J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2015.22.4.315
Volume 22, Number 4 (November 2015), Pages 315–331

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SOME OPIAL-TYPE INEQUALITIES APPLICABLE TO DIFFERENTIAL EQUATIONS INVOLVING IMPULSES

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Abstract. The purpose of this paper is to obtain Opial-type inequalities that are useful to study various qualitative properties of certain differential equations involving impulses. After we obtain some Opial-type inequalities, we apply our results to certain differential equations involving impulses.

1. Introduction

Opial-type inequalities are very useful to study various qualitative properties of differential equations. For a good reference of the work on such inequalities together with various applications, we recommend the monograph [1]. In this paper we obtain some Opial-type inequalities that involve Stieltjes derivatives which are applicable to differential equations with impulses. Differential equations involving impulses arise in various real world phenomena, we refer to the monograph [8].

2. Preliminaries

To obtain our results in this paper we need some preliminaries.

Let $\mathbf{R}$ be the set of all real numbers. Assume that $[a, b] \subset \mathbf{R}$ is a bounded interval. A function $f : [a, b] \longrightarrow \mathbf{R}$ is called *regulated* on $[a, b]$ if both

$$f(s+) = \lim_{\eta \to 0+} f(s + \eta), \text{ and } f(t-) = \lim_{\eta \to 0+} f(t - \eta)$$

exist for every point $s \in [a, b), t \in (a, b]$, respectively. Let $G([a, b])$ be the set of all regulated functions on $[a, b]$. For $f \in G([a, b])$ we define $f(a-) = f(a), f(b+) = f(b)$. For convenience we define

$$\Delta^+ f(s) = f(s+) - f(s), \quad \Delta^- f(s) = f(s) - f(s-) \text{ and } \Delta f(s) = f(s+) - f(s-).$$

Received by the editors March 09, 2015. Accepted September 07, 2015
2010 *Mathematics Subject Classification.* 26D10, 26D15.
Key words and phrases. Stieltjes derivatives, Opial-type inequalities, differential equations involving impulses.

Remark 2.1. Let $f \in G([a,b])$. Since both $f(s+)$ and $f(s-)$ exist for every $s \in [a,b]$ it is obvious that f is bounded on $[a,b]$, and since f is the uniform limit of step functions, f is Borel measurable (see [3, Theorem 3.1.]).

For a closed interval $I = [c,d]$, we define $f(I) = f(d) - f(c)$.

A *tagged* interval $(\tau, [c,d])$ in $[a,b]$ consists of an interval $[c,d] \subset [a,b]$ and a point $\tau \in [c,d]$.

Let $I_i = [c_i, d_i] \subset [a,b], i = 1, ..., m$. We say that the intervals I_i are pairwise non-overlapping if

$$int(I_i) \cap int(I_j) = \emptyset$$

for $i \neq j$ where $int(I)$ denotes the interior of an interval I.

A finite collection $\{(\tau_i, I_i) : i = 1, 2, ..., m\}$ of pairwise non-overlapping tagged intervals is called a *tagged partition of* $[a,b]$ if $\cup_{i=1}^m I_i = [a,b]$. A positive function δ on $[a,b]$ is called a *gauge* on $[a,b]$.

From now on we use notation $\overline{1,m} = 1, ..., m$.

Definition 2.2 ([6, 9])**.** Let δ be a gauge on $[a,b]$. A tagged partition $P = \{(\tau_i, [t_{i-1}, t_i]) : t_{i-1} < t_i, i = \overline{1,m}\}$ of $[a,b]$ is said to be δ*−fine* if for every $i = \overline{1,m}$ we have

$$\tau_i \in [t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Moreover if a $\delta-$fine partition P satisfies the implications

$$\tau_i = t_{i-1} \Rightarrow i = 1, \quad \tau_i = t_i \Rightarrow i = m,$$

then it is called a δ^*-fine partition of $[a,b]$.

The following lemma implies that for a gauge δ on $[a,b]$ there exists a δ^*-fine partition of $[a,b]$. This also implies the existence of a $\delta-$fine partition of $[a,b]$.

Lemma 2.3 ([6])**.** *Let δ be a gauge on $[a,b]$ and a dense subset $\Omega \subset (a,b)$ be given. Then there exists a δ^*-fine partition $P = \{(\tau_i, [t_{i-1}, t_i]) : i = \overline{1,m}\}$ of $[a,b]$ such that $t_i \in \Omega$ for $i = \overline{1, m-1}$.*

We now give a formal definition of two types of the Kurzweil integrals.

Definition 2.4 ([6, 9])**.** Assume that $f, g : [a,b] \longrightarrow \mathbf{R}$ are given. We say that $f\,\mathrm{d}g$ is *Kurzweil integrable* (or shortly, K-*integrable*) on $[a,b]$ and $v \in \mathbf{R}$ is its integral if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\left| \sum_{i=1}^{m} f(\tau_i) g(I_i) - v \right| \leq \varepsilon,$$

provided $P = \{(\tau_i, I_i) : i = \overline{1,m}\}$ is a $\delta-$fine tagged partition of $[a,b]$. In this case we define $v = \int_a^b f(s)\,\mathrm{d}g(s)$ (or, shortly,$v = \int_a^b f\,\mathrm{d}g$).

If, in the above definition, $\delta-$fine is replaced by δ^*-*fine*, then we say that $f\,\mathrm{d}g$ is *Kurzweil* integrable*(or, shortly, K*-*integrable*) on $[a,b]$ and we define $v = (K^*)\int_a^b f\,\mathrm{d}g$.

Remark 2.5. By the above definition it is obvious that K-integrability implies K*-integrability.

The following results are needed in this paper. For other properties of the K-integrals, see, e.g., [2, 7, 9, 10].

In this paper $BV([a,b])$ denotes the set of all functions that are of bounded variation on $[a,b]$.

