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FIXED POINTS AND ADDITIVE ρ-FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we solve the additive ρ-functional equations

$$\begin{aligned} & f(x+y+z)-f(x)-f(y)-f(z) \\ & = \rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right), \end{aligned} \tag{0.1}$$

where ρ is a fixed number with $\rho \neq 1, 2$, and

$$\begin{aligned} & f(x+y+z)-f(x)-f(y)-f(z) \\ & = \rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right), \end{aligned} \tag{0.2}$$

where ρ is a fixed number with $\rho \neq 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [14] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

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We recall a fundamental result in fixed point theory.

Theorem 1.1 ([2, 5]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [8] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 10, 11, 12]).

In Section 2, we solve the additive functional equation (0.1) and prove the Hyers-Ulam stability of the additive functional equation (0.1) in Banach spaces.

In Section 3, we solve the additive functional equation (0.2) and prove the Hyers-Ulam stability of the additive functional equation (0.2) in Banach spaces.

Throughout this paper, assume that X is a normed space and that Y is a Banach space.

2. Additive ρ-functional Equation (0.1)

Let ρ be a number with $\rho \neq 1, 2$.

We solve and investigate the additive ρ-functional equation (0.1) in normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} & f(x+y+z) - f(x) - f(y) - f(z) \\ & = \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $-2f(0) = -\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y+z) - f(x) - f(y) - f(z) &= \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \\ &= \rho(f(x+y+z) - f(x) - f(y) - f(z)) \end{aligned}$$

and so $f(x+y+z) = f(x) + f(y) + f(z)$ for all $x, y, z \in X$. Since $f(0) = 0$,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. $\square$

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$. and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\| f(x+y+z) - f(x) - f(y) - f(z) - \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \leq \varphi(x, y, z) \tag{2.4}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x, 0) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0) \tag{2.6}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}, 0\right) \le \frac{L}{2}\varphi(x, x, 0)$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi\left(x, x, 0\right),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [9]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi\left(x, x, 0\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\| 2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right) \right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \\ &\le 2\varepsilon\frac{L}{2}\varphi\left(x, x, 0\right) \le L\varepsilon\varphi\left(x, x, 0\right) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \le \frac{L}{2}\varphi(x, x, 0)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(x\right) = 2A\left(\frac{x}{2}\right) \tag{2.7}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \quad \leq \quad \mu\varphi(x, x, 0)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\left\|A(x+y+z) - A(x) - A(y) - A(z) - \rho\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z)\right)\right\| \\
&= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right.\right. \\
&\left.\left. -\rho\left(2f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right)\right)\right\| \leq \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0
\end{aligned}$$

for all $x, y, z \in X$. So

$$A(x+y) - A(x) - A(y) - A(z) = \rho\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z)\right)$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. $\square$

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\left\|f(x+y+z) - f(x) - f(y) - f(z) - \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \\
&\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad (2.8)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{1}{2}\varphi(x, x, 0) \tag{2.9}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

It follows from (2.9) that $d(f, Jf) \leq \frac{1}{2}$. So

$$\|f(x) - A(x)\| \leq \frac{1}{2(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.8). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

3. Additive ρ-functional Equation (0.2)

Let ρ be a number with $\rho \neq 1$.

We solve and investigate the additive ρ-functional equation (0.2) in normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies*

$$f(x+y+z) - f(x) - f(y) - f(z) = \rho \left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z) \right) \tag{3.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $-2f(0) = -2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (3.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z)$$

for all $x, y, z \in X$. and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\left\| f(x+y+z) - f(x) - f(y) - f(z) - \rho \left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z) \right) \right\| \\ &\leq \varphi(x, y, z) \end{aligned} \tag{3.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}\varphi\left(x, x, 0\right)$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (3.3), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x, 0) \tag{3.4}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let* $r > 1$ *and* θ *be nonnegative real numbers, and let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\begin{aligned} &\left\|f(x+y+z) - f(x) - f(y) - f(z) - \rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right)\right\| \\ (3.5) \qquad &\le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 3.4. *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi\left(x, y, z\right) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (3.3). *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)}\varphi\left(x, x, 0\right)$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.5). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

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