbb{M}_{m,n}$ that preserve Boolean rank as follows:

Theorem 1.1. *For a linear operator T on $\mathbb{M}_{m,n}$, the following are equivalent*:

(i) *T preserves Boolean rank*;

(ii) *T preserves Boolean ranks* 1 *and* 2;

(iii) *T is a (P,Q)-operator.*

The characterization of linear operators on vector space of matrices which leave functions, sets or relations invariant began over a century ago when in 1897 Fröbenius [7] characterized the linear operators that leave the determinant function invariant. Since then, several researchers have investigated the preservers of nearly every function, set and relation on matrices over fields. See [6, 7] for an excellent survey of Linear Preserver Problems through 2001. For Boolean matrix and Boolean rank are important research topics on matrix theory. See [4, 5] for detailed contents and applications on Boolean matrix theory.

Recently Beasley and Song ([3]) have obtained a new characterization of Theorem 1.1: For a linear operator T on $\mathbb{M}_{m,n}$, T preserves Boolean rank if and only if T preserves Boolean ranks 1 and k, where $1 < k \leq m$. They also have obtained characterizations of the linear transformations that preserve term rank between different matrix spaces over semirings containing the binary Boolean algebra in [2].

In this paper, we extend Theorem 1.1 to any two consecutive Boolean rank preservers. Furthermore we obtain other characterizations.

2. PRELIMINARIES

The matrix O is an arbitrary zero matrix and $J_{m,n}$ is the $m \times n$ matrix all of whose entries are 1. A matrix in $\mathbb{M}_{m,n}$ is called a *cell* if it has exactly one 1 entry. We denote the cell whose one 1 entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Further we let $\mathcal{E}_{m,n}$ be the set of all cells in $\mathbb{M}_{m,n}$. That is, $\mathcal{E}_{m,n} = \{E_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n\}$.

If A and B are Boolean matrices in $\mathbb{M}_{m,n}$, we say that A *dominates* B (written $B \sqsubseteq A$ or $A \sqsupseteq B$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}$. For Boolean matrices A and B in $\mathbb{M}_{m,n}$ with $B \sqsubseteq A$, we define $A \setminus B$ to be the Boolean matrix C such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$ for all i and j.

Lemma 2.1 ([1]). *If T is a linear operator on $\mathbb{M}_{m,n}$, then T is invertible if and only if T permutes $\mathcal{E}_{m,n}$.*

A Boolean matrix $L \in \mathbb{M}_{m,n}$ is called a *line matrix* if $L = \sum_{l=1}^{n} E_{i,l}$ or $L = \sum_{s=1}^{m} E_{s,j}$ for some $i \in \{1, \dots, m\}$ or for some $j \in \{1, \dots, n\}$: $R_i = \sum_{l=1}^{n} E_{i,l}$ is the ith *row matrix* and $C_j = \sum_{s=1}^{m} E_{s,j}$ is the jth *column matrix.*

For a linear operator T on $\mathbb{M}_{m,n}$, we say that T *preserves line matrices* if $T(L)$ is a line matrix for every line matrix L.

Lemma 2.2. *Let T be an invertible linear operator on $\mathbb{M}_{m,n}$. Then T preserves line matrices if and only if T is a (P, Q)-operator.*

Proof. By Lemma 2.1, T permutes $\mathcal{E}_{m,n}$ and hence $T(J_{m,n}) = J_{m,n}$. Let T preserve all line matrices. Now we will claim that either

(1) T maps $\{R_1, \dots, R_m\}$ onto $\{R_1, \dots, R_m\}$ and
maps $\{C_1, \dots, C_n\}$ onto $\{C_1, \dots, C_n\}$, or
(2) T maps $\{R_1, \dots, R_n\}$ onto $\{C_1, \dots, C_n\}$ and
maps $\{C_1, \dots, C_n\}$ onto $\{R_1, \dots, R_n\}$.

If $m \ne n$, (1) is satisfied since T is invertible and preserves all line matrices.

Thus we assume that $m = n$. Suppose that the claim is not true. Then there are distinct row matrices R_i and R_j (or column matrices C_i and C_j) such that $T(R_i)$ is a row matrix and $T(R_j)$ is a column matrix. But then $T(J_{m,n}) = T(R_1) + \cdots T(R_i) + \cdots + T(R_j) + \cdots + T(R_n)$ cannot dominate $J_{m,n}$. This contradicts $T(J_{m,n}) = J_{m,n}$. Hence the claim is true.

Case (1): We note that $T(R_i) = R_{\alpha(i)}$ for all i and $T(C_j) = C_{\beta(j)}$ for all j, where α and β are permutations of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Then for any cell $E_{i,j}$, we have $T(E_{i,j}) = E_{\alpha(i),\beta(j)}$. Let P and Q be the permutation matrices corresponding to α and β, respectively. Then for any Boolean matrix

$X = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} E_{i,j} \in \mathbb{M}_{m,n}$, we have

$$T(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} E_{\alpha(i),\beta(j)} = PXQ.$$

Hence T is a (P, Q)-operator.