Theorem 2.6 ([11, 2.15. Theorem]). *Assume that $f \in G([a,b])$ and $g \in BV([a,b])$. Then both $f\,\mathrm{d}g$ and $g\,\mathrm{d}f$ are K-integrable on $[a,b]$ and*

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) + \sum_{t\in[a,b]} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].$$

Remark 2.7. In the above theorem, the sum $\sum_{t\in[a,b]}[\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)]$ is actually a countable sum because every regulated function has only countable discontinuities.

Theorem 2.8 ([10, p. 40, 4.25. Theorem]). *Let $h \in BV([a,b])$, $g : [a,b] \longrightarrow \mathbf{R}$ and $f : [a,b] \longrightarrow \mathbf{R}$. If the integral $\int_a^b g\,\mathrm{d}h$ exists and f is bounded on $[a,b]$, then the integral $\int_a^b f(s)\,\mathrm{d}\left[\int_a^s g(v)\,\mathrm{d}h(v)\right]$ exists if and only if the integral $\int_a^b f(s)g(s)\,\mathrm{d}h(s)$ exists and in this case we have*

$$\int_a^b f(s)\,\mathrm{d}\left[\int_a^s g(v)\,\mathrm{d}h(v)\right] = \int_a^b f(s)g(s)\,\mathrm{d}h(s).$$

Theorem 2.9 ([10, p. 34, 4.13. Corollary]). *Assume that $f \in G([a,b])$ and $g \in BV([a,b])$. Then we have for every $t \in [a,b]$*

$$\lim_{\eta\to 0+} \int_a^{t\pm\eta} f(s)\,\mathrm{d}g(s) = \int_a^t f(s)\,\mathrm{d}g(s) \pm f(t)\Delta^{\pm} g(t).$$

The following result is the Hölder's inequality for K-integral. In this paper we frequently use this inequality.

Theorem 2.10. (Hölder's inequality) *Assume that $f, g \in G([a,b])$ and h is a nondecreasing function defined on $[a,b]$. Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\int_a^b |fg| \, \mathrm{d}h \leq \left(\int_a^b |f|^p \, \mathrm{d}h \right)^{\frac{1}{p}} \left(\int_a^b |g|^q \, \mathrm{d}h \right)^{\frac{1}{q}}. \tag{2.1}$$

Proof. The proof of this theorem is very similar to the proof of the classical Hölder's inequality. So we omit the proof. □

3. Stieltjes Derivatives

In this section we state the results in [4, 5] that are essential to obtain our main results.

Throughout this section, we assume that $f \in G([a,b])$ and g is a nondecreasing function on $[a,b]$.

We say that the function g is *not locally constant* at $t \in (a,b)$ if there exists $\eta > 0$ such that g is not constant on $(t-\varepsilon, t+\varepsilon)$ for every $0 < \varepsilon < \eta$. We also say that the function g is *not locally constant* at a and b, respectively if there exist $\eta, \eta^* > 0$ such that g is not constant on $[a, a+\varepsilon), (b-\varepsilon^*, b]$ respectively, for every $\varepsilon \in (0,\eta), \varepsilon^* \in (0, \eta^*)$.

Definition 3.1 ([4])**.** If g is not locally constant at $t \in (a,b)$, we define

$$\frac{\mathrm{d}f(t)}{\mathrm{d}g(t)} = \lim_{\eta,\delta \to 0+} \frac{f(t+\eta) - f(t-\delta)}{g(t+\eta) - g(t-\delta)},$$

provided that the limit exists.

If g is not locally constant at $t = a$ and $t = b$ respectively, we define

$$\frac{\mathrm{d}f(a)}{\mathrm{d}g(a)} = \lim_{\eta \to 0+} \frac{f(a+\eta) - f(a)}{g(a+\eta) - g(a)}, \quad \frac{\mathrm{d}f(b)}{\mathrm{d}g(b)} = \lim_{\delta \to 0+} \frac{f(b) - f(b-\delta)}{g(b) - g(b-\delta)},$$

respectively, provided that the limits exist. Frequently we use $f'_g(t)$ instead of $\frac{\mathrm{d}f(t)}{\mathrm{d}g(t)}$.

If both f and g are constant on some neighborhood of t, then we define $f'_g(t) = 0$.

Remark 3.2. It is obvious that if g is not continuous at t then $f'_g(t)$ exists. Thus if $f'_g(t)$ does not exist then g is continuous at t. $f'_g(t)$ is called a *Stieltjes derivative* of f with respect to g.

Theorem 3.3 ([4])**.** *Assume that if g is not locally constant at $t \in [a,b]$. If f is continuous at t or g is not continuous at t, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}g(t)} \int_a^t f(s)\,\mathrm{d}g(s) = f(t).$$

K*-integrals recover Stieltjes derivatives.

Theorem 3.4 ([4]). *Assume that if g is constant on some neighborhood of t then there is a neighborhood of t where both f and g are constant. Suppose that $f_g'(t)$ exists at every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*) \int_a^b f_g'(s)\,\mathrm{d}g(s) = f(b) - f(a).$$

Lemma 3.5 ([4]). *Assume that if g is constant on some neighborhood of t then there is a neighborhood of t such that both f_1 and f_2 are constant there. If both $\frac{\mathrm{d}f_1(t)}{\mathrm{d}g(t)}$ and $\frac{\mathrm{d}f_2(t)}{\mathrm{d}g(t)}$ exist and $f_1, f_2 \in G([a,b])$, then we have*

$$\frac{\mathrm{d}[f_1(t)f_2(t)]}{\mathrm{d}g(t)} = \frac{\mathrm{d}f_1(t)}{\mathrm{d}g(t)} f_2(t+) + f_1(t-)\frac{\mathrm{d}f_2(t)}{\mathrm{d}g(t)}.$$

Similarly to the Riemann integral we have the following integration by parts formula.

