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JACOBI-TRUDI TYPE FORMULA FOR PARABOLICALLY SEMISTANDARD TABLEAUX

Jee-Hye Kim

Abstract. The notion of a parabolically semistandard tableau is a generalisation of Young tableau, which explains combinatorial aspect of various Howe dualities of type A. We prove a Jacobi-Trudi type formula for the character of parabolically semistandard tableaux of a given generalised partition shape by using non-intersecting lattice paths.

1. Introduction

A Schur polynomial is a symmetric polynomial, which plays an important role in algebraic combinatorics and representation theory (we refer the reader to [6, 15, 16] for general exposition on Schur polynomials). Let $x_1, \ldots, x_n$ be mutually commuting n variables. Let $s_\lambda(x_1, \ldots, x_n)$ be the Schur polynomial corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. It is well-known that $s_\lambda(x_1, \ldots, x_n)$ is the character of a complex irreducible polynomial representation of the general linear group $GL_n(\mathbb{C})$ whose highest weight corresponds to λ. There are several equivalent definitions of $s_\lambda(x_1, \ldots, x_n)$. One is the celebrated Weyl character formula, which has been extended to the case of a symmetrizable Kac-Moody algebra [9]. There is a combinatorial formula, where $s_\lambda(x_1, \ldots, x_n)$ is given as the weight generating function of the set of Young tableaux of shape λ. Another well-known one is the Jacobi-Trudi formula

$$s_\lambda(x_1, \ldots, x_n) = \det(h_{\lambda_i - i + j}(x_1, \ldots, x_n))_{1 \le i,j \le n},$$

where $h_k(x_1, \ldots, x_n)$ is the kth complete symmetric polynomial. While the above formula is originally due to Jacobi, Gessel and Viennot introduced a new interesting

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proof in terms of non-intersecting lattice paths [7], which has resulted in various generalizations and applications in combinatorics.

In [12], Kwon introduced a new combinatorial object, which we call parabolically semistandard tableaux, in order to understand the combinatorial aspect of Howe duality of type A [8]. For a generalized partition λ of length n, the weight generating function $\mathcal{S}_\lambda$ of parabolically semistandard tableaux of shape λ gives the character of an irreducible representation of a general linear Lie (super)algebra $\mathfrak{g}$, which arises from $(\mathfrak{g}, GL_n(\mathbb{C}))$-duality on various Fock spaces. The character $\mathcal{S}_\lambda$ includes a usual Schur polynomial as a special case, and it also has a Weyl-Kac type character formula, and a Jacobi-Trudi type formula (see also [13]).

The goal of this paper is to show a Jacobi-Trudi type formula for $\mathcal{S}_\lambda$ by using non-intersecting lattice paths. Since a parabolically semistandard tableau is roughly speaking a pair (S,T) of skew-shaped Young tableaux with a common inner shape and each component corresponds to an n-tuple of non-intersecting lattice paths, the pair (S,T) corresponds to an n-tuple of non-intersecting zigzag-shaped lattice paths which is obtained by gluing non-intersecting paths associated to S and T. This is our key observation. Then we apply the arguments similar to [7] to obtain a Jacobi-Trudi type formula for $\mathcal{S}_\lambda$.

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2. Parabolically Semistandard Tableaux

2.1. Young tableaux Let us briefly recall necessary background on Young tableaux (see [6] for more details). We denote by $\mathbb{Z}$ and $\mathbb{Z}_{>0}$ the set of integers and positive integers, respectively. A *partition* is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\sum_{i\geq 1} \lambda_i$ is finite. We say that λ is a partition of n if $\sum_{i\geq 1} \lambda_i = n$ and denote by $\ell(\lambda)$ the number of positive entries of λ. Let $\mathscr{P}$ be the set of all partitions, and put $\mathscr{P}_n = \{\, \lambda \in \mathscr{P} \,|\, \ell(\lambda) \leq n \,\}$ for $n \geq 1$.

A *Young diagram* is a collection of boxes arranged in left-justified row, with weakly decreasing number of boxes in each row from top to bottom. A Young diagram determines a unique partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, where λ_i is the number of boxes in the ith row of the diagram. From now on, we identify a Young diagram with its partition.

Example 2.1. The Young diagram corresponding to the partition $\lambda = (5, 3, 3, 1)$ is

5
3
3
1

Let $\lambda \in \mathscr{P}$ be given. A *Young tableau* T is a filling of λ or the boxes in its Young diagram with positive integers such that the entries are weakly increasing from left to right in each row, and strictly increasing from top to bottom in each column. We say that λ is the *shape* of T, and write $\mathrm{sh}(T) = \lambda$.

Example 2.2. For $\lambda = (5, 3, 3, 1)$

1	2	2	3	3
3	3	4		
4	4	5		
5				

is a Young tableau of shape λ.

For $\mu \in \mathscr{P}$ with $\lambda \supset \mu$ (that is, $\lambda_i \geq \mu_i$ for all i), λ/μ denotes the *skew Young diagram.* A *skew Young tableau* is a filling of a skew Young diagram λ/μ with positive integers in the same way as in the case of Young tableaux.

Example 2.3. For $\lambda/\mu = (5, 3, 3, 1)/(2, 1)$,

		1	2	4
	3	3		
1	4	5		
2				

is a skew Young tableau of shape λ/μ.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of formal commuting variables. For a Young tableau T, we put $\mathbf{x}^T = \prod_{i \geq 1} x_i^{m_i}$, where m_i is the number of times i occurs in T. For T in Example 2.2, we have $\mathbf{x}^T = x_1 x_2^2 x_3^4 x_4^3 x_5^2$. Let $s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$ be the *Schur function* corresponding to $\lambda \in \mathscr{P}$, where the sum is over all Young tableaux T of $sh(T) = \lambda$. For $k \geq 0$, let $h_k(\mathbf{x}) = s_{(k)}(\mathbf{x})$, which is called the kth *complete symmetric function.* For $\mu \in \mathscr{P}$, we put $h_\mu(\mathbf{x}) = h_{\mu_1}(\mathbf{x}) h_{\mu_2}(\mathbf{x}) \ldots$.

