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ON MATRIX POLYNOMIALS ASSOCIATED WITH HUMBERT POLYNOMIALS

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Abstract. The principal object of this paper is to study a class of matrix polynomials associated with Humbert polynomials. These polynomials generalize the well known class of Gegenbauer, Legendre, Pincherl, Horadam, Horadam-Pethe and Kinney polynomials. We shall give some basic relations involving the Humbert matrix polynomials and then take up several generating functions, hypergeometric representations and expansions in series of matrix polynomials.

1. Introduction and Notations

Gould [6] (see also [11]) presented a systematic study of an interesting generalization of Humbert, Gegenbauer and several other polynomial systems defined by

$$(c - mxt + yt^m)^{-p} = \sum_{n=0}^{\infty} P_n(m, x, y, p, c)\, t^n \tag{1.1}$$

where m is a positive integer, $|t| < 1$ and other parameters are unrestricted in general. For the table of main special cases of (1.1), including Gegenbauer, Legendre, Tchebycheff, Pincherle, Kinney and Humbert polynomials, see Gould [6]. In [10] Milovanovic and Dordevic considered the polynomials $\left\{P_{n,m}^{\lambda}\right\}_n^{\infty}$ defined by the generating function

$$(1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} P_{n,m}^{\lambda}(x)\, t^n, \tag{1.2}$$

where $m \in \mathbb{N} := \{1, 2, 3, ...\}$, $|t| < 1$ and $\lambda > -\frac{1}{2}$.

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The explicit form of the polynomial $P_{n,m}^{\lambda}(x)$ is

$$P_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} \frac{(-1)^k (\lambda)_{n-(m-1)k} (2x)^{n-mk}}{k!(n-mk)!}, \tag{1.3}$$

where the Pochhammer symbol is defined by $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+n-1), (\forall n \geq 1)$ and $(\lambda)_0 = 1.\Gamma(.)$: is the familiar Gamma function.
Note that

$$P_{n,2}^{\lambda}(x) = C_n^{\lambda}(x),$$

where $C_n^{\lambda}(x)$ are Gegenbauer polynomials [12]. The set of polynomials denoted by $S_n^{\nu}(x)$ considered by Sinha [17]

$$\left(1 - 2xt + t^2(2x-1)\right)^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x)\, t^n \tag{1.4}$$

is precisely a generalization of $S_n^{\nu}(x)$ defined and studied by Shreshtha [16]. In [14] the authors investigated Gegenbauer matrix polynomials defined by

$$\left(1 - 2xt + t^2\right)^{-A} = \sum_{n=0}^{\infty} C_n^{A}(x)\, t^n, \tag{1.5}$$

where A is a positive stable matrix in the complex space $\mathbb{C}^{N\times N}$, $\mathbb{C}$ bing the set of complex numbers, of all square matrices of common order N. The explicit representation of the Gegenbauer matrix polynomials $C_n^{A}(x)$ has been given in [14, p. 104 (15)] in the form

$$C_n^{A}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (A)_{n-k} (2x)^{n-2k}}{k!(n-2k)!}. \tag{1.6}$$

In the last decade the study of matrix polynomials has been made more systematic with the consequence that many basic results of scalar orthogonality have been extended to the matrix case (see, for example [1]-[5] and [13]). We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, \ldots, A_n \ldots$, are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots + A_1 x + A_0,$$

a matrix polynomial of degree n in x. If $A + nI$ is invertible for every integer $n \geq 0$ then

$$(A)_n = A(A+I)(A+2I)\cdots(A+(n-1)I); n \geq 1; (A)_0 = I \tag{1.7}$$

Thus we have

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-n)_k}{n!}I = \frac{(-nI)_k}{n!}; 0 \leq k \leq n. \tag{1.8}$$

The hypergeometric matrix function

$$ {}_2F_1\left[A, B; C; z\right] = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_n(B)_n[(C)_n]^{-1}z^n \tag{1.9}$$

where A, B and C are matrices in $\mathbb{C}^{N\times N}$ such that $C + nI$ is invertible for integer $n \geq 0$ and $|z| < 1$. The generalized hypergeometric matrix function (see (1.9)) is given in the form:

$$\begin{aligned} &{}_pF_q\left[A_1, A_2, ..., A_p; C_1, C_2, ..., C_q; z\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(A_1)_n(A_2)_n \cdots (A_p)_n[(C_1)_n]^{-1}....[(C_q)_n]^{-1}z^n \end{aligned} \tag{1.10}$$