Theorem 3.6. (Integration by Parts) *Assume that functions $f, g, h \in G([a,b])$ are all left-continuous and h is nondecreasing. Suppose that both $f_h'(t)$ and $g_h'(t)$ exist for every $t \in [a,b]$ and $f_h', g_h' \in G([a,b])$. Then we have*

(3.1)
$$\int_a^b f_h' g\,\mathrm{d}h = f(b)g(b) - f(a)g(a) - \int_a^b f g_h'\,\mathrm{d}h + \sum_{a\le t\le b} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].$$

Proof. By Theorem 2.8 and Theorem 3.4 we have

$$\int_a^b f\,\mathrm{d}g = \int_a^b f(s)\,\mathrm{d}\left[\int_a^s g_h'\,\mathrm{d}h\right] = \int_a^b f g_h'\,\mathrm{d}h.$$

So by Theorem 2.6 we get

$$\begin{aligned}
&\int_a^b f_h' g \,\mathrm{d}h = \int_a^b g(s)\,\mathrm{d}\left[\int_a^s f_h'\,\mathrm{d}h\right] = \int_a^b g\,\mathrm{d}f \\
&= f(b)g(b) - f(a)g(a) - \int_a^b f\,\mathrm{d}g + \sum_{a\le t\le b}[\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)] \\
&= f(b)g(b) - f(a)g(a) - \int_a^b f g_h'\,\mathrm{d}h + \sum_{a\le t\le b}[\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].
\end{aligned}$$

This completes the proof. □

Let

$$a \le t_1 < t_2 < \cdots < t_m < b.$$

The Heaviside function $H_\tau : \mathbf{R} \longrightarrow \{0,1\}$ is defined by

$$H_\tau(t) = \begin{cases} 0, & \text{if } t \le \tau \\ 1, & \text{if } t > \tau. \end{cases}$$

Using the Heaviside function H_τ, we define function $\phi : [a,b] \longrightarrow \mathbf{R}$ by

$$\phi(t) = t + \sum_{k=1}^m H_{t_k}(t), \quad t \in [a,b]. \tag{3.2}$$

Remark 3.7. It is obvious that the function ϕ is strictly increasing and of bounded variation on $[a,b]$, and left-continuous on $[a,b]$.

Lemma 3.8 ([5]). *Assume that $f \in G([a,b])$ and $f'(t)$ exists for $t \neq t_k, k = \overline{1,m}$ Then we have*

$$\begin{aligned}
&(a) \quad f_\phi'(t) = f'(t), \quad f_\phi'(t_k) = f(t_k+) - f(t_k-), \\
&(b) \quad \int_a^t f\,\mathrm{d}\phi = \int_a^t f(s)\,\mathrm{d}s + \sum_{a\le t_k<t} f(t_k).
\end{aligned}$$

4. Opial-type Integral Inequalities involving Stieltjes Derivatives

In this section we obtain some Opial-type integral inequalities involving Stieltjes derivatives. The Opial-type inequalities have many interesting applications in the theory of differential equations(see, e.g., [1]).

Throughout this paper we always assume that

$$a \le t_1 < t_2 < \cdots < t_m < b,$$

and that a function $\alpha : [a, b] \longrightarrow \mathbf{R}$ is strictly increasing on $[a, b]$, and continuous at $t \neq t_k$, and $\Delta\,\alpha(t_k) \neq 0$, for every $k = \overline{1, m}$.

Remark 4.1. Note that strictly increasing implies nondecreasing, and a nondecreasing function is regulated.

Let $PC([a, b]) = \{u \in G([a, b]) : u \text{ is continuous at every } t \neq t_k, k = \overline{1, m}\}$.

From now on we always assume that $u, u'_\alpha \in PC([a, b])$, and we define

$$u_+(t) = u(t+), u_-(t) = u(t-), \forall t \in [a, b].$$

The following result is an Opial-type inequality with Stieltjes derivatives.

Theorem 4.2. *Assume that* $u(a) = u(b) = 0$. *If both* u *and* α *are left-continuous on* $[a, b]$, *then we have*

$$\int_a^b (|u| + |u_+|)|u'_\alpha| \,\mathrm{d}\alpha \leq K_\alpha \int_a^b (u'_\alpha)^2 \,\mathrm{d}\alpha, \tag{4.1}$$

where $K_\alpha = \inf_{h\in[a,b]} \max\{\,\alpha(h) - \,\alpha(a),\, \alpha(b) - \,\alpha(h)\}$.

Proof. Let for $t \in [a, b]$,

$$y(t) = \int_a^t |u'_\alpha| \,\mathrm{d}\alpha, \quad z(t) = \int_t^b |u'_\alpha| \,\mathrm{d}\alpha.$$

By Theorem 2.9, the functions, y and z are left-continuous on $[a, b]$. Also by Theorem 3.3, we have

$$y'_\alpha(t) = |u'_\alpha(t)| = -z'_\alpha(t)$$

and we have by Theorem 3.4 and $u(a) = u(b) = 0$

$$|u(t)| \leq y(t), \quad |u(t)| \leq z(t),$$

for $t \in [a, b]$. So by Theorem 3.4, Lemma 3.5, and using Hölder's inequality, we get

$$\begin{aligned}\int_a^h (|u| + |u_+|)|u'_\alpha| \,\mathrm{d}\alpha &\leq \int_a^h (y + y_+)y'_\alpha \,\mathrm{d}\alpha = \int_a^h (y^2)'_\alpha \,\mathrm{d}\alpha \\ &= y^2(h) = [\,\alpha(h) - \,\alpha(a)] \int_a^h (u'_\alpha)^2 \,\mathrm{d}\alpha,\end{aligned}$$

and similarly we obtain

$$\begin{aligned}&\int_h^b (|u| + |u_+|)|u'_\alpha| \,\mathrm{d}\alpha \leq -\int_h^b (z + z_+)z'_\alpha \,\mathrm{d}\alpha = -\int_h^b (z^2)'_\alpha \,\mathrm{d}\alpha = z^2(h) \\ &\leq [\,\alpha(b) - \,\alpha(h)] \int_h^b (u'_\alpha)^2 \,\mathrm{d}\alpha.\end{aligned} \tag{4.2}$$

So we have

$$\int_a^b (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha = \int_a^h (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha + \int_h^b (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha$$
$$\leq [\, \alpha(h) - \alpha(a)] \int_a^h (u'_\alpha)^2 \, \mathrm{d}\alpha + [\, \alpha(b) - \alpha(h)] \int_h^b (u'_\alpha)^2 \, \mathrm{d}\alpha$$
$$\leq K_\alpha \int_a^b (u'_\alpha)^2 \, \mathrm{d}\alpha.$$

The proof is complete. □

A slightly more general result is as follows.