There is another well-known equivalent definition of a Schur function called the *Jacobi-Trudi formula*, which expresses a Schur function as a determinant, and hence as a linear combination of $h_\mu(\mathbf{x})$'s for $\mu \in \mathscr{P}$ (cf. [6]).

Theorem 2.4. *For $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$,*

$$s_\lambda(\mathbf{x}) = \det(h_{\lambda_i - i + j}(\mathbf{x}))_{1 \leq i,j \leq n},$$

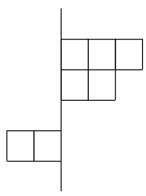
where we assume that $h_{-k}(\mathbf{x}) = 0$ for $k \geq 1$.

2.2. Parabolically semistandard tableaux Let $\mathcal{A}$ be a linearly ordered countable set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For $a \in \mathcal{A}$, a is called *even* (resp. *odd*) if $a \in \mathcal{A}_0$ (resp. $a \in \mathcal{A}_1$). Let λ/μ be a skew Young diagram. A tableau T obtained by filling λ/μ with entries in $\mathcal{A}$ is called *$\mathcal{A}$-semistandard* if the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), and the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row). We say that λ/μ is the *shape of* T, and write $\mathrm{sh}(T) = \lambda/\mu$. We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all $\mathcal{A}$-semistandard tableaux of shape λ/μ. We set $\mathscr{P}_{\mathcal{A}} = \{\, \lambda \in \mathscr{P} \,|\, SST_{\mathcal{A}}(\lambda) \neq \emptyset \,\}$. Let $\mathbf{x}_{\mathcal{A}} = \{\, x_a \,|\, a \in \mathcal{A} \,\}$ be a set of formal commuting variables indexed by $\mathcal{A}$. For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, put $\mathbf{x}_{\mathcal{A}}^T = \prod_{a \in \mathcal{A}} x_a^{m_a}$, where m_a is the number of occurrences of a in T. We define the character of $SST_{\mathcal{A}}(\lambda/\mu)$ to be $s_{\lambda/\mu}(\mathbf{x}_{\mathcal{A}}) = \sum_{T \in SST_{\mathcal{A}}(\lambda/\mu)} \mathbf{x}_{\mathcal{A}}^T$.

We assume that $\mathbb{Z}_{>0}$ is given with a usual linear ordering and all entries even. When $\mathcal{A} = \mathbb{Z}_{>0}$, an $\mathcal{A}$-semistandard tableau is a (skew) Young tableau, and $s_\lambda(\mathbf{x}_{\mathcal{A}})$ is the Schur function associated to $\lambda \in \mathscr{P}$.

Let $\mathbb{Z}_+^n = \{\, \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \,|\, \lambda_1 \geq \cdots \geq \lambda_n \,\}$ be the set of all *generalized partitions* of length n. We may identify λ with a *generalized Young diagram* as in the following example.

Example 2.5. The generalized partition $\lambda = (3, 2, 0, -2) \in \mathbb{Z}_+^4$ corresponds to



Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two disjoint linearly ordered $\mathbb{Z}_2$-graded countable sets. Now, let us introduce our main combinatorial object.

Definition 2.6 ([12])**.** For $\lambda \in \mathbb{Z}_+^n$, a *parabolically semistandard tableau of shape λ* (with respect to $(\mathcal{A}, \mathcal{B})$) is a pair of tableaux (T^+, T^-) such that

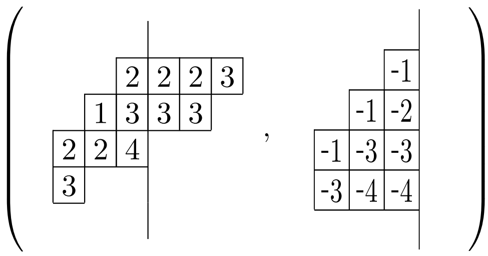
$$T^+ \in SST_{\mathcal{A}}((\lambda + (d^n))/\mu), \quad T^- \in SST_{\mathcal{B}}((d^n)/\mu),$$

for some integer $d \geq 0$ and $\mu \in \mathscr{P}_n$ satisfying (1) $\lambda + (d^n) \in \mathscr{P}_n$, and (2) $\mu \subset (d^n), \mu \subset \lambda + (d^n)$. We denote by $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ the set of all parabolically semistandard tableaux of shape λ with respect to $(\mathcal{A}, \mathcal{B})$.

Roughly speaking, a parabolically semistandard tableau of shape λ is a pair of $\mathcal{A}$-semistandard tableau and $\mathcal{B}$-semistandard tableau whose shapes are not necessarily fixed ones but satisfy certain conditions determined by λ.

Example 2.7. Suppose that $\mathcal{A} = \mathbb{Z}_{>0} = \{ 1 < 2 < 3 < \dots \}$ and $\mathcal{B} = \mathbb{Z}_{<0} = \{ -1 < -2 < -3 < \dots \}$ with all entries even. Then

$(T^+, T^-) =$



$\in SST_{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}((3,2,0,-2))$

where the vertical lines in T^+ and T^- correspond to the one in the generalized partition $(3,2,0,-2)$. In this case, we have $\mathrm{sh}(T^+) = \big((3,2,0,-2) + (3^4)\big)/(2,1,0,0)$, and $\mathrm{sh}(T^-) = (3^4)/(2,1,0,0)$.

For $\lambda \in \mathbb{Z}^n_+$, we define the *character of* $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ to be

$$S_\lambda^{\mathcal{A}/\mathcal{B}} = \sum_{(T^+,T^-) \in SST_{\mathcal{A}/\mathcal{B}}(\lambda)} \mathbf{x}_{\mathcal{A}}^{T^+} (\mathbf{x}_{\mathcal{B}}^{T^-})^{-1}. \tag{1}$$

We put $\mathscr{P}_{\mathcal{A}/\mathcal{B},n} = \{ \lambda \in \mathbb{Z}^n_+ \mid SST_{\mathcal{A}/\mathcal{B}}(\lambda) \neq \emptyset \}$. For $k \in \mathbb{Z}$, we put $S_k^{\mathcal{A}/\mathcal{B}} = S_{(k)}^{\mathcal{A}/\mathcal{B}}$.