For the purpose of this work we recall the following relations [12]:

$$(1-x)^{-A} = \sum_{n=0}^{\infty}(A)_n\frac{x^n}{n!},\ |x| < 1, \tag{1.11}$$

and

$$(t-\nu)^n = \sum_{k=0}^{n} \frac{n!t^k(-\nu)^{n-k}}{k!(n-k)!}. \tag{1.12}$$

Also, we recall that if $A(k, n)$ and $B(k, n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geq 0$ and $k \geq 0$ then it follows that [18]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k), \tag{1.13}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+2k). \tag{1.14}$$

For m a positive integer, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k, n-(m-1)k), \tag{1.15}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n+mk). \tag{1.16}$$

The primary goal of this work is to introduce and study a new class of matrix polynomials, namely the Humbert Matrix polynomials $P^A_{n,m}(x, y; a, b, c)$, which is general enough to account for many of polynomials involved in generalized potential problems (see [9]-[11]). This is interesting since, as will be shown, the matrix polynomials $P^{A}_{n,m}(x,y;a,b,c)$ is an extension to the matrix framework of the classical families of the polynomials mentioned above.

2. Humbert Matrix Polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. We define the Humbert matrix polynomials by means of the generating relation

$$(c-axt+bt^{m}(2y-1))^{-A}=\sum_{n=0}^{\infty}P^{A}_{n,m}(x,y;a,b,c)\,t^{n} \tag{2.1}$$

where m is a positive integer and other parameters are unrestricted in general. Based on (1.11) and (1.12), formula (2.1) can be written in the form

$$\sum_{n=0}^{\infty}P^{A}_{n,m}(x,y;a,b,c)\,t^{n}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k}c^{-A-nI}(A)_{n}}{k!(n-k)!}(ax)^{n-k}\left[b(2y-1)\right]^{k}t^{n+(m-1)k}$$

which, in view of (1.15), gives us

$$\begin{aligned}&\sum_{n=0}^{\infty}P^{A}_{n,m}(x,y;a,b,c)\,t^{n}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^{k}c^{-A-(n-(m-1)k)I}(A)_{n+(1-m)k}}{k!(n-mk)!}(ax)^{n-mk}\left[b(2y-1)\right]^{k}t^{n}.\end{aligned} \tag{2.2}$$

By equating the coefficients of t^{n} in (2.2), we obtain an explicit representation for the polynomials $P^{A}_{n,m}(x,y;a,b,c)$ in the form

$$P^{A}_{n,m}(x,y;a,b,c)=\sum_{k=0}^{[n/m]}\frac{(-1)^{k}c^{-A-(n-(m-1)k)I}(A)_{n+(1-m)k}}{k!(n-mk)!}(ax)^{n-mk}\left[b(2y-1)\right]^{k}. \tag{2.3}$$

Again, starting from (2.1), it is easily seen that

$$\sum_{n=0}^{\infty}P^{A}_{n,m}(x,y;a,b,c)\,t^{n}=c^{-A}\left[1-\frac{axt}{2c}\right]^{-2A}\left[1-\frac{\frac{a^{2}x^{2}t^{2}}{4c^{2}}-\frac{b}{c}t^{m}(2y-1)}{(1-\frac{axt}{2c})^{2}}\right]^{-A},$$

which, with the help of the results (1.11) and (1.12), gives

(2.4)

$$\sum_{n=0}^{\infty} P_{n,m}^{A}(x,y;a,b,c)\,t^{n}$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\left[\frac{n}{m-1}\right]}\sum_{k=\left[\frac{n-(m-2)s}{2}\right]}^{\infty}\frac{c^{-A-(n-(m-2)s)I}(-1)^{n}(-(n-k))_{(m-1)s}(A)_{n-k-(m-2)s}}{(2k-n+(m-2)s)!s!(n-k)!}$$
$$(2A+2(n-k-(m-2)s)I)_{2k-n+(m-2)s}\left(\frac{-ax}{2}\right)^{n-(m-2)s}\left(\frac{4bc}{a^{2}x^{2}}(2y-1)\right)^{s}t^{n},$$
$$m>1.$$

By equating the coefficients of t^n in (2.4), we obtain another explicit representation for the polynomials $P_{n,m}^{A}(x,y;a,b,c)$ as follows:

