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INTERSECTION-SOFT IDEALS IN CI-ALGEBRAS

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Abstract. The notion of intersection-soft ideal of CI-algebras is introduced, and related properties are investigated. A characterization of an intersection-soft ideal is provided, and a new intersection-soft ideal from the old one is established.

1. Introduction

Mathematics requires that all mathematical notions (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers and recently computer scientists as well as other researcher have become interested in vague concepts [1-4, 10-12]. One of them, Hájek [3] introduced a BL-algebra which is an algebraic structure for many valued logic. Many researchers investigated the various algebraic structures as MV-algebras, BCK-algebras, BE-algebras and CI-algebras [1-6, 12]. As a generalization of a BCK-algebra, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. The notion of CI-algebras is introduced by Meng [8] as a generalization of BE-algebras. Ideal theory and properties in CI-algebras are studied by Kim [5].

On the hand, rough set theory was introduced by Pawlak [11,12] to generalize the classical set theory. Rough approximations are defined by the equivalence relation. There has been a rapid growth in interest in rough set theory in recent years. Its applications are decision system modeling and analysis of complex systems, neural networks, evolutionary computing, data mining and knowledge discovery, pattern recognition, machine learning, business and finance, chemistry, computer engineering, environment, medicine, etc. As a generalization of a rough set, Molodtsov [9]

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introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. In [7], Lee applied soft set theory to CI-algebras.

In this paper, we introduce the notion of int-soft ideal in CI-algebras, and investigate related properties. We provide a characterization of an int-soft ideal. We make a new int-soft ideal from the old one.

2. PRELIMINARIES

An algebra $(X; *, 1)$ of type $(2, 0)$ is called a *CI-algebra* if it satisfies the following properties:

(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$. A CI-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies:

$$(\forall x, y, z \in X)\left((y * z) * ((x * y) * (x * z)) = 1\right). \tag{2.1}$$

A CI-algebra $(X; *, 1)$ is said to be *self-distributive* if it satisfies:

$$(\forall x, y, z \in X)\left(x * (y * z) = (x * y) * (x * z)\right). \tag{2.2}$$

Note that every self-distributive CI-algebra is a transitive CI-algebra (see [5]).

A non-empty subset I of a CI-algebra $(X; *, 1)$ is called an *ideal* of X (see [5]) if it satisfies:

(I1) $(\forall x, y \in X)\left(y \in I \;\Rightarrow\; x * y \in I\right)$,
(I2) $(\forall x, a, b \in X)\left(a, b \in I \;\Rightarrow\; (a * (b * x)) * x \in I\right)$.

Molodtsov [9] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subset E$.

A pair $\left(\tilde{f}, A\right)$ is called a *soft set* (see [9]) over U, where $\tilde{f}$ is a mapping given by

$$\tilde{f} : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $\left(\tilde{f}, A\right)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [9].

For a soft set $\left(\tilde{f}, X\right)$ over U and a subset γ of U, the *γ-inclusive set* of $\left(\tilde{f}, X\right)$, denoted by $(\tilde{f};\gamma)^{\supseteq}$, is defined to be the set

$$(\tilde{f};\gamma)^{\supseteq} := \left\{x \in X \mid \gamma \subseteq \tilde{f}(x)\right\}.$$

3. Intersection-soft Ideals

In what follows, denote by $S(U,X)$ the set of all soft sets of X over U where X is a CI-algebra unless otherwise specified.

Definition 3.1. A soft set $\left(\tilde{f},X\right) \in S(U,X)$ is called an *intersection-soft ideal* (briefly, *int-soft ideal*) (of X) over U if it satisfies the following conditions:

$$(\forall x,y \in X)\left(\tilde{f}(y) \subseteq \tilde{f}(x*y)\right), \tag{3.1}$$

$$(\forall x,y,z \in X)\left(\tilde{f}((x*(y*z))*z) \supseteq \tilde{f}(x) \cap \tilde{f}(y)\right). \tag{3.2}$$

Example 3.2. Let $X = \{1,a,b,c,d,0\}$ be a CI-algebra with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

(1) Let $\left(\tilde{f},X\right) \in S(U,X)$ be given as follows:

$$\left(\tilde{f},X\right) = \{(1,\gamma_1),(a,\gamma_1),(b,\gamma_1),(c,\gamma_2),(d,\gamma_2),(0,\gamma_2)\}$$

where γ_1 and γ_2 are subsets of U with $\gamma_2 \nsubseteq \gamma_1$. Then $\left(\tilde{f},X\right)$ is an int-soft ideal over U.