Case (2): We note that $m = n$, $T(R_i) = C_{\alpha(i)}$ for all i and $T(C_j) = R_{\beta(j)}$ for all j, where α and β are permutations of $\{1, \ldots, n\}$. By a parallel argument similar to Case (1), we obtain that $T(X)$ is of the form $T(X) = PX^tQ$, and thus T is a (P, Q)-operator. The converse is obvious. □

For nonzero $A \in \mathbb{M}_{m,n}$, it is well known ([1]) that $b(A)$ is the least integer k such that A is the sum of k Boolean matrices of Boolean rank 1. This establishes the following:

Lemma 2.3. *For Boolean matrices A and B in $\mathbb{M}_{m,n}$, we have*

$$b(A + B) \leq b(A) + b(B).$$

Theorem 2.4. *Let T be an invertible linear operator on $\mathbb{M}_{m,n}$ and $1 \leq k \leq m$. Then T preserves Boolean rank k if and only if T is a (P, Q)-operator.*

Proof. By Lemma 2.1, T permutes $\mathcal{E}_{m,n}$. Assume that T preserves Boolean rank k. Now, we will show that T preserves line matrices, and then T is a (P, Q)-operator by Lemma 2.2. For the case of $k = 1$, it is clear that T preserves line matrices since the Boolean rank of every line matrix is 1. Thus we assume that $k \geq 2$. Suppose that T does not preserve a line matrix. Then there are two distinct cells $E_{i,j}$ and $E_{s,t}$ that are not dominated by the same line matrix such that $T(E_{i,j})$ and $T(E_{s,t})$ are dominated by the same line matrix. Without loss of generality, we assume that $T(E_{1,1} + E_{2,2}) = E_{1,1} + E_{1,2}$. So, we have a contradiction for the case of $k = 2$. Hence we assume that $k \geq 3$. Then for the Boolean matrix $A = E_{3,3} + \cdots + E_{k,k}$, we have $b(E_{1,1} + E_{2,2} + A) = k$. But by Lemma 2.3,

$$b(T(E_{1,1} + E_{2,2} + A)) \leq b(T(E_{1,1} + E_{2,2})) + b(T(A)) \leq 1 + (k - 2) = k - 1,$$

a contradiction to the fact that T preserves Boolean rank k. Hence T preserves line matrices. The converse is obvious. □

3. CHARACTERIZATIONS OF BOOLEAN RANK PRESERVERS

An operator T on $\mathbb{M}_{m,n}$ is *singular* if $T(X) = O$ for some nonzero $X \in \mathbb{M}_{m,n}$;

otherwise T is *nonsingular*. In fact, if T is a singular linear operator on $\mathbb{M}_{m,n}$, then we can easily check that $T(E) = O$ for some cell E. Further, if T is a (P, Q)-operator on $\mathbb{M}_{m,n}$, then T is nonsingular.

Example 3.1. For $1 \leq k \leq m$, let $A = E_{1,1} + E_{2,2} + \cdots + E_{k,k} \in \mathbb{M}_{m,n}$. Define an operator T on $\mathbb{M}_{m,n}$ by $T(O) = O$ and $T(X) = A$ for all nonzero $X \in \mathbb{M}_{m,n}$. Clearly, T is linear, nonsingular and preserves Boolean rank k, while T does not preserve Boolean rank.

The number of nonzero entries of a Boolean matrix $A \in \mathbb{M}_{m,n}$ is denoted by $\sharp(A)$.

Lemma 3.2. *Let $1 \leq k < m$ and $1 \leq l \leq m$. Assume that T is a linear operator on $\mathbb{M}_{m,n}$. If*

(i) *T preserves Boolean rank k and $k+1$, or*

(ii) *T strongly preserves Boolean rank l,*

then T is nonsingular.

Proof. If T is singular, then $T(E) = O$ for some cell E. Hence we have a contradiction for the case of $k = l = 1$. Thus we assume that $k, l \geq 2$. Now, choose Boolean matrices A and B with $E \sqsubseteq A$ and $E \sqsubseteq B$ such that $b(A) = \sharp(A) = k+1$ and $b(B) = \sharp(B) = l$. It follows that $b(A \setminus E) = k$ and $b(B \setminus E) = l - 1$. But then $T(A) = T(A \setminus E) + T(E) = T(A \setminus E)$ contradicts the condition (i) and $T(B) = T(B \setminus E) + T(E) = T(B \setminus E)$ contradicts the condition (ii). Hence T is nonsingular. $\square$

Lemma 3.3. *Let T be a linear operator on $\mathbb{M}_{m,n}$. If*

(i) *T preserves Boolean rank k and $k+1$ with $1 \leq k \leq m-1$, or*

(ii) *T strongly preserves Boolean rank k with $1 \leq k \leq m$,*

then T maps cells to cells.