(2.5)

$$P_{n,m}^{A}(x,y;a,b,c)$$
$$=\sum_{s=0}^{\left[\frac{n}{m-1}\right]}\sum_{k=\left[\frac{n-(m-2)s}{2}\right]}^{\infty}\frac{c^{-A-(n-(m-2)s)!}(-1)^{n}(-(n-k))_{(m-1)s}(A)_{n-k-(m-2)s}}{(2k-n+(m-2)s)!s!(n-k)!}$$
$$(2A+2(n-k-(m-2)s)I)_{2k-n+(m-2)s}\left(\tfrac{-ax}{2}\right)^{n-(m-2)s}\left(\tfrac{4bc}{a^{2}x^{2}}(2y-1)\right)^{s},$$
$$m>1.$$

According to the relation

$$\begin{aligned}(A)_{n-k-(m-2)s}&(2A+2(n-k-(m-2)s)I)_{2k-n+(m-2)s}\\ &=\frac{(2A)_{n-(m-2)s}}{2^{2k}}\left[\left(A+\frac{1}{2}I\right)_{n-k-(m-2)s}\right]^{-1},\end{aligned}\tag{2.6}$$

Equation (2.5) can be written in the form

(2.7)

$$P_{n,m}^{A}(x,y;a,b,c)$$
$$=\sum_{s=0}^{\left[\frac{n}{m-1}\right]}\sum_{k=\left[\frac{n-(m-2)s}{2}\right]}^{\infty}\frac{c^{-A-(n-(m-2)s)I}(-1)^{n}(-(n-k))_{(m-1)s}(2A)_{n-(m-2)s}}{(2k-n+(m-2)s)!s!(n-k)!2^{2k}}$$
$$\left[(A+\tfrac{1}{2}I)_{n-k-(m-2)s}\right]^{-1}\left(\tfrac{-ax}{2}\right)^{n-(m-2)s}\left(\tfrac{4bc}{a^{2}x^{2}}(2y-1)\right)^{s},\ m>1,$$

where $A+\frac{1}{2}I+(n-k(m-2)s)I$ and $2A+(n-(m-2)s)I$ are invertible.

Now, we mention some interesting special cases of our results of this section. First, if in (2.3) and (2.5) we let $y=0, a=m$ and $c=1=-b$, we get

$$h_{n,m}^{A}(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^{k}(A)_{n+(1-m)k}}{k!(n-mk)!}(mx)^{n-mk},\tag{2.8}$$

and

$$h_{n,m}^{A}(x) = \sum_{s=0}^{\left[\frac{n}{m-1}\right]} \sum_{k=\left[\frac{n-(m-2)s}{2}\right]}^{\infty} \frac{(-1)^n(-(n-k))_{(m-1)s}(A)_{n-k-(m-2)s}}{(2k-n+(m-2)s)!s!(n-k)!} (2A+2(n-k-(m-2)s)I)_{2k-n+(m-2)s}\left(\frac{-mx}{2}\right)^{n-ms}, \; m>1. \tag{2.9}$$

respectively, where $h_{n,m}^{A}$ is the matrix version of Humbert polynomials $h_{n,m}^{\nu}$ (see [11]).

Next, for $m=3$, Equations (2.8) and (2.9) further reduce to following explicit representations:

$$P_n^A(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k(A)_{n-2k}(3x)^{n-3s}}{k!(n-3k)!} \tag{2.10}$$

and

$$P_n^A(x) = \sum_{s=0}^{\left[\frac{n}{2}\right]} \sum_{k=\left[\frac{n-s}{2}\right]}^{\infty} \frac{(-1)^n(-(n-k))_{2s}(A)_{n-k-s}}{(2k-n+s)!s!(n-k)!} (2A+2(n-k-s)I)_{2k-n+s}\left(\frac{3x}{2}\right)^{n-3s}, \; m>1, \tag{2.11}$$

respectively, where $P_n^A(x)$ is the matrix version of Pincherle polynomials $P_n(x)$ [11]. Moreover, in view of the relationship (see Equations (1.5) and (2.1))

$$P_{n,2}^{A}(x,0;2,-1,1) = C_n^A(x), \tag{2.12}$$

equation (2.3) reduces to finite series representation for the matrix Gegenbauer polynomials $C_n^A(x)$ as follows:

$$C_n^A(x) = (2A)_n \sum_{k=0}^{[n/2]} \frac{1}{2^{2k}k!(n-2k)!}\left[\left(A+\frac{1}{2}I\right)\right]^{-1}\left(x^2-1\right)_k x^{n-2k}. \tag{2.13}$$

Note that equation (2.12) is a known result (see [14, p. 109 (40)]).