(2) For $U = \mathbb{Z}$(the set of integers), let $(\tilde{g},X) \in S(U,X)$ be given as follows:

$$(\tilde{g},X) = \{(1,2\mathbb{Z}),(a,2\mathbb{Z}),(b,2\mathbb{N}),(c,2\mathbb{N}),(d,2\mathbb{N}),(0,2\mathbb{N})\}$$

Then $(\tilde{g},X)$ is not an int-soft ideal over U since

$$\tilde{g}(a) \cap \tilde{g}(a) = 2\mathbb{Z} \nsubseteq \tilde{g}\left((a*(a*b))*b\right) = \tilde{g}(b) = 2\mathbb{N}.$$

Proposition 3.3. *Every int-soft ideal $\left(\tilde{f},X\right)$ over U satisfies the following assertion:*

$$(\forall x,y \in X)\left(\tilde{f}(y) \supseteq \tilde{f}(x*y) \cap \tilde{f}(x)\right). \tag{3.3}$$

Proof. Using (CI2), (CI1) and (3.2), we have

$$\begin{aligned}\tilde{f}(y) &= \tilde{f}(1 * y) = \tilde{f}\left(((x * y) * (x * y)) * y\right) \\ &\supseteq \tilde{f}(x * y) \cap \tilde{f}(x).\end{aligned}$$

for all $x, y \in X$. □

Lemma 3.4. *Every int-soft ideal $\left(\tilde{f}, X\right)$ over U satisfies the following assertion:*

$$(\forall x \in X)\left(\tilde{f}(1) \supseteq \tilde{f}(x)\right). \tag{3.4}$$

Proof. Using (CI1) and (3.1), we have $\tilde{f}(1) = \tilde{f}(x * x) \supseteq \tilde{f}(x)$ for all $x \in X$. □

Proposition 3.5. *Every int-soft ideal $\left(\tilde{f}, X\right)$ over U satisfies the following assertion:*

$$(\forall x, y \in X)\left(\tilde{f}((x * y) * y) \supseteq \tilde{f}(x)\right). \tag{3.5}$$

Proof. Taking $y = 1$ and $z = y$ in (3.2) and using (CI2) and Lemma 3.4, we get

$$\tilde{f}((x * y) * y) = \tilde{f}((x * (1 * y)) * y) \supseteq \tilde{f}(x) \cap \tilde{f}(1) = \tilde{f}(x)$$

for all $x, y \in X$. □

Corollary 3.6. Every int-soft ideal $\left(\tilde{f}, X\right)$ over U satisfies the following assertion:

$$(\forall x, y \in X)\left(x * y = 1 \;\Rightarrow\; \tilde{f}(x) \subseteq \tilde{f}(y)\right). \tag{3.6}$$

Proof. Let $x, y \in X$ be such that $x * y = 1$. Then

$$\tilde{f}(y) = \tilde{f}(1 * y) = \tilde{f}((x * y) * y) \supseteq \tilde{f}(x)$$

by (CI2) and (3.5). □

If a soft set $\left(\tilde{f}, X\right) \in S(U, X)$ satisfies the condition (3.6), we say that $\left(\tilde{f}, X\right)$ is *order preserving*. Hence every int-soft ideal is order preserving.

Proposition 3.7. *If X is a transitive CI-algebra, then every int-soft ideal $\left(\tilde{f}, X\right)$ over U satisfies the condition*

$$(\forall x, y, z \in X)\left(\tilde{f}(x * z) \supseteq \tilde{f}(x * (y * z)) \cap \tilde{f}(y)\right). \tag{3.7}$$

Proof. Let $\left(\tilde{f}, X\right)$ be an int-soft ideal over U. Since X is transitive, we have

$$((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$$

for all $x, y, z \in X$. It follows from (CI2), (3.2) and (3.5) that

$$\begin{aligned}\tilde{f}(x * z) &= \tilde{f}(1 * (x * z)) \\ &= \tilde{f}\left(((y * z) * z) * ((x * (y * z)) * (x * z)) * (x * z)\right) \\ &\supseteq \tilde{f}((y * z) * z) \cap \tilde{f}(x * (y * z)) \\ &\supseteq \tilde{f}(x * (y * z)) \cap \tilde{f}(y).\end{aligned}$$

Therefore (3.7) is valid. □

Corollary 3.8. *If X is a self-distributive CI-algebra, then every int-soft ideal $\left(\tilde{f}, X\right)$ over U satisfies the condition* (3.7).