Theorem 4.3. *Assume that $u(b) = 0$. If both u and α are left-continuous on $[a, b]$, then we have*

$$\int_a^b (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha \leq [\, \alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, \mathrm{d}\alpha. \tag{4.3}$$

Proof. From (4.2) we have

$$\int_h^b (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha \leq [\, \alpha(b) - \alpha(h)] \int_h^b (u'_\alpha)^2 \, \mathrm{d}\alpha \leq [\, \alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, \mathrm{d}\alpha.$$

So we get

$$\int_a^b (|u| + |u_+|)|u'_\alpha| \, \mathrm{d}\alpha \leq [\, \alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, \mathrm{d}\alpha.$$

This gives (4.3). The proof is complete. □

More generally we have the following result.

Theorem 4.4. *Let $p \geq 0, q \geq 1, r \geq 0, m \geq 1$ be real numbers and let $f \in PC([a, b])$ be a positive function on $[a, b]$ with $\inf_{s\in[a,b]} f(s) > 0$. Assume that both functions u and α are left-continuous on $[a, b]$. If $u(b) = 0$, then we have*

$$\int_a^b f \; |u|^{m(p+q)}|u'_\alpha|^{mr} \, \mathrm{d}\alpha \leq [(p + q + r)^m I(m, f)]^{p+q} \int_a^b f \; |u'_\alpha|^{m(p+q+r)} \, \mathrm{d}\alpha, \tag{4.4}$$

where $I(m, f) = \int_a^b f\gamma \, \mathrm{d}\, \alpha$, $\gamma(t) = \left[\int_t^b (f)^{\frac{1}{1-m}} \, \mathrm{d}\alpha\right]^{m-1}_,$ for $m \neq 1$, and $\gamma(t) = \left[\inf_{s\in[t,b]} f(s)\right]^{-1}_,$ for $m = 1$.

Proof. Let for $t \in [a, b]$,

$$z(t) = \int_t^b |u'_\alpha| \, \mathrm{d}\alpha.$$

Then by Theorem 3.4, $|u(t)| \le z(t)$ and by Theorem 2.9, z is left-continuous, and non-increasing on $[a,b]$.

If $t \ne t_k, k = \overline{1,m}$, then by Theorem 3.3, $z'_\alpha(t)$ exists, and by Theorem 2.9, z is continuous at t. Using the Mean Value Theorem and by the definition of the Stieltjes derivatives, if z is not locally constant at t, then we have,

$$\begin{aligned}(z^{p+q})'_\alpha(t) &= \lim_{\delta,\eta\to 0+} \frac{z^{p+q}(t+\eta) - z^{p+q}(t-\delta)}{z(t+\eta) - z(t-\delta)} \frac{z(t+\eta) - z(t-\delta)}{\alpha(t+\eta) - \alpha(t-\delta)} \\ &= \lim_{\delta,\eta\to 0+} (p+q)\omega^{p+q-1} z'_\alpha(t), \text{ where } z(t+\eta) \le \omega \le z(t-\delta) \\ &= (p+q) z^{p+q-1}(t) z'_\alpha(t).\end{aligned}$$

If z is constant on some neighborhood of t, then since $(z^{p+q})'_\alpha(t) = 0 = z'_\alpha(t)$, the above equality is also true. If $t = t_k, k = \overline{1,m}$, since z is non-increasing on $[a,b]$, and $z'_\alpha = -|u'_\alpha| \le 0$, and by the Mean Value Theorem, and by the definition of the Stieltjes derivatives, we have,

$$\begin{aligned}(z^{p+q})'_\alpha(t_k) &= [z^{p+q}(t_k+) - z^{p+q}(t_k-)]/[\alpha(t_k+) - \alpha(t_k-)] \\ &= (p+q)\omega^{p+q-1}[z(t_k+) - z(t_k-)]/[\alpha(t_k+) - \alpha(t_k-)] \\ &= (p+q)\omega^{p+q-1} z'_\alpha(t_k) \ge (p+q) z^{p+q-1}(t_k) z'_\alpha(t_k),\end{aligned}$$

where $z(t_k+) \le \omega \le z(t_k) = z(t_k-)$. Thus we have

$$-(z^{p+q})'_\alpha(t) \le -(p+q) z^{p+q-1}(t) z'_\alpha(t), \ \forall t \in [a,b]. \tag{4.5}$$

Let $\beta(t) = \int_a^t f \, \mathrm{d}\alpha$. Then by hypotheses, β is strictly increasing on $[a,b]$.