2.3. Irreducible characters Let us briefly recall a representation theoretic meaning of parabolically semistandard tableaux. For an arbitrary $\mathbb{Z}_2$-graded linearly ordered set $\mathcal{C}$, we denote by $V_{\mathcal{C}}$ a superspace with basis $\{ v_c \mid c \in \mathcal{C} \}$, and let $\mathfrak{gl}(V_{\mathcal{C}})$ be the general linear Lie superalgebra spanned by $E_{cc'}$ for $c, c' \in \mathcal{C}$. Here $E_{cc'}$ is the matrix where the entry at (c, c')-position is 1 and 0 elsewhere.

Let $\mathfrak{g} = \mathfrak{gl}(V_{\mathcal{C}})$ with $\mathcal{C} = \mathcal{B} * \mathcal{A}$, where $\mathcal{B} * \mathcal{A}$ is the $\mathbb{Z}_2$-graded set $\mathcal{A} \sqcup \mathcal{B}$ with the extended linear ordering defined by $y < x$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Let

$$\mathscr{F} = S(V_{\mathcal{A}} \oplus V_{\mathcal{B}}^{\vee})$$

be the super symmetric algebra generated by $V_{\mathcal{A}} \oplus V_{\mathcal{B}}^{\vee}$, where $V_{\mathcal{B}}^{\vee}$ is the restricted dual space of $V_{\mathcal{B}}$. One can define a semisimple action of $\mathfrak{g}$ on $\mathscr{F}$, and a semisimple action of $GL_n(\mathbb{C})$ on $\mathscr{F}^{\otimes n}$ for $n \geq 1$ so that we have the following multiplicity-free decomposition as a $(\mathfrak{g}, GL_n(\mathbb{C}))$-module,

$$\mathscr{F}^{\otimes n} \cong \bigoplus_{\lambda \in H_n} L(\lambda) \otimes L_n(\lambda), \tag{2}$$

for a subset H_n of $\mathbb{Z}_+^n$, where $L_n(\lambda)$ is the irreducible $GL_n(\mathbb{C})$-module with highest weight $\lambda \in H_n$, and $L(\lambda)$ is an irreducible $\mathfrak{g}$-module corresponding to $L_n(\lambda)$ (see the arguments in [4, Sections 5.1 and 5.4]). We define the character $\mathrm{ch}L(\lambda)$ to be the trace of the operator $\prod_{c \in \mathcal{C}} x_c^{E_{cc}}$ on $L(\lambda)$ for $\lambda \in H_n$. Finally from a Cauchy type identity for parabolically semistandard tableaux [12, Theorem 4.1], we can conclude the following (cf. [14, Theorem 2.3]).

Theorem 2.8. *For $n \geq 1$, we have*

$$\mathscr{F}^{\otimes n} \cong \bigoplus_{\lambda \in \mathscr{P}_{\mathcal{A}/\mathcal{B},n}} L(\lambda) \otimes L_n(\lambda),$$

as a $(\mathfrak{g}, GL_n(\mathbb{C}))$-module, that is, $H_n = \mathscr{P}_{\mathcal{A}/\mathcal{B},n}$, and the irreducible character $\mathrm{ch}L(\lambda)$ is given by $S_\lambda^{\mathcal{A}/\mathcal{B}}$ for $\lambda \in \mathscr{P}_{\mathcal{A}/\mathcal{B},n}$.

Recall that when $\mathcal{A}$ is finite with $\mathcal{A} = \mathcal{A}_0$ or $\mathcal{A}_1$ and $\mathcal{B} = \emptyset$, the decomposition in Theorem 2.8 is the classical $(GL_\ell(\mathbb{C}), GL_n(\mathbb{C}))$-Howe duality on the symmetric algebra or exterior algebra generated by $\mathbb{C}^\ell \otimes \mathbb{C}^n$, where $\ell = |\mathcal{A}|$ (cf. [8]). Moreover, the decomposition in Theorem 2.8 includes other Howe dualities of type A which have been studied in [1, 2, 3, 5, 8, 10, 11] under suitable choices of $\mathcal{A}$ and $\mathcal{B}$ (see [12] for more details).

3. JACOBI-TRUDI FORMULA

3.1. Lattice paths

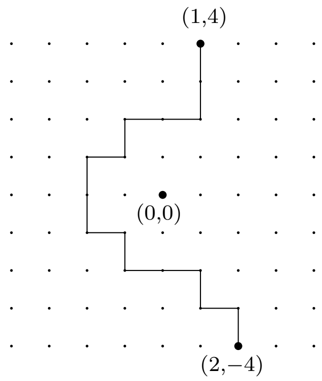
Definition 3.1. A *lattice path* is a sequence

$$p = v_1 ... v_r$$

of points $v_1, ..., v_r$ in $\mathbb{Z} \times \mathbb{Z}$ with $v_i = (a_i, b_i)$ such that $b_1 < 0 < b_r$, and

$$v_{i+1} - v_i = \begin{cases} (0,1) \text{ or } (-1,0), & \text{for } 1 \leq i < r \text{ with } b_i, b_{i+1} < 0, \\ (0,1) \text{ or } (1,0), & \text{for } 1 \leq i < r \text{ with } b_i, b_{i+1} > 0, \\ (0,1), & \text{for } 1 \leq i < r \text{ with } b_i = 0 \text{ or } b_{i+1} = 0. \end{cases}$$