3. Hypergeometric Matrix Representations

Starting from (2.3) and using the results

$$(-nI)_{mk} = (-1)^{mk}\frac{n!}{(n-mk)!}I = m^{mk}\prod_{i=1}^{m}\left(\frac{-n+i-1}{m}I\right)_k \tag{3.1}$$

and

(3.2)
$$(A)_{n-(m-1)k} = (-1)^{(m-1)k}(A)_n \left[(m-1)^{(m-1)k} \prod_{j=1}^{m-1} \left(\frac{(-A-nI)+jI}{m-1} \right)_k \right]^{-1},$$

where
$$0 \le (m-1)k \le n,$$

we get

(3.3)
$$\begin{aligned} P_{n,m}^{A}(x,y;a,b,c) = {} & \frac{(A)_n c^{-A-nI}}{n!}(ax)^n \sum_{k=0}^{\infty} \prod_{i=1}^{m} \left(\frac{-n+i-1}{m} I \right)_k \\ & \left[\prod_{j=1}^{m-1} \left(\frac{(-A-nI)+jI}{m-1} \right)_k \right]^{-1} \frac{c^{(m-1)k} m^{mk} \left[b(2y-1) \right]^k}{k!(m-1)^{(m-1)k}(ax)^{mk}}, \end{aligned}$$

which, in view of (1.16), gives us the following hypergeometric matrix representation:

(3.4)
$$\begin{aligned} P_{n,m}^{A}(x,y;a,b,c) = {} & \frac{(A)_n c^{-A-nI}}{n!}(ax)^n {}_mF_{m-1} \left[\frac{-n}{m} I, \frac{-n+1}{m} I, ..., \right. \\ & \left. \frac{-n+m-1}{m} I; \frac{-A+(n-1)I}{m-1}, ..., \frac{-A-(n-m+1)I)}{m-1}; \frac{c^{m-1} m^m \left[b(2y-1) \right]^k}{(m-1)^{m-1}(ax)^m} \right], \end{aligned}$$

where $A+nI$ and $\frac{-A-(n-m+1)I}{m-1}$ are invertible. According to the relationship (2.12), Equation (3.4), yields the following known representation for the Gegenbauer matrix polynomials C_n^A (see [14, p. 109 (39)]):

(3.5)
$$C_n^A(x) = \frac{(A)_n (2x)^n}{n!} {}_2F_1 \left[\frac{-n}{2} I, \frac{1-n}{2} I; I-A-nI; x^{-2} \right].$$

Next, if in (3.4) we put $a = m, c = 1 = -b$ and $y = 0$, we get the following representation for the matrix Humbert polynomials $h_{n,m}^A(x)$:

(3.6)
$$\begin{aligned} h_{n,m}^{A}(x) = {} & \frac{(A)_n}{n!}(mx)^n {}_mF_{m-1} \left[\frac{-n}{m} I, \frac{-n+1}{m} I, ..., \right. \\ & \left. \frac{-n+m-1}{m} I; \frac{-A-(n-1)I}{m-1}, ..., \frac{-A-(n-m+1)I)}{m-1}; \frac{1}{(m-1)^{m-1}(x)^m} \right]. \end{aligned}$$

4. More Generating Functions

By proceeding in a fashion similar to that in Section 2, in this section we aim at establishing the following additional generating functions for the Humbert matrix polynomilas $P_{n,m}^{A}(x,y;a,b,c)$:

(4.1)
$$\sum_{n=0}^{\infty} P_{n,m}^{A}(x,y;a,b,c)\left[(A)_n\right]^{-1}t^n = \sum_{n=0}^{\infty}\frac{c^{-A-nI}(axt)^n}{n!}$$
$$ {}_1F_m\left[A+nI;\frac{A+nI}{m},\frac{A+(n+1)I}{m},.....,\frac{A+(n+m-1)I}{m};\frac{-bt^m(2y-1)}{m^m}\right],$$

(4.2)
$$\begin{aligned}&\sum_{n=0}^{\infty}(B)_nP_{n,m}^{A}(x,y;a,b,c)\left[(A)_n\right]^{-1}t^n\\ &=\sum_{n=0}^{\infty}\frac{c^{-A-nI}(axt)^n(B)_n}{n!}{}_{m+1}F_m\Bigg[A+nI,\frac{B+nI}{m},\frac{B+(n+1)I}{m}\\ &....,\frac{B+(n+m-1)I}{m};\frac{A+nI}{m},\frac{A+(n+1)I}{m},\\ &.....,\frac{A+(n+m-1)I}{m};\frac{-bt^m(2y-1)}{c}\Bigg],\end{aligned}$$