Proposition 3.9. *If a soft set $\left(\tilde{f}, X\right) \in S(U, X)$ satisfies two conditions* (3.4) *and* (3.7)*, then $\left(\tilde{f}, X\right)$ is order preserving.*

Proof. Let $x, y \in X$ be such that $x * y = 1$. Then

$$\tilde{f}(y) = \tilde{f}(1 * y) \supseteq \tilde{f}(1 * (x * y)) \cap \tilde{f}(x) = \tilde{f}(1 * 1) \cap \tilde{f}(x) = \tilde{f}(x)$$

by (CI1), (CI2), (3.7) and (3.4). Therefore $\left(\tilde{f}, X\right)$ is order preserving. □

Theorem 3.10. *If $\left(\tilde{f}, X\right)$ is an int-soft ideal over U, then the set*

$$\mathcal{I} := \left\{x \in X \mid \tilde{f}(x) = \tilde{f}(1)\right\}$$

is an ideal of X.

Proof. Let $x \in X$ and $a \in \mathcal{I}$. Then $\tilde{f}(a) = \tilde{f}(1)$, and so

$$\tilde{f}(x * a) \supseteq \tilde{f}(a) = \tilde{f}(1) \tag{3.8}$$

by (3.1). Combining this and (3.4), we have $\tilde{f}(x * a) = \tilde{f}(1)$, that is, $x * a \in \mathcal{I}$. For any $x, a, b \in X$, if $a, b \in \mathcal{I}$, then $\tilde{f}(a) = \tilde{f}(1) = \tilde{f}(b)$. It follows from (3.2) that

$$\tilde{f}((a * (b * x)) * x) \supseteq \tilde{f}(a) \cap \tilde{f}(b) = \tilde{f}(1)$$

and so that $\tilde{f}((a * (b * x)) * x) = \tilde{f}(1)$. Thus $(a * (b * x)) * x \in \mathcal{I}$. Therefore $\mathcal{I}$ is an ideal of X. □

We provide characterizations of an int-soft ideal.

Theorem 3.11. *A soft set $\left(\tilde{f}, X\right) \in S(U, X)$ is an int-soft ideal over U if and only if the γ-inclusive set $(\tilde{f}; \gamma)^{\supseteq}$ is an ideal of X for all $\gamma \in P(U)$ with $(\tilde{f}; \gamma)^{\supseteq} \neq \emptyset$.*

The ideal $(\tilde{f};\gamma)^{\supseteq}$ in Theorem 3.11 is called the *inclusive ideal* of X.

Proof. Assume that $\left(\tilde{f}, X\right)$ is an int-soft ideal over U. Let $\gamma \in P(U)$ be such that $(\tilde{f};\gamma)^{\supseteq} \neq \emptyset$. Let $x \in X$ and $a \in (\tilde{f};\gamma)^{\supseteq}$. Then $\tilde{f}(a) \supseteq \gamma$. It follows from (3.1) that $\tilde{f}(x*a) \supseteq \tilde{f}(a) \supseteq \gamma$. Hence $x*a \in (\tilde{f};\gamma)^{\supseteq}$. Let $x \in X$ and $a, b \in (\tilde{f};\gamma)^{\supseteq}$. Then $\tilde{f}(a) \supseteq \gamma$ and $\tilde{f}(b) \supseteq \gamma$. Using (3.2), we have $\tilde{f}((a*(b*x))*x) \supseteq \tilde{f}(a) \cap \tilde{f}(b) \supseteq \gamma$, and thus $(a*(b*x))*x \in (\tilde{f};\gamma)^{\supseteq}$. Therefore $(\tilde{f};\gamma)^{\supseteq}$ is an ideal of X for all $\gamma \in P(U)$ with $(\tilde{f};\gamma)^{\supseteq} \neq \emptyset$.