Example 3.2. The following path



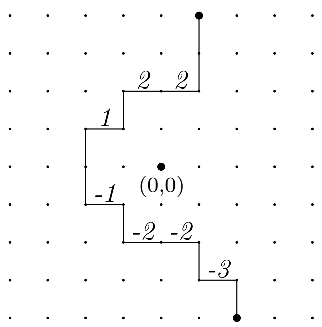
is the lattice path

$$p = (2,-4)(2,-3)(1,-3)\dots(-2,-1)(-2,0)(-2,1)\dots(1,2)(1,3)(1,4).$$

We denote by $\mathcal{P}$ the set of lattice paths. Let $p = v_1 ... v_r \in \mathcal{P}$ be given with $v_i = (a_i, b_i)$ for $1 \le i \le r$. We often identify p with its extended lattice path $v_0 v_1 ... v_r v_{r+1}$, where $v_0 = (a_1, -\infty)$ and $v_{r+1} = (a_r, \infty)$. Here we regard $(a_1, -\infty)$ as a point below (a_1, y) for all $y \le b_1$, and (a_r, ∞) as a point above (a_r, y) for all $y \ge b_r$. We also write $p : v_0 \longrightarrow v_{r+1}$. For $0 \le i \le r$, let $v_i v_{i+1}$ denote the line segment joining v_i and v_{i+1}, where we understand $v_0 v_1$ (resp. $v_r v_{r+1}$) as an half-infinite line joining (a_1, b_1) and $(a_1, -\infty)$ (resp. (a_r, b_r) and (a_r, ∞)). Let $\mathbf{z} = \{\, z_i \,|\, i \in \mathbb{Z}^\times \,\}$ be a set of formal commuting variables, where $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$. We consider a weight monomial

$$\mathbf{z}^p = \prod_{v_i v_{i+1}:\ \text{horizontal}} z_{b_i}.$$

Example 3.3. For a lattice path

$p =$ 

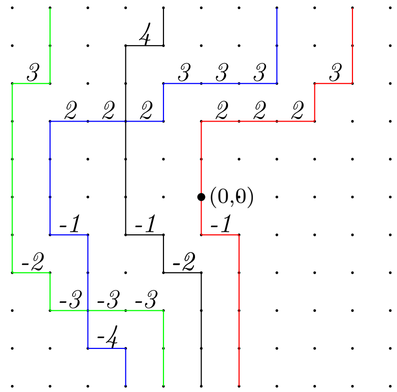
its weight monomial is $\mathbf{z}^p = z_1 z_2^2 z_{-1} z_{-2}^2 z_{-3}$ (the numbers on the horizontal line segments denote their y-coordinates in $\mathbb{Z} \times \mathbb{Z}$).

Fix a positive integer n. Let S_n be the group of permutations on n letters. Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ be given with $\alpha_1 > \ldots > \alpha_n$ and $\beta_1 > \ldots > \beta_n$. We define

$$\mathcal{P}(\alpha,\beta) = \left\{ \mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{P}^n \;\middle|\; \begin{array}{l} \text{there exists } \pi \in S_n \text{ such that} \\ p_i : (\alpha_i, -\infty) \to (\beta_{\pi(i)}, \infty) \text{ for } 1 \le i \le n \end{array} \right\}.$$

Put $\mathbf{z}^{\mathbf{p}} = \prod_i \mathbf{z}^{p_i}$, and $(-1)^{\mathbf{p}} = sgn(\pi)$ for $\mathbf{p} \in \mathcal{P}(\alpha,\beta)$ with its associated permutation $\pi \in S_n$.

Example 3.4. Let $n = 4$, $\alpha = (1, 0, -1, -2)$ and $\beta = (4, 2, -1, -4)$. Then

$\mathbf{p} =$ 

$\in \mathcal{P}(\alpha,\beta)$

with the associated permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

A weight monomial of $\mathbf{p}$ is

$$\begin{aligned} \mathbf{z}^{\mathbf{p}} &= (z_{-1} z_2^3 z_3)(z_{-2} z_{-1} z_4)(z_{-3}^3 z_{-2} z_3)(z_{-4} z_{-1} z_2^3 z_3^3) \\ &= z_{-4} z_{-3}^3 z_{-2}^2 z_{-1}^3 z_2^6 z_3^5 z_4 \end{aligned}$$

and $(-1)^{\mathbf{p}} = sgn(\pi) = 1$.

Let us define a map

$$\phi : \mathcal{P}(\alpha,\beta) \longrightarrow \mathcal{P}(\alpha,\beta)$$

as follows; for $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}(\alpha,\beta)$

(1) If $p_i \cap p_j = \emptyset$ for all $1 \le i \ne j \le n$, then $\phi(\mathbf{p}) = \mathbf{p}$.

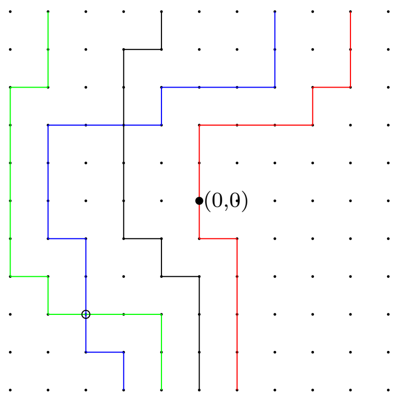
(2) Otherwise, we choose the largest i such that p_i has an intersection point w with p_j for some $i > j$, and assume that w is the first intersection point appearing in p_i from the bottom. Then we define $\phi(\mathbf{p})$ to be the n-tuple of

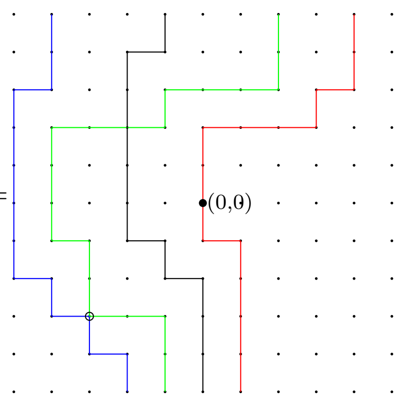
paths obtained from **p** by replacing

$$\begin{cases} p_i = (u_1 \cdots \ u_p)(u_{p+1} \cdots \ u_r) \\ p_j = (v_1 \cdots \ v_q)(v_{q+1} \cdots \ v_s) \end{cases} \quad \textit{with} \quad \begin{cases} p_i = (u_1 \cdots \ u_p)(v_{q+1} \cdots \ v_s) \\ p_j = (v_1 \cdots \ v_q)(u_{p+1} \cdots \ u_r) \end{cases} ,$$

where $u_p = v_q = w$.