Since

$$\begin{aligned}(z^{p+q})'_\beta(t) &= \lim_{\delta,\eta\to 0+} \frac{z^{p+q}(t+\eta) - z^{p+q}(t-\delta)}{\alpha(t+\eta) - \alpha(t-\delta)} \frac{\alpha(t+\eta) - \alpha(t-\delta)}{\beta(t+\eta) - \beta(t-\delta)} \\ &= \frac{(z^{p+q})'_\alpha(t)}{\beta'_\alpha(t)} = \frac{(z^{p+q})'_\alpha(t)}{f(t)}, \text{by Theorem 3.3,}\end{aligned}$$

we have by Theorem 3.4 and (4.5), since $z(b) = 0$ and $z'_\alpha \le 0$,

$$\begin{aligned}z^{p+q}(t) &= -\int_t^b (z^{p+q})'_\beta \, \mathrm{d}\beta = -\int_t^b f^{-1} (z^{p+q})'_\alpha \, \mathrm{d}\beta \\ &\le (p+q) \int_t^b f^{-1} z^{p+q-1} (-z'_\alpha) \, \mathrm{d}\beta = (p+q) \int_t^b f^{-1} z^{p+q-1} |z'_\alpha| \, \mathrm{d}\beta.\end{aligned}$$

Using Hölder's inequality with indices $m, \frac{m}{m-1}$, we have

$$z^{m(p+q)}(t) \le (p+q)^m \gamma(t) \int_a^b z^{m(p+q-1)} |z'_\alpha|^m \, \mathrm{d}\beta, \quad \forall t \in [a,b]. \tag{4.6}$$

Integrating (4.6) on $[a,b]$ and using Hölder's inequality with indices $q, \frac{q}{q-1}$, and considering $\int_a^b \gamma \,\mathrm{d}\beta = \int_a^b f\gamma \,\mathrm{d}\alpha$ by Theorem 2.8, we get

$$\tag{4.7} \begin{aligned} &\int_a^b z^{m(p+q)}\,\mathrm{d}\beta \\ &\le (p+q)^m I(m,f) \int_a^b \left(z^{\frac{mp}{q}} |z'_\alpha|^m\right) \cdot z^{m(p+q-1)-\frac{mp}{q}}\,\mathrm{d}\beta \\ &\le (p+q)^m I(m,f) \left(\int_a^b z^{mp}|z'_\alpha|^{mq}\,\mathrm{d}\beta\right)^{\frac{1}{q}} \left(\int_a^b z^{m(p+q)}\,\mathrm{d}\beta\right)^{\frac{q-1}{q}}. \end{aligned}$$

If $\int_a^b z^{m(p+q)}\,\mathrm{d}\beta = 0$, then

$$\tag{4.8} \int_a^b z^{m(p+q)}\,\mathrm{d}\beta \le [(p+q)^m I(m,f)]^q \int_a^b z^{mp}|z'_\alpha|^{mq}\,\mathrm{d}\beta$$

is obviously true, otherwise, dividing both sides of (4.7) by $\left(\int_a^b z^{m(p+q)}\,\mathrm{d}\beta\right)^{\frac{q-1}{q}}$ and then taking the qth power on both sides of the resulting inequality we get also (4.8).

Using the Hölder's inequality with indices $\frac{q+r}{r}, \frac{q+r}{q}$, we have, by (4.8),

(4.9)

$$\begin{aligned} &\int_a^b z^{m(p+q)}|z'_\alpha|^{mr}\,\mathrm{d}\beta \\ &= \int_a^b [z^{m(pr/(q+r))}|z'_\alpha|^{mr}] \cdot [z^{m(p+q)-m(pr/(q+r))}]\,\mathrm{d}\beta \\ &\le \left[\int_a^b z^{mp}|z'_\alpha|^{m(q+r)}\,\mathrm{d}\beta\right]^{r/(q+r)} \left[\int_a^b z^{m(p+q+r)}\,\mathrm{d}\beta\right]^{q/(q+r)} \\ &\le \left[\int_a^b z^{mp}|z'_\alpha|^{m(q+r)}\,\mathrm{d}\beta\right]^{\frac{r}{q+r}} \left[[(p+q+r)^m I(m,f)]^{q+r} \int_a^b z^{mp}|z'_\alpha|^{m(q+r)}\,\mathrm{d}\beta\right]^{\frac{q}{q+r}} \\ &= [(p+q+r)^m I(m,f)]^q \int_a^b z^{mp}|z'_\alpha|^{m(q+r)}\,\mathrm{d}\beta. \end{aligned}$$

Using Hölder's inequality with indices $\frac{p+q}{p}, \frac{p+q}{q}$, we get by (4.9)

(4.10)

$$\begin{aligned} &\int_a^b z^{m(p+q)}|z'_\alpha|^{mr}\,\mathrm{d}\beta \le [(p+q+r)^m I(m,f)]^q \int_a^b z^{mp}|z'_\alpha|^{m(q+r)}\,\mathrm{d}\beta, \\ &\le [(p+q+r)^m I(m,f)]^q \int_a^b \left[z^{mp}|z'_\alpha|^{m(rp/(p+q))}\right] \cdot \left[|z'_\alpha|^{m(q+r)-m(rp/(p+q))}\right]\,\mathrm{d}\beta \end{aligned}$$

$$\leq [(p+q+r)^m I(m,f)]^q \left[\int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, \mathrm{d}\beta\right]^{\frac{p}{p+q}} \left[\int_a^b |z'_\alpha|^{m(p+q+r)} \, \mathrm{d}\beta\right]^{\frac{q}{p+q}}.$$

If $\int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, \mathrm{d}\beta = 0$, then the inequality

$$\int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, \mathrm{d}\beta \leq [(p+q+r)^m I(m,f)]^{p+q} \int_a^b |z'_\alpha|^{m(p+q+r)} \, \mathrm{d}\beta \tag{4.11}$$

is obviously true, otherwise, dividing both sides of (4.10) by $\left[\int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, \mathrm{d}\beta\right]^{\frac{p}{p+q}}$ and then taking the $\frac{p+q}{q}$th power on both sides of the resulting inequality we get also (4.11). Since $|u| \leq z$ and $|u'_\alpha| = |z'_\alpha|$ we have

$$\begin{aligned}
&\int_a^b f \, |u|^{m(p+q)} |u'_\alpha|^{mr} \, \mathrm{d}\alpha = \int_a^b |u|^{m(p+q)} |u'_\alpha|^{mr} \, \mathrm{d}\beta \leq \int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, \mathrm{d}\beta \\
&\leq [(p+q+r)^m I(m,f)]^{p+q} \int_a^b |z'_\alpha|^{m(p+q+r)} \, \mathrm{d}\beta, \quad \text{by (4.11)} \\
&\leq [(p+q+r)^m I(m,f)]^{p+q} \int_a^b f \, |u'_\alpha|^{m(p+q+r)} \, \mathrm{d}\alpha.
\end{aligned}$$