Conversely, suppose that $(\tilde{f};\gamma)^{\supseteq}$ is an ideal of X for all $\gamma \in P(U)$ with $(\tilde{f};\gamma)^{\supseteq} \neq \emptyset$. For any $a \in X$, let $\tilde{f}(a) = \gamma$. Then $a \in (\tilde{f};\gamma)^{\supseteq}$. Since $(\tilde{f};\gamma)^{\supseteq}$ is an ideal of X, we have $x*a \in (\tilde{f};\gamma)^{\supseteq}$ for all $x \in X$. Thus $\tilde{f}(x*a) \supseteq \gamma = \tilde{f}(a)$ for all $x, a \in X$. For any $x, y \in X$, let $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in (\tilde{f};\gamma)^{\supseteq}$ which implies that $(x*(y*z))*z \in (\tilde{f};\gamma)^{\supseteq}$ for all $z \in X$. Hence

$$\tilde{f}((x*(y*z))*z) \supseteq \gamma = \gamma_x \cap \gamma_y = \tilde{f}(x) \cap \tilde{f}(y)$$

for all $x, y, z \in X$. Thus $\left(\tilde{f}, X\right)$ is an int-soft ideal over U. □

Theorem 3.12. *For any soft set* $\left(\tilde{f}, X\right) \in S(U,X)$*, let* $\left(\tilde{f}^*, X\right) \in S(U,X)$ *be defined by*

$$\tilde{f}^* : X \to P(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in (\tilde{f};\gamma)^{\supseteq}, \\ \delta & \text{otherwise} \end{cases}$$

where γ *and* δ *are subsets of* U *with* $\delta \not\subseteq \tilde{f}(x)$*. If* $\left(\tilde{f}, X\right)$ *is an int-soft ideal over* U*, then so is* $\left(\tilde{f}^*, X\right)$.

Proof. It is straightforward by Theorem 3.11. □

Theorem 3.13. *Every ideal of* X *can be realized as an inclusive ideal of some int-soft ideal over* X.

Proof. Let I be an ideal of X. Define a soft set $\left(\tilde{f}, X\right) \in S(U,X)$ as follows:

$$\tilde{f} : X \to P(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in I, \\ \emptyset & \text{otherwise} \end{cases}$$

where γ is a nonempty subset of U. Let $x, y \in X$. If $y \in I$, then $x*y \in I$ and so $\tilde{f}(y) = \gamma = \tilde{f}(x*y)$. If $y \notin I$, then $\tilde{f}(y) = \emptyset \subseteq \tilde{f}(x*y)$. For any $x, a, b \in X$, let $a, b \in I$. Then $(a*(b*x))*x \in I$ and thus $\tilde{f}(a) \cap \tilde{f}(b) = \gamma = \tilde{f}((a*(b*x))*x)$. If $a \notin I$ or $b \notin I$, then $\tilde{f}(a) = \emptyset$ or $\tilde{f}(b) = \emptyset$. Hence $\tilde{f}(a) \cap \tilde{f}(b) = \emptyset \subseteq \tilde{f}((a*(b*x))*x)$. Obviously, $(\tilde{f};\gamma)^{\supseteq} = I$. This completes the proof. □

For any $a, b \in X$, consider the following set:

$$C(a,b) := \{x \in X \mid a * (b * x) = 1\}.$$

Theorem 3.14. *Every int-soft ideal* $\left(\tilde{f}, X\right)$ *over* U *satisfies the following assertion:*

$$(\forall a, b \in X)(\forall \gamma \in P(U)) \left(a, b \in (\tilde{f}; \gamma)^{\supseteq} \;\Rightarrow\; C(a,b) \subseteq (\tilde{f}; \gamma)^{\supseteq}\right). \tag{3.9}$$

Proof. Assume that $\left(\tilde{f}, X\right)$ is an int-soft ideal over U. Let $a, b \in X$ be such that $a, b \in (\tilde{f}; \gamma)^{\supseteq}$. Then $\tilde{f}(a) \supseteq \gamma$ and $\tilde{f}(b) \supseteq \gamma$. If $y \in C(a,b)$, then $a * (b * y) = 1$ and so

$$\tilde{f}(y) = \tilde{f}(1 * y) = \tilde{f}((a * (b * y)) * y) \supseteq \tilde{f}(a) \cap \tilde{f}(b) \supseteq \gamma \tag{3.10}$$

by (CI2) and (3.2). Hence $y \in (\tilde{f}; \gamma)^{\supseteq}$. Therefore $C(a,b) \subseteq (\tilde{f}; \gamma)^{\supseteq}$. □

Corollary 3.15. *For every int-soft ideal* $\left(\tilde{f}, X\right)$ *over* U, *we have*

$$(\forall \gamma \in P(U)) \left((\tilde{f}; \gamma)^{\supseteq} \neq \emptyset \;\Rightarrow\; (\tilde{f}; \gamma)^{\supseteq} = \bigcup_{a, b \in (\tilde{f}; \gamma)^{\supseteq}} c(a,b) \right).$$

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