Example 3.5. Let **p** be as in Example 3.4. Then

$\mathbf{p} =$ 

$\phi(\mathbf{p}) =$ 

By definition of ϕ , we can check that for $\mathbf{p} \in \mathcal{P}(\alpha, \beta)$

(1) $\phi(\mathbf{p}) = \mathbf{p}$ if and only if **p** has no intersection point,
(2) $\phi^2(\mathbf{p}) = \mathbf{p}$,
(3) $\mathbf{z}^{\phi(\mathbf{p})} = \mathbf{z}^{\mathbf{p}}$,
(4) $(-1)^{\phi(\mathbf{p})} = -(-1)^{\mathbf{p}}$.

We put

$$\mathcal{P}_0(\alpha, \beta) = \{\, \mathbf{p} \,|\, \mathbf{p} \in \mathcal{P}(\alpha, \beta),\ \phi(\mathbf{p}) = \mathbf{p} \,\} \tag{3}$$

the set of fixed points in $\mathcal{P}(\alpha,\beta)$ under ϕ, or the subset of $\mathbf{p}$ in $\mathcal{P}(\alpha,\beta)$ with no intersection point.

For $\lambda \in \mathbb{Z}_+^n$, we define

$$\mathcal{S}_\lambda = \sum_{\mathbf{p}\in\mathcal{P}_0(\delta,\lambda+\delta)} \mathbf{z}^{\mathbf{p}}, \tag{4}$$

where $\delta = (0,-1,\ldots,-n+1)$ and $\lambda+\delta = (\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1)$. For $k \in \mathbb{Z}$, we put $\mathcal{S}_k = \mathcal{S}_{(k)}$. Then we have the following Jacobi-Trudi formula for $\mathcal{S}_\lambda$, which is an analogue of [12] for our zigzag-shaped lattice paths.

Proposition 3.6. *For $\lambda \in \mathbb{Z}_+^n$, we have*

$$\mathcal{S}_\lambda = \det\left(\mathcal{S}_{\lambda_i - i + j}\right)_{1\le i,j\le n}.$$

Proof. Recall from (4) that for $1 \le i,j \le n$

$$\mathcal{S}_{\lambda_i-i+j} = \sum_{p:(-j+1,-\infty)\to(\lambda_i-i+1,\infty)} \mathbf{z}^p,$$

where we shift the x-coordinates in $\mathbf{p}$ by $-j+1$. Thus

$$\begin{aligned}
\det(\mathcal{S}_{\lambda_i-i+j})_{1\le i,j\le n} &= \sum_{\pi\in S_n} sgn(\pi)\mathcal{S}_{\lambda_{\pi(1)}-\pi(1)+1}\cdots\mathcal{S}_{\lambda_{\pi(n)}-\pi(n)+n} \\
&= \sum_{\pi\in S_n} \sum_{\substack{\mathbf{p}\in\mathcal{P}(\delta,\lambda+\delta) \\ p_i:(-i+1,-\infty)\to(\lambda_{\pi(i)}-\pi(i)+1,\infty)}} sgn(\pi)\mathbf{z}^{\mathbf{p}} \\
&= \sum_{\mathbf{p}\in\mathcal{P}(\delta,\lambda+\delta)} (-1)^{\mathbf{p}}\mathbf{z}^{\mathbf{p}} \\
&= \sum_{\mathbf{p}\in\mathcal{P}_0(\delta,\lambda+\delta)} (-1)^{\mathbf{p}}\mathbf{z}^{\mathbf{p}} + \sum_{\mathbf{p}\notin\mathcal{P}_0(\delta,\lambda+\delta)} (-1)^{\mathbf{p}}\mathbf{z}^{\mathbf{p}}.
\end{aligned}$$

Since $\phi(\mathbf{p}) \ne \mathbf{p}$ for $\mathbf{p} \notin \mathcal{P}_0(\delta,\lambda+\delta)$ and $(-1)^{\phi(\mathbf{p})} = -(-1)^{\mathbf{p}}$, we have

$$\sum_{\mathbf{p}\notin\mathcal{P}_0(\delta,\lambda+\delta)} (-1)^{\mathbf{p}}\mathbf{z}^{\mathbf{p}} = 0.$$

Also note that $(-1)^{\mathbf{p}} = 1$ for $\mathbf{p} \in \mathcal{P}_0(\delta,\lambda+\delta)$. Therefore, we have

$$\det(\mathcal{S}_{\lambda_i-i+j})_{1\le i,j\le n} = \sum_{\mathbf{p}\in\mathcal{P}_0(\delta,\lambda+\delta)} \mathbf{z}^{\mathbf{p}} = \mathcal{S}_\lambda.$$

□

3.2. Non-intersecting paths and Young tableaux Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ be such that $\alpha_1 > \ldots > \alpha_n$, $\beta_1 > \ldots > \beta_n$ and $\alpha_i \le \beta_i$ for all i. Consider an n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$ of non-intersecting (extended) lattice paths where

$$p_i : (\alpha_i, 0) \longrightarrow (\beta_i, \infty) \tag{5}$$

for $1 \le i \le n$. Note that p_i is a lattice path starting from a point $(\alpha_i, 0)$, which is a upper half of a lattice path defined in Definition 3.1. Put $\delta = (0, -1, \ldots, -n+1)$. Choose $d \ge 0$ satisfying

$$\alpha - \delta + (d^n), \quad \beta - \delta + (d^n) \in \mathscr{P}_n.$$

If we put $\mu = \alpha - \delta + (d^n)$ and $\lambda = \beta - \delta + (d^n)$, then λ/μ is a skew Young diagram.

Now, associated to $\mathbf{p}$, we define a tableau T of shape λ/μ with entries in $\mathbb{Z}_{>0}$ as follows. For $1 \le i \le n$ with $\alpha_i < \beta_i$ and $1 \le j \le \beta_i - \alpha_i$, we fill the box in the ith row and jth column of λ/μ with k if

$$p_i = (\alpha_i, 0) \ldots (\alpha_i + j - 1, k)(\alpha_i + j, k) \ldots (\beta_i, \infty).$$

The following lemma is well-known [7]. But we give a detailed proof for the readers' convenience.