This gives (4.4). The proof is complete. □

5. Some Applications to certain Differential Equations involving Impulses

In this section we always assume that both functions u and u' are left- continuous on $[a,b]$, and that $\alpha = \phi$ (see (3.2)). Consider the following impulsive differential equation: for $k = \overline{1,m}$,

$$\begin{gathered}
u'' + q_1(t)u = 0, \ t \neq t_k, \\
\Delta u'(t_k) = a_k u'(t_k), \\
\Delta u(t_k) = b_k u'(t_k) \ , b_k \neq 0,
\end{gathered} \tag{5.1}$$

where $q_1 \in PC([a,b])$. Now we define

$$u''_\alpha(t) = (u')'_\alpha(t).$$

Since by Lemma 3.8 for $k = \overline{1,m}$

$$u''_\alpha(t) = \begin{cases} u''(t), & t \neq t_k \\ \Delta u'(t_k), & t = t_k, \end{cases}$$

the equation (5.1) implies the following equation:

$$u''_\alpha + p(t)u' + q(t)u = 0, \tag{5.2}$$

where

$$p(t) = \begin{cases} 0, & t \neq t_k \\ -a_k, & t = t_k, \end{cases} \quad q(t) = \begin{cases} q_1(t), & t \neq t_k \\ 0, & t = t_k, \ k = \overline{1,m}. \end{cases} \tag{5.3}$$

We need the following result.

Lemma 5.1. *If the function u satisfies the equation (5.1) and $c \in [a,b]$, then we have*

$$\int_a^c |uu'| \, \mathrm{d}\alpha = \int_a^c |uu'_\alpha| \, \mathrm{d}\alpha + \sum_{a \le t_k < c} (1 - |b_k|)|u(t_k)u'(t_k)|, \tag{5.4}$$

$$\sum_{a \le t_k < c} |b_k||u(t_k)u'(t_k)| \le \int_a^c |uu'_\alpha| \, \mathrm{d}\alpha, \tag{5.5}$$

$$\int_a^c u'u'_\alpha \, \mathrm{d}\alpha = \int_a^c |u'_\alpha|^2 \, \mathrm{d}\alpha + \sum_{a \le t_k < c} (b_k - b_k^2)|u'(t_k)|^2, \tag{5.6}$$

$$\sum_{a \le t_k < c} b_k^2 |u'(t_k)|^2 \le \int_a^c |u'_\alpha|^2 \, \mathrm{d}\alpha. \tag{5.7}$$

Proof. In the proof, we frequently use Lemma 3.8, $u'(t) = u'_\alpha(t)$. $t \neq t_k$, $\Delta u'(t_k) = u''_\alpha(t_k) = a_k u'(t_k)$, and $\Delta u(t_k) = u'_\alpha(t_k) = b_k u'(t_k), \ k = \overline{1,m}$.

$$\int_a^c |uu'| \, \mathrm{d}\alpha = \int_a^c |u(s)u'(s)| \, \mathrm{d}s + \sum_{a \le t_k < c} |u(t_k)u'(t_k)|$$
$$= \int_a^c |u(s)u'_\alpha(s)| \, \mathrm{d}s + \sum_{a \le t_k < c} |u(t_k)u'(t_k)|$$
$$= \int_a^c |u(s)u'_\alpha(s)| \, \mathrm{d}s + \sum_{a \le t_k < c} |u(t_k)u'_\alpha(t_k)| - \sum_{a \le t_k < c} |u(t_k)u'_\alpha(t_k)| + \sum_{a \le t_k < c} |u(t_k)u'(t_k)|$$
$$= \int_a^c |uu'_\alpha| \, \mathrm{d}\alpha - \sum_{a \le t_k < c} |u(t_k)u'_\alpha(t_k)| + \sum_{a \le t_k < c} |u(t_k)u'(t_k)|$$
$$= \int_a^c |uu'_\alpha| \, \mathrm{d}\alpha - \sum_{a \le t_k < c} |b_k||u(t_k)u'(t_k)| + \sum_{a \le t_k < c} |u(t_k)u'(t_k)|$$
$$= \int_a^c |uu'_\alpha| \, \mathrm{d}\alpha + \sum_{a \le t_k < c} (1 - |b_k|)|u(t_k)u'(t_k)|.$$

This gives (5.4). And

$$
\begin{aligned}
&\sum_{a\le t_k<c} |u(t_k)u'(t_k)| \le \int_a^c |u(s)u'(s)|\,\mathrm{d}s + \sum_{a\le t_k<c} |u(t_k)u'(t_k)| \\
&= \int_a^c |u(s)u'_\alpha(s)|\,\mathrm{d}s + \sum_{a\le t_k<c} |u(t_k)u'_\alpha(t_k)| - \sum_{a\le t_k<c} |u(t_k)u'_\alpha(t_k)| + \sum_{a\le t_k<c} |u(t_k)u'(t_k)| \\
&= \int_a^c |uu'_\alpha|\,\mathrm{d}\alpha - \sum_{a\le t_k<c} |b_k||u(t_k)u'(t_k)| + \sum_{a\le t_k<c} |u(t_k)u'(t_k)|.
\end{aligned}
$$