(4.3)
$$\sum_{n=0}^{\infty} P_{n,m}^{A}(x,y;a,b,c)\left[(2A)_n\right]^{-1}t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\frac{(-1)^{n+k}c^{-A}(-2k)_{n+s}(-n)_k}{2^{2k}n!(2k)!s!}$$
$$\left[\left(A+\frac{1}{2}I\right)_{n-k+s}(2A+(n-s)I)_{ms}\right]^{-1}\left(\frac{axt}{2c}\right)^n\left[\frac{2bt^{m-1}(2y-1)}{ax}\right]^s,$$

(4.4)
$$\sum_{n=0}^{\infty}(B)_nP_{n,m}^{A}(x,y;a,b,c)\left[(2A)_n\right]^{-1}t^n=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\frac{(-1)^{n+k}c^{-A}(-2k)_{n+s}(-n)_k}{2^{2k}n!(2k)!s!}$$
$$\left[\left(A+\frac{1}{2}I\right)_{n-k+s}(2A+(n-s)I)_{ms}\right]^{-1}(B)_{n+(m-1)s}\left(\frac{axt}{2c}\right)^n\left[\frac{2bt^{m-1}(2y-1)}{ax}\right]^s,$$

where $A+nI,\ B+nI,\ 2A+(n+2k)I+((m-2)s)I,\ B+(n+2k)I\ ,\frac{A+(n+m-1)I}{m}$ and $\frac{B+(n+m-1)I}{m}$ are invertible matrices.

Derivation of the results (4.1) to (4.4). Starting from (2.3) and using the results (1.14) and (3.1), we get

(4.5)
$$\sum_{n=0}^{\infty} P_{n,m}^{A}(x,y;a,b,c)\left[(A)_n\right]^{-1}t^n = \sum_{n=0}^{\infty}\frac{c^{-A-nI}(axt)^n}{n!}\sum_{k=0}^{\infty}\frac{(A+nI)_k}{k!}$$
$$\left[\left(\frac{A+nI}{m}\right)_k\left(\frac{A+(n+1)I}{m}\right)_k\cdots\left(\frac{A+(n+m-1)I}{m}\right)_k\right]^{-1}\left(\frac{-bt^m(2y-1)}{cm^m}\right)^k,$$

which, on using the definition of the generalized matrix hypergeometric series (1.10), gives us the generating function (4.1). This completes the proof of (4.1).

If B is a positve stable matrix in the complex space $\mathbb{C}^{N\times N}$ of all square matrices of common order N, then following the method of derivation of equation (4.1) , we can easily establish relation (4.2).

Again, starting from (2.5), and employing the results (2.6) and (1.16), we can derive the result (4.3). The proof of Equation (4.4) is similar to that of (4.3). Therefore, we skip the details.

It is easy to observe that the main results (4.1) to (4.4) give a number of generating functions of matrix version polynomials, for example, the matrix polynomials $P^{\lambda}_{n,m}(x)$ (see (1.2)), the matrix versions of Pincherle, Humbert, Sinha, Sheshtha, Kinney, Horadam and Horadam-Pethe polynomials (see [13]).

5. Expansions

Expansion for the matrix polynomials $P^{A}_{n,m}(x,y;a,b,c)$ in series of Legendre, Hermite, Gegenbauer and Laguerre polynomials relevant to our present investigation are given as follows:

(5.1)
$$\begin{aligned}&P^{A}_{n,m}(x,y;a,b,c)\\ &=\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\sum_{j=0}^{\left[\frac{n+s}{2}\right]}\frac{c^{-A-nI}(-1)^{n+k}(-n)_k(-2k)_{n+s}(n+1)_s\left[2(n+s)-4j+1\right]}{s!j!(2k)!2^{2k}\left(\frac{3}{2}\right)_{n+s-j}}\\ &\quad(2A)_{n+s}\left[(A+\frac{1}{2}I)_{n+s-k}\right]^{-1}\left(\frac{4b}{a^2x^2}(2y-1)\right)^{s}\ P_{n+s-2j}\left(\frac{ax}{4}\right),\end{aligned}$$