Lemma 3.7. *Under the above assumptions, T is $\mathbb{Z}_{>0}$-semistandard or a Young tableau of shape λ/μ.*

Proof. Fix $1 \le i \le n$. Let $T_{i,j}$ denote the jth (non-empty) entry of T (from the left) in the ith row (from the top) for $1 \le j \le \beta_i - \alpha_i$.

It is clear that the entries of T in each row are weakly increasing from left to right since the y-coordinates of each path $p_i : (\alpha_i, 0) \longrightarrow (\beta_i, \infty)$ are weakly increasing from bottom to top. Hence it is enough to show that the entries of T in each column are strictly increasing from top to bottom.

Fix $1 \le i < n$. Suppose first that $\alpha_i - \alpha_{i+1} = \ell \ge 1$. Then $\mu_i - \mu_{i+1} = \{\alpha_i - (-i+1) + d\} - \{\alpha_i - (-i) + d\} = (\alpha_i - \alpha_{i+1}) - 1 = \ell - 1$. This implies that $T_{i,j}$ and $T_{i+1,j+(\ell-1)}$ are in the same column in T for all j such that $T_{i,j}$ and $T_{i+1,j+(\ell-1)}$ are non-empty. The jth and $(j+\ell-1)$th horizontal line segments of p_i and p_{i+1} are given by

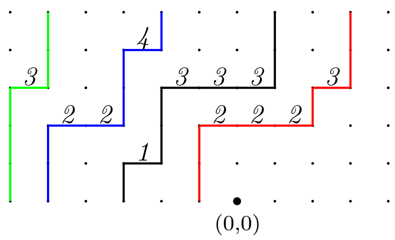
$$\begin{aligned}
&(\alpha_i + j - 1, k)(\alpha_i + j, k), \\
&(\alpha_{i+1} + j - 1 + (\ell - 1), k')(\alpha_{i+1} + j + (\ell - 1), k') \\
&\qquad\qquad = (\alpha_i + j - 2, k')(\alpha_i + j - 1, k')
\end{aligned}$$

respectively, where $k = T_{i,j}$ and $k' = T_{i+1,j+(\ell-1)}$. If $k \geq k'$, then the paths p_i and p_{i+1} necessarily have an intersection point, which is a contradiction. Therefore, $T_{i,j} < T_{i+1,j+(\ell-1)}$. □

Note that the shape of T does not depend on the choice of d, and the correspondence $\mathbf{p} \mapsto T$ gives a bijection between the set of non-intersecting paths satisfying (5) and $SST_{\mathbb{Z}_{>0}}(\lambda/\mu)$.

Example 3.8. Consider a quadruple of non-intersecting paths $\mathbf{p} = (p_1, p_2, p_3, p_4)$ with $\alpha = (-1, -3, -5, -6)$ and $\beta = (3, 1, -2, -5)$

$\mathbf{p} =$



Then the associated Young tableau is

T =

		2	2	2	3
	1	3	3	3	
2	2	4			
3					

Now, consider parabolically semistandard tableaux, where $\mathcal{A} = \mathbb{Z}_{>0} = \{1 < 2 < 3 < \dots\}$ and $\mathcal{B} = \mathbb{Z}_{<0} = \{-1 < -2 < -3 < \dots\}$ with all entries even. Note that the linear ordering on $\mathcal{B}$ is a reverse ordering of the usual one. Then we have

Proposition 3.9. *For $\lambda \in \mathbb{Z}_+^n$, there exists a bijection*

$$\psi : \mathcal{P}_0(\delta, \lambda + \delta) \longrightarrow SST_{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}(\lambda).$$

Proof. Let $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_0(\delta, \lambda + \delta)$ be given with

$$p_i = (-i+1, -\infty) \dots (\gamma_i, 0) \dots (\lambda_i - i + 1, \infty)$$

for some $\gamma_i \in \mathbb{Z}$ $(1 \leq i \leq n)$. Then we put $\mathbf{p}^+ = (p_1^+, \dots, p_n^+)$, where $p_i^+ = (\gamma_i, 0) \dots (\lambda_i - i + 1, \infty)$, an upper half of p_i with the vertices having non-negative second components, and put $\mathbf{p}^- = (p_1^-, \dots, p_n^-)$, where $p_i^- = (-i+1, -\infty)\dots(\gamma_i, 0)$, the lower half of p_i.

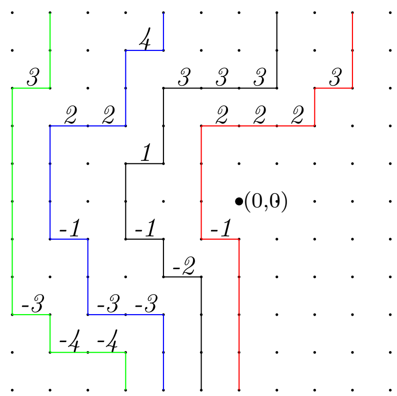
Choose $d \geq 0$ such that $\gamma - \delta + (d^n)$, $(\lambda + \delta) - \delta + (d^n) \in \mathscr{P}_n$. First, as in Lemma 3.7, we may associate a Young tableau T^+ of shape $(\lambda + (d^n))/\mu$ where $\mu = \gamma - \delta + (d^n)$.

Let $\mathbf{p}^{-*} = (p_1^{-*}, \dots, p_n^{-*})$, where p_i^{-*} is obtained by reversing the order of the vertices in p_i^- and changing the sign of their second components. By the same argument, we may associate a Young tableau of shape $(d^n)/\mu$, and then replace an entry k with $-k$ once again to get a $\mathbb{Z}_{<0}$-semistandard tableau T^- of $(d^n)/\mu$.