This gives (5.5). And

$$
\begin{aligned}
&\int_a^c u'u'_\alpha\,\mathrm{d}\alpha = \int_a^c u'(s)u'_\alpha(s)\,\mathrm{d}s + \sum_{a\le t_k<c} u'(t_k)u'_\alpha(t_k) \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}s + \sum_{a\le t_k<c} u'(t_k)u'_\alpha(t_k) \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}s + \sum_{a\le t_k<c} |u'_\alpha(t_k)|^2 - \sum_{a\le t_k<c} |u'_\alpha(t_k)|^2 + \sum_{a\le t_k<c} u'(t_k)u'_\alpha(t_k) \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}\alpha - \sum_{a\le t_k<c} b_k^2|u'(t_k)|^2 + \sum_{a\le t_k<c} b_k|u'(t_k)|^2 \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}\alpha + \sum_{a\le t_k<c} (b_k - b_k^2)|u'(t_k)|^2.
\end{aligned}
$$

This gives (5.6). Also

$$
\begin{aligned}
&\sum_{a\le t_k<c} |u'(t_k)|^2 \le \int_a^c |u'(s)|^2\,\mathrm{d}s + \sum_{a\le t_k<c} |u'(t_k)|^2 \\
&= \int_a^c |u'_\alpha(s)|^2\,\mathrm{d}s + \sum_{a\le t_k<c} |u'(t_k)|^2 \\
&= \int_a^c |u'_\alpha(s)|^2\,\mathrm{d}s + \sum_{a\le t_k<c} |u'_\alpha(t_k)|^2 - \sum_{a\le t_k<c} |u'_\alpha(t_k)|^2 + \sum_{a\le t_k<c} |u'(t_k)|^2 \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}\alpha - \sum_{a\le t_k<c} |u'_\alpha(t_k)|^2 + \sum_{a\le t_k<c} |u'(t_k)|^2 \\
&= \int_a^c |u'_\alpha|^2\,\mathrm{d}\alpha - \sum_{a\le t_k<c} b_k^2|u'(t_k)|^2 + \sum_{a\le t_k<c} |u'(t_k)|^2.
\end{aligned}
$$

This gives (5.7). The proof is complete. $\square$

Theorem 5.2. *Assume that u satisfies the equation (5.1) and $u'(a)=0, u(a)\neq 0$. If we have*

$$1 >[\,\alpha(b)-\,\alpha(a)]\left[\max_{a\le s\le b}|Q(s)|+\max_{a\le t_k\le b}|a_k|\left(1+\max_{a\le t_k\le b}\frac{|1-|b_k||}{|b_k|}\right)\right] + \max_{a\le t_k\le b}\frac{|1-b_k+a_k|}{|b_k|}, \tag{5.8}$$

where $Q(t)=\int_a^t q\,\mathrm{d}\alpha$, then $u(t)\neq 0$ for every $t\in[a,b]$.

Proof. Assume that there is a number $c\in(a,b]$ with $u(c)=0$. Then multiplying both sides of (5.2) by u and integrating we have

$$\int_a^c uu''_\alpha\,\mathrm{d}\alpha+\int_a^c puu'\,\mathrm{d}\alpha+\int_a^c qu^2\,\mathrm{d}\alpha=0. \tag{5.9}$$

Using Theorem 3.3, Lemma 3.5 and Theorem 3.6, and $u(c)=Q(a)=0$, we get, since, by Theorem 2.9 and Remark 3.7, Q is left-continuous on $[a,b]$, and $\Delta\,\alpha(t_k)=\Delta^+\,\alpha(t_k)=1, q(t_k)=0,\ k=\overline{1,m}$,

(5.10)
$$\begin{aligned}
&\int_a^c qu^2\,\mathrm{d}\alpha=\int_a^c Q'_\alpha u^2\,\mathrm{d}\alpha\\
&=\left[Qu^2\right]_a^c-\int_a^c Q(u^2)'_\alpha\,\mathrm{d}\alpha-\sum_{a\le t_k<c}\Delta^+Q(t_k)\Delta^+u^2(t_k),\ \text{since }\Delta^-\mathrm{Q}(\mathrm{t_k})=0,\\
&=-\int_a^c Q(u^2)'_\alpha\,\mathrm{d}\alpha-\sum_{a\le t_k<c}q(t_k)(u^2)'_\alpha(t_k),\ \text{by Theorem2.9}\\
&=-\int_a^c Q(u+u_+)u'_\alpha\,\mathrm{d}\alpha.
\end{aligned}$$

Since both u and u' are left-continuous

$$\begin{aligned}
\Delta^+u'(t_k)&=\Delta u'(t_k)=a_ku'(t_k),\\
\Delta^+u(t_k)&=\Delta u(t_k)=b_ku'(t_k).
\end{aligned}$$

By Lemma 3.8 and Lemma 5.1, we get, since $u(c)=u'(a)=0$,

$$\int_a^c uu''_\alpha\,\mathrm{d}\alpha=\int_a^c u(u')'_\alpha\,\mathrm{d}\alpha \tag{5.11}$$

$$
\begin{aligned}
&= \big[uu'\big]_a^c - \int_a^c u'_\alpha u' \,\mathrm{d}\alpha - \sum_{a\le t_k<c} \Delta^+ u(t_k)\Delta^+ u'(t_k), \text{ since } \Delta^- \mathrm{u(t_k)} = 0 \\
&= -\int_a^c u'_\alpha u' \,\mathrm{d}\alpha - \sum_{a\le t_k<c} a_k b_k |u'(t_k)|^2 \\
&= -\int_a^c |u'_\alpha|^2 \,\mathrm{d}\alpha - \sum_{a\le t_k<c} (b_k - b_k^2)|u'(t_k)|^2 - \sum_{a\le t_k<c} a_k b_k |u'(t_k)|^2 \\
&= -\int_a^c |u'_\alpha|^2 \,\mathrm{d}\alpha - \sum_{a\le t_k<c} b_k(1 - b_k + a_k)|u'(t_k)|^2.
\end{aligned}
$$

By (5.9), (5.10) and (5.11), we have

$$
\begin{aligned}
&\int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha + \int_a^c Q(u+u_+)u'_\alpha \,\mathrm{d}\alpha \\
&- \int_a^c puu' \,\mathrm{d}\alpha + \sum_{a\le t_k<c} b_k(1 - b_k + a_k)|u'(t_k)|^2 = 0.
\end{aligned}
$$