Proof. If T preserves Boolean rank k and $k+1$ with $1 \leq k \leq m-1$, or T strongly preserves Boolean rank k with $1 \leq k \leq m$, then T is nonsingular by Lemma 3.2. Suppose that T does not map cells to cells, in particular suppose that $T(E)$ dominates two cells for some cell E. By permuting we may assume that $T(E) \sqsupseteq E + F$ for some cell $F \neq E$.

If E and F are in the same row, we may assume by permuting that $E = E_{1,k+1}$ and $F = E_{1,k}$. If E and F are in the same column, we may assume by permuting that $E = E_{k+1,1}$ and $F = E_{k,1}$. If E and F are in different rows and different columns, we may assume by permuting that $E = E_{1,k+1}$ and $F = E_{2,k-1}$. For $1 \le r \le m$, let $W_r = [w_{i,j}^{(r)}]$ where $w_{i,j}^{(r)} = 0$ if and only if $i + j \le r$. Then $b(W_r) = r$. Since $E \sqsubseteq W_{k+1}$ and $F \not\sqsubseteq W_{k+1}$, we have that $b(W_{k+1} + E) = k+1$ and $b(W_{k+1} + F) = k$.

Let $L = T^d$ where d is chosen so that L is idempotent ($L^2 = L$). Then, L preserves Boolean ranks k and $k+1$ for case (i), or L strongly preserves Boolean rank k for case (ii) and $L(E) \sqsupseteq E + F$.

Now, since $L(E) = F + X$ for some Boolean matrix X,

$$L(E) + F = (X + F) + F = X + F = L(E)$$

and since L is idempotent,

$$\begin{aligned} L(E) = L^2(E) &= L(L(E)) = L(L(E) + F) \\ &= L^2(E) + L(F) = L(E) + L(F) = L(E + F). \end{aligned}$$

That is, $L(E+F) = L(E)$. Thus if Y is any Boolean matrix which dominates E, we have that $L(Y + F) = L(Y)$ since $L(Y + F) = L(Y + E + F) = L(Y) + L(E + F) = L(Y) + L(E) = L(Y + E) = L(Y)$. Thus,

$$L(W_{k+1}) = L(W_{k+1} + F).$$

However, $b(W_{k+1}) = k + 1$, $b(W_{k+1} + F) = k$ and L preserves both Boolean rank k and $k+1$ for case (i) or L strongly preserves Boolean rank k for case (ii). Thus, we have

$$k + 1 = b(L(W_{k+1})) = b(L(W_{k+1} + F)) = k,$$

which is a contradiction for the both cases (i) and (ii). Therefore T maps cells to cells. □

Theorem 3.4. *Let T be a linear operator on $\mathbb{M}_{m,n}$. Then T preserves Boolean rank if and only if*

(i) *T preserves Boolean rank k and $k+1$ with $1 \le k \le m-1$, or*
(ii) *T strongly preserves Boolean rank k with $1 \le k \le m$.*

Proof. Let T preserve Boolean ranks k and $k+1$ or T strongly preserves Boolean rank k. Then T maps cells to cells by Lemma 3.3. Now, suppose that T is not invertible. Then $T(E) = T(F)$ for some distinct cells E and F by Lemma 2.1. If $b(E+F) = 2$, choose a Boolean matrix $A \in \mathbb{M}_{m,n}$ with $b(A) = \sharp(A) = k-1$ such that $b(E+A) = k$ and $b(E + F + A) = k+1$. But then $k+1 = b(T(E+F+A)) = b(T(E+A)) = k$, a contradiction for both cases (i) and (ii). For the case of $b(E + F) = 1$, we may assume, without loss of generality, that $E = E_{1,1}$ and $F = E_{1,2}$. Let $B =$

$E_{2,1} + E_{2,2} + E_{3,3} + \cdots + E_{k+1,k+1}$. Then $b(E + F + B) = k$ and $b(E + B) = k + 1$. But then $k = b(T(E + F + B)) = b(T(E + B)) = k + 1$, a contradiction for both cases (i) and (ii). Thus T is invertible. By Theorem 2.4, T is a (P, Q)-operator and hence T preserves Boolean rank by Theorem 1.1. The converse is obvious. □

Recently Beasley and Song ([3]) showed that for a linear operator T on $\mathbb{M}_{m,n}$, T preserves Boolean rank if and only if T preserves Boolean ranks 1 and k, where $2 \leq k \leq m$.

Now we summarize our results by:

Theorem 3.5. *Let T be a linear operator on $\mathbb{M}_{m,n}$. Then the following are equivalent*:

(i) *T preserves Boolean rank*;

(ii) *T preserves Boolean ranks k and $k + 1$, where $1 \leq k \leq m - 1$*;

(iii) *T preserves Boolean ranks 1 and k, where $2 \leq k \leq m$*;

(iv) *T strongly preserves Boolean rank k, where $1 \leq k \leq m$*;

(v) *T is a (P, Q)-operator.*

As a concluding remark, we suggest to prove the following conjecture:

Conjecture 3.6. *Let T be a linear operator on $\mathbb{M}_{m,n}$. Then T preserves Boolean rank if and only if T preserves any two Boolean ranks h and k with $1 \leq h < k \leq m \leq n$.*

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