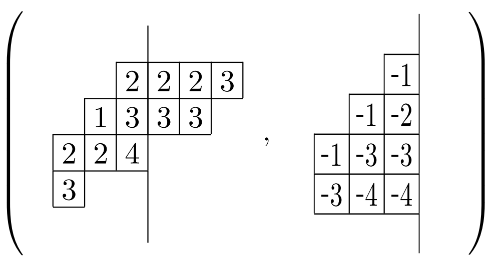
We define a map $\psi : \mathcal{P}_0(\delta, \lambda+\delta) \to SST_{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}(\lambda)$ by $\psi(\mathbf{p}) = (T^+, T^-)$. Since the correspondence $\mathbf{p} \mapsto (T^+, T^-)$ is reversible, ψ is a bijection. $\square$

Remark 3.10. The bijection ψ in Proposition 3.9 preserves weight in the following sense: If $(T^+, T^-) = \psi(\mathbf{p})$ for $\mathbf{p} \in \mathcal{P}_0(\delta, \lambda+\delta)$, then $\mathbf{z}^{\mathbf{p}} = \mathbf{x}_{\mathbb{Z}_{>0}}^{T^+}\left(\mathbf{x}_{\mathbb{Z}_{<0}}^{T^-}\right)^{-1}$, where we assume that $z_k = x_k$ and $z_{-k} = x_{-k}^{-1}$ for $k \geq 1$.

Example 3.11. Let $\mathbf{p} \in \mathcal{P}_0(\delta, \lambda+\delta)$ be a 4-tuple of lattice paths with $\delta = (0, -1, -2, -3)$ and $\lambda + \delta = (3, 1, -2, -5)$ as follows.

$\mathbf{p} =$ 

Then

$(T^+, T^-) =$  $\in SST_{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}((3, 2, 0, -2)).$

Now, we are in a position to prove our main theorem.

Theorem 3.12. *For $\lambda \in \mathbb{Z}_+^n$, we have*

$$S_\lambda^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}} = \det(S_{\lambda_i-i+j}^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}})_{1\leq i,j\leq n}.$$

Proof. Let us assume that $z_k = x_k$ and $z_{-k} = x_{-k}^{-1}$ for $k \geq 1$. Then we have $\mathcal{S}_l = S_l^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}$ for $l \in \mathbb{Z}$. So by Proposition 3.6, we have

$$\mathcal{S}_\lambda = \det(\mathcal{S}_{\lambda_i-i+j})_{1\leq i,j\leq n} = \det(S_{\lambda_i-i+j}^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}})_{1\leq i,j\leq n}. \tag{6}$$

On the other hand, by Proposition 3.9 and Remark 3.10 we have

$$\mathcal{S}_\lambda = S_\lambda^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}}. \tag{7}$$

Combining (6) and (7), we obtain

$$S_\lambda^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}} = \det(S_{\lambda_i-i+j}^{\mathbb{Z}_{>0}/\mathbb{Z}_{<0}})_{1\leq i,j\leq n}.$$

This completes the proof. □

3.3. General cases for $\mathcal{A}$ and $\mathcal{B}$ In this subsection, we prove that Theorem 3.12 can be naturally extended to the case of $S_\lambda^{\mathcal{A}/\mathcal{B}}$, where $\mathcal{A} = \mathbb{Z}_{>0} = \{ 1 < 2 < 3 < \dots \}$ and $\mathcal{B} = \mathbb{Z}_{<0} = \{ -1 < -2 < -3 < \dots \}$ with arbitrary $\mathbb{Z}_2$-gradings.

For this, we consider a lattice path $p = v_1 ... v_r$ of points $v_1, ..., v_r$ in $\mathbb{Z} \times \mathbb{Z}$ with $v_i = (s_i, t_i)$ satisfying the following conditions:

(1) $t_1 < 0 < t_r$,

(2) if $t_i \neq 0$ and $t_i \in \mathcal{A}_0 \sqcup \mathcal{B}_0$, then

$$v_{i+1} - v_i = \begin{cases} (0,1) \text{ or } (-1,0), & \text{for } 1 \leq i < r \text{ with } t_i, t_{i+1} < 0, \\ (0,1) \text{ or } (1,0), & \text{for } 1 \leq i < r \text{ with } t_i, t_{i+1} > 0, \end{cases}$$

(3) if $t_i \neq 0$ and $t_i \in \mathcal{A}_1 \sqcup \mathcal{B}_1$, then

$$v_{i+1} - v_i = \begin{cases} (0,1) \text{ or } (-1,1), & \text{for } 1 \leq i < r \text{ with } t_i, t_{i+1} < 0, \\ (0,1) \text{ or } (1,1), & \text{for } 1 \leq i < r \text{ with } t_i, t_{i+1} > 0, \end{cases}$$

(4) if $t_i = 0$, then $v_{i+1} - v_i = (0, 1)$.

We may define the notion of an extended path in the same way as in Section 3.1, and accordingly $\mathcal{P}(\alpha, \beta)$, the involution ϕ, and $\mathcal{P}_0(\alpha, \beta)$. For an (extended) path p and $\mathbf{z} = \{ z_i \,|\, i \in \mathbb{Z}^\times \}$ the set of formal commuting variables, we put

$$\mathbf{z}^p = \prod_{\substack{v_i v_{i+1}:\text{ horizontal}\\ \text{or diagonal}}} z_{t_i},$$

where $p = (s_1, -\infty)v_1 \ldots v_r(s_r, \infty)$ with $v_i = (s_i, t_i)$ for $1 \le i \le r$. Then we define

$$\mathcal{S}_\lambda = \sum_{\mathbf{p} \in \mathcal{P}_0(\delta, \lambda+\delta)} \mathbf{z}^{\mathbf{p}},$$

and we have

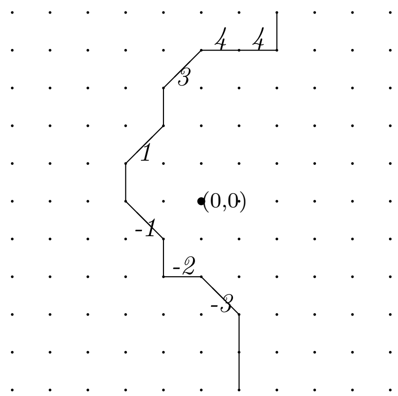
$$\mathcal{S}_\lambda = \det\left(\mathcal{S}_{\lambda_i - i + j}\right)_{1 \le i,j \le n} \tag{8}$$

by the same arguments as in Proposition 3.6.