Hence by Theorem 4.3 and Lemma 5.1, we get

$$
\begin{aligned}
\int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha &\le \int_a^c |Q|(|u| + |u_+|)|u'_\alpha| \,\mathrm{d}\alpha + \int_a^c |p||uu'| \,\mathrm{d}\alpha \\
&\quad + \sum_{a\le t_k<c} |b_k||1 - b_k + a_k||u'(t_k)|^2 \\
&\le \max_{a\le s\le c} |Q(s)|[\,\alpha(c) - \alpha(a)] \int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha + \max_{a\le t_k\le c} |p(t_k)| \int_a^c |uu'| \,\mathrm{d}\alpha \\
&\quad + \sum_{a\le t_k<c} \frac{|1 - b_k + a_k|}{|b_k|} b_k^2 |u'(t_k)|^2 \\
&\le \max_{a\le s\le c} |Q(s)|[\,\alpha(c) - \alpha(a)] \int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha + \max_{a\le t_k\le c} |a_k| \int_a^c |uu'| \,\mathrm{d}\alpha \\
&\quad + \max_{a\le t_k\le c} \frac{|1 - b_k + a_k|}{|b_k|} \sum_{a\le t_k<c} b_k^2 |u'(t_k)|^2 \\
&\le \max_{a\le s\le c} |Q(s)|[\,\alpha(c) - \alpha(a)] \int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha \\
&\quad + \max_{a\le t_k\le c} |a_k| \left(\int_a^c |uu'_\alpha| \,\mathrm{d}\alpha + \sum_{a\le t_k<c} |1 - |b_k|||u(t_k)u'(t_k)| \right) \\
&\quad + \max_{a\le t_k\le c} \frac{|1 - b_k + a_k|}{|b_k|} \int_a^c (u'_\alpha)^2 \,\mathrm{d}\alpha
\end{aligned} \tag{5.12}
$$

$$\leq \max_{a\leq s\leq c}|Q(s)|[\,\alpha(c)-\,\alpha(a)]\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$+\max_{a\leq t_k\leq c}|a_k|\left(\int_a^c |uu'_\alpha|\,\mathrm{d}\alpha+\sum_{a\leq t_k<c}\frac{|1-|b_k||}{|b_k|}|b_k||u(t_k)u'(t_k)|\right)$$
$$+\max_{a\leq t_k\leq c}\frac{|1-b_k+a_k|}{|b_k|}\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$\leq \max_{a\leq s\leq c}|Q(s)|[\,\alpha(c)-\,\alpha(a)]\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$+\max_{a\leq t_k\leq c}|a_k|\left(\int_a^c |uu'_\alpha|\,\mathrm{d}\alpha+\max_{a\leq t_k\leq c}\frac{|1-|b_k||}{|b_k|}\int_a^c |uu'_\alpha|\,\mathrm{d}\alpha\right)$$
$$+\max_{a\leq t_k\leq c}\frac{|1-b_k+a_k|}{|b_k|}\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$\leq \max_{a\leq s\leq c}|Q(s)|[\,\alpha(c)-\,\alpha(a)]\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$+\max_{a\leq t_k\leq c}|a_k|[\,\alpha(c)-\,\alpha(a)]\left(1+\max_{a\leq t_k\leq c}\frac{|1-|b_k||}{|b_k|}\right)\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$$
$$+\max_{a\leq t_k\leq c}\frac{|1-b_k+a_k|}{|b_k|}\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha.$$

If
$$0=\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha=\int_a^c (u'_\alpha)^2(s)\,\mathrm{d}s+\sum_{a\leq t_k<c}(u'_\alpha)^2(t_k),$$
then, since $0=\int_a^c (u'_\alpha)^2(s)\,\mathrm{d}s=\int_a^c (u')^2(s)\,\mathrm{d}s$, $u'(t)=0, \forall t\in[a,b]-\{t_k : t_k<c\}$ and $u'_\alpha(t_k)=u(t_k+)-u(t_k)=0$. This implies that u is a constant on $[a,c]$. So $u(c)=u(a)\neq 0$. But this is a contradiction to $u(c)=0$. Hence we conclude that $\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha>0$.

In (5.12), canceling $\int_a^c (u'_\alpha)^2\,\mathrm{d}\alpha$, we get a contradiction to (5.8). This completes the proof. $\square$

In the following result we apply Theorem 4.4.

Theorem 5.3. *Let $q\in PC([a,b])$ and let $\alpha=\phi$ (see* (3.2)*).* If $u\in PC([a,b])$ is left-continuous and a nontrivial solution of the following equation:
$$(u'_\alpha)^m+\frac{q(t)u^{m+1}}{1+|u|+(u')^2}=0,\quad u(b)=0,\ (m=1,3,5,...),$$
then we have
$$1\leq I(m,1)\max_{a\leq s\leq b}|q(s)|. \tag{5.13}$$

Proof. Substituting $f \equiv 1, p = 0, q = 1, r = 0$ into Theorem 4.4, then we have

$$\int_a^b |u|^m \, \mathrm{d}\alpha \le I(m,1) \int_a^b |u'_\alpha|^m \, \mathrm{d}\alpha.$$

So we have

$$\begin{aligned}\int_a^b |u'_\alpha|^m \, \mathrm{d}\alpha &\le \int_a^b \frac{|q||u|^{m+1}}{1+|u|+(u')^2} \, \mathrm{d}\alpha \le \int_a^b |q||u|^m \, \mathrm{d}\alpha \\ &\le I(m,1) \max_{a\le s\le b} |q(s)| \int_a^b |u'_\alpha|^m \, \mathrm{d}\alpha.\end{aligned}$$

Canceling $\int_a^b |u'_\alpha|^m \, \mathrm{d}\alpha$, we get (5.13). □

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