(5.2)
$$\begin{aligned}&P^{A}_{n,m}(x,y;a,b,c)\\ &=\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\sum_{j=0}^{\left[\frac{n+s}{2}\right]}\frac{c^{-A-nI}(-1)^{n+k}(-n)_k(-2k)_{n+s}(n+1)_s\left[\nu+n+s-2j\right]}{s!j!(2k)!2^{2k}(\nu)_{n+s-j+1}}\\ &\quad(2A)_{n+s}\left[(A+\frac{1}{2}I)_{n+s-k}\right]^{-1}\left(\frac{4b}{a^2x^2}(2y-1)\right)^{s}\ C^{\nu}_{n+s-2j}\left(\frac{ax}{4}\right),\end{aligned}$$

(5.3)
$$\begin{aligned}&P^{A}_{n,m}(x,y;a,b,c)\\ &=\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\sum_{j=0}^{\left[\frac{n+s}{2}\right]}\frac{c^{-A-nI}(-1)^{n+k+j}(-n)_k(-2k)_{n+s}(n+1)_s}{s!j!(2k)!2^{n+2k}(n+s-2j)!}\end{aligned}$$

$$(2A)_{n+s}\left[(A+\frac{1}{2}I)_{n+s-k}\right]^{-1}\left(\frac{2b}{a^2x^2}(2y-1)\right)^s H_{n+s-2j}\left(\frac{ax}{2}\right),$$

$$\begin{aligned}(5.4)\quad & P^A_{n,m}(x,y;a,b,c)\\ &=\sum_{s=0}^{\infty}\sum_{k=\left[\frac{n+s}{2}\right]}^{\infty}\sum_{j=0}^{n+s}\frac{c^{-A-nI}(-1)^{n+k}(-n)_k(-2k)_{n+s}(n+1)_s(1+\alpha)_{n+s}}{s!j!(2k)!2^{2k}(1+\alpha)_j(n+s-j)!}\\ &\quad (2A)_{n+s}\left[(A+\frac{1}{2}I)_{n+s-k}\right]^{-1}\left(\frac{4b}{a^2x^2}(2y-1)\right)^s L^{(\alpha)}_j\left(\frac{ax}{2}\right),\end{aligned}$$

where $2A+(n+s)I$ and $A+(n+s-k)I+\frac{1}{2}I$ are invertible matrices.

Derivation of the results (5.1) to (5.4). On inserting the result (see [12, p. 181 (4)])

$$(5.5)\qquad \frac{(ax)^n}{n!}=\sum_{s=0}^{\left[\frac{n}{2}\right]}\frac{(2n-4s+1)}{s!\left(\frac{3}{2}\right)_{n-s}}P_{n-2s}\left(\frac{ax}{2}\right)$$

in relation (2.7) , we get

$$\begin{aligned}& P^A_{n,m}(x,y;a,b,c)\\ &=\sum_{s=0}^{\left[\frac{n}{m-1}\right]}\sum_{k=\left[\frac{n-(m-2)s}{2}\right]}^{\infty}\sum_{j=0}^{\left[\frac{n-(m-2)s}{2}\right]}\frac{c^{-A-(n-(m-2)s)I}(-(n-k))_{(m-1)s}}{(-1)^{ms}(2k-n+(m-2)s)!}\\ &\quad\frac{(2A)_{n-(m-2)s}(n-(m-2)s))!\,[2(n-(m-2)s)-4j+1]}{s!j!(n-k)!2^{2k}\left(\frac{3}{2}\right)_{n-(m-2)s-j}}\\ &\quad\left[(A+\frac{1}{2}I)_{n-k-(m-2)s}\right]^{-1}\left(\frac{4bc}{a^2x^2}(2y-1)\right)^s P_{n-(m-2)ss-2j}\left(\frac{ax}{4}\right),\end{aligned}$$

which on using the result (1.16),and simplifying gives us (5.1). Similarly, the results (5.2), (5.3) and (5.4) are obtained by using the known results [12, p. 283 (36), p. 194 (4), p. 207 (2)]

$$(5.6)\qquad \frac{(2x)^n}{n!}=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\nu+n-2k)C^{\nu}_{n-2k}(x)}{k!(\nu)_{n+1-k}},$$

$$(5.7)\qquad \frac{x^n}{n!}=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{H_{n-2k}(x)}{2^n k!(n-2k)!},$$

and

$$(5.8)\qquad \frac{x^n}{n!}=\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n L^{(\alpha)}_k(x)}{(n-k)!(1+\alpha)_k}$$

respectively, instead of (5.5).

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