Example 3.13. Suppose that $\mathbb{Z}_2$-gradings on $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A}_0 &= \{\, 2, 4, 6, \ldots \}, \quad \mathcal{B}_0 = \{\, -2, -4, -6, \ldots \}, \\ \mathcal{A}_1 &= \{\, 1, 3, 5, \ldots \}, \quad \mathcal{B}_1 = \{\, -1, -3, -5 \ldots \}. \end{aligned} \tag{9}$$

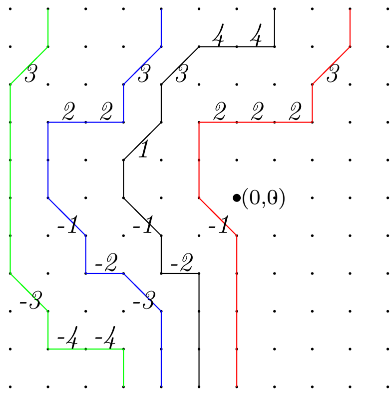
For a lattice path

$p =$ 

its weight monomial is $\mathbf{z}^p = z_1 z_3 z_4^2 z_{-1} z_{-2} z_{-3}$ (the numbers on the horizontal or the diagonal denote their y-coordinates in $\mathbb{Z} \times \mathbb{Z}$).

Now, for $\lambda \in \mathbb{Z}_+^n$, there is also a weight-preserving bijection from $\mathcal{P}_0(\delta, \lambda + \delta)$ to $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ (see Lemma 3.7 and Proposition 3.9).

Example 3.14. We assume that $\mathcal{A}$ and $\mathcal{B}$ are as in (9). Let $\mathbf{p} \in \mathcal{P}_0(\delta, \lambda + \delta)$ be a 4-tuple of lattice paths with $\delta = (0, -1, -2, -3)$ and $\lambda + \delta = (3, 1, -2, -5)$ as follows.

$\mathbf{p} =$ 

Then it corresponds to

$$(T^+, T^-) = \left(\begin{array}{cccccc} & & 2 & 2 & 2 & 3 \\ & 1 & 3 & 4 & 4 & \\ 2 & 2 & 3 & & & \\ 3 & & & & & \end{array} \quad , \quad \begin{array}{ccc} & & -1 \\ & -1 & -2 \\ -1 & -2 & -3 \\ -3 & -4 & -4 \end{array} \right) \in SST_{\mathcal{A}/\mathcal{B}}((3,2,0,-2))$$

Therefore, combining with (8), we obtain the Jacobi-Trudi type formula for $S_\lambda^{\mathcal{A}/\mathcal{B}}$.

Theorem 3.15. *For $\lambda \in \mathbb{Z}_+^n$, we have*

$$S_\lambda^{\mathcal{A}/\mathcal{B}} = \det(S_{\lambda_i - i + j}^{\mathcal{A}/\mathcal{B}})_{1 \le i,j \le n}.$$

Remark 3.16. One can also prove Theorem 3.15 when $\mathcal{A}$ and $\mathcal{B}$ are arbitrary two disjoint linearly ordered $\mathbb{Z}_2$-graded sets, by slightly modifying the notion of extended paths.

REFERENCES

1. S.-J. Cheng & N. Lam: Infinite-dimensional Lie superalgebras and hook Schur functions. *Comm. Math. Phys.* **238** (2003) 95-118.
2. S.-J. Cheng, N. Lam & R.B. Zhang: Character formula for infinite-dimensional unitarizable modules of the general linear superalgebra. *J. Algebra* **273** (2004) 780-805.
3. S.-J. Cheng & W. Wang: Lie subalgebras of differential operators on the super circle. *Publ. Res. Inst. Math. Sci.* **39** (2003) 545-600.
4. ______: Dualities and Representations of Lie Superalgebras. *Graduate Studies in Mathematics* **144**. American Mathematical Society, Providence, RI, 2012.

5. I.B. Frenkel: Representations of Kac-Moody algebras and dual resonance models in Applications of group theory in physics and mathematical physics. *Lectures in Appl. Math.* **21** 325-353, AMS, Providence, 1985.
6. W. Fulton: *Young tableaux, with Application to Representation theory and Geometry.* Cambridge Univ. Press, 1997.
7. I. Gessel & G. Viennot: Binomial determinants, paths, and hook length formulae. *Adv. Math.* **58** (1985) 300-321.
8. R. Howe: Remarks on classical invariant theory. *Trans. Amer. Math. Soc.* **313** (1989) 539-570.
9. V.G. Kac: *Infinite-dimensional Lie algebras.* Third edition, Cambridge University Press, Cambridge, 1990.
10. V.G. Kac & A. Radul: Representation theory of the vertex algebra $W_{1+\infty}$. *Transform. Groups* **1** (1996) 41-70.
11. M. Kashiwara & M. Vergne: *On the Segal-Shale-Weil representations and harmonic polynomials. Invent. Math.* **44** (1978) 1-47.
12. J.-H. Kwon: *Rational semistandard tableaux and character formula for the Lie superalgebra* $\widehat{\mathfrak{gl}}_{\infty|\infty}$. *Adv. Math.* **217** (2008) 713–739.
13. ______: A combinatorial proof of a Weyl type formula for hook Schur polynomials. *J. Algber. Comb.* **28** (2008) 439-459.
14. J.-H. Kwon & E.-Y. Park: Duality on Fock spaces and combinatorial energy functions. *J. Combin. Theory Ser. A* **134** (2015) 121-146.
15. I.G. Macdonald: *Symmetric functions and Hall polynomials.* Oxford University Press, 2nd ed., 1995.
16. R.P. Stanley: *Enumerative Combinatorics.* vol. 2, Cambridge University Press, 1998.

Department of Mathematics, Sungkyunkwan University 2066 Seobu-ro, Jangan-gu, Suwon, Korea
Email address: iblue044@skku.edu, 99407044@hanmail.net