

BOUNDEDNESS IN NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS VIA t_∞ -SIMILARITY

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ABSTRACT. In this paper, we investigate bounds for solutions of nonlinear functional differential systems using the notion of t_∞ -similarity.

1. INTRODUCTION

Integral inequalities play a vital role in the study of boundedness and other qualitative properties of solutions of differential equations. In particular, Bihari's integral inequality is continuous to be an effective tool to study sophisticated problems such as stability, boundedness, and uniqueness of solutions. The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are two useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : the use of integral inequalities, the method of variation of constants formula.

The notion of h -stability (hS) was introduced by Pinto [13,14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. He obtained a general variational h -stability and some properties about asymptotic behavior of solutions of differential systems called h -systems. Choi and Ryu [3], and Choi et al. [4] investigated h -stability and bounds for solutions of the perturbed functional differential systems. Also, Goo [6,7,8] and Goo et al. [9] studied boundedness of solutions for the perturbed functional differential systems.

The aim of this paper is to obtain boundedness for solutions of nonlinear functional differential systems under suitable conditions on perturbed term.

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2. PRELIMINARIES

We consider the nonlinear nonautonomous differential system

$$(2.1) \quad x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and \mathbb{R}^n is the Euclidean n -space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed functional differential systems of (2.1)

$$(2.2) \quad y' = f(t, y) + \int_{t_0}^t g(s, y(s)) ds + h(t, y(t), Ty(t)), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$(2.3) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0$$

and

$$(2.4) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We recall some notions of h -stability [14].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an h -system if there exist a constant $c \geq 1$, and a positive continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c |x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called (hS) h -stable if there exists $\delta > 0$ such that (2.1) is an h -system for $|x_0| \leq \delta$ and h is bounded.

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [5].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$(2.5) \quad \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [5, 10].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞ -similarity.

We give some related properties that we need in the sequel.

Lemma 2.4 ([14]). *The linear system*

$$(2.6) \quad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (respectively h -stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on \mathbb{R}^+ such that

$$(2.7) \quad |\phi(t, t_0)| \leq c h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.6).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$(2.8) \quad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.5 ([2]). *Let x and y be a solution of (2.1) and (2.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

Theorem 2.6 ([3]). *If the zero solution of (2.1) is hS , then the zero solution of (2.3) is hS .*

Theorem 2.7 ([4]). *Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.3) is hS , then the solution $z = 0$ of (2.4) is hS .*

Lemma 2.8. (Bihari – type inequality) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s)w(u(s))ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda(s)ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s)ds \in \text{dom} W^{-1} \right\}.$$

Lemma 2.9 ([6]). *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \leq & c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau ds \\ & + \int_{t_0}^t \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau) ds \in \text{dom} W^{-1} \right\}.$$

We obtain the following corollary from Lemma 2.9.

Corollary 2.10. *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}.$$

3. MAIN RESULTS

In this section, we investigate boundedness for solutions of the nonlinear perturbed differential systems via t_∞ -similarity.

For the proof we need the following lemma.

Lemma 3.1. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$\begin{aligned} u(t) \leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau)u(\tau) \\ + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)u(r)dr)d\tau ds + \int_{t_0}^t \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$(3.1) \quad \begin{aligned} u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau \right. \\ \left. + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau) ds \right], \end{aligned}$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.8 and

$$\begin{aligned} b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau \right. \\ \left. + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}. \end{aligned}$$

Proof. Defining

$$z(t) = c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau)u(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)u(r)dr)d\tau ds \\ + \int_{t_0}^t \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau ds,$$

then we have $z(t_0) = c$ and

$$z'(t) = \lambda_1(t)u(t) + \lambda_2(t) \int_{t_0}^t (\lambda_3(s)u(s) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)u(\tau)d\tau)ds \\ + \lambda_6(t) \int_{t_0}^t \lambda_7(s)w(u(s))ds \\ \leq (\lambda_1(t) + \lambda_2(t) \int_{t_0}^t (\lambda_3(s) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau)ds + \lambda_6(t) \int_{t_0}^t \lambda_7(s)ds)w(z(t)),$$

$t \geq t_0$, since $z(t)$ and $w(u)$ are nondecreasing, $u \leq w(u)$, and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$(3.2) \quad z(t) \leq c + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau \\ + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau)w(z(s))ds.$$

It follows from Lemma 2.8 that (3.2) yields the estimate (3.1). \square

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h .

(H3) $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$.

Theorem 3.2. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies*

$$(3.3) \quad |g(t, y(t))| \leq a(t)|y(t)| + b(t) \int_{t_0}^t k(s)|y(s)|ds$$

and

$$(3.4) \quad |h(t, y(t), Ty(t))| \leq c(t)(|y(t)| + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^t q(s)w(|y(s)|)ds,$$

where $t \geq t_0 \geq 0$ and $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^\tau k(r)dr)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau)ds \right],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^\tau k(r)dr)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.3) is hS. Using (H1), by Theorem 2.7, the solution $z = 0$ of (2.4) is hS. By Lemma 2.4, Lemma 2.5, together with (3.3), and (3.4), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \left(\int_{t_0}^s (a(\tau)|y(\tau)| + b(\tau) \int_{t_0}^\tau k(r)|y(r)|dr)d\tau + c(s)(|y(s)| + \int_{t_0}^s q(\tau)w(|y(\tau)|)d\tau) \right) ds. \end{aligned}$$

Applying (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \left(c(s) \frac{|y(s)|}{h(s)} \right. \\ &\quad \left. + \int_{t_0}^s (a(\tau) \frac{|y(\tau)|}{h(\tau)} + b(\tau) \int_{t_0}^\tau k(r) \frac{|y(r)|}{h(r)} dr)d\tau + c(s) \int_{t_0}^s q(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Lemma 3.1 that we have

$$\begin{aligned} |y(t)| &\leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^\tau k(r)dr)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau)ds \right], \end{aligned}$$

where $c = c_1|y_0|h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. \square

Remark 3.3. Letting $c(t) = 0$ in Theorem 3.2, we obtain the similar result as that of Theorem 3.3 in [7].

Theorem 3.4. Let $a, b, c, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies

$$(3.5) \quad \int_{t_0}^t |g(s, y(s))| ds \leq a(t)w(|y(t)|), |h(t, y(t), Ty(t))| \leq b(t)w(|y(t)|) + c(t)|Ty(t)|$$

and

$$(3.6) \quad |Ty(t)| \leq \int_{t_0}^t q(s)w(|y(s)|) ds,$$

where $a, b, c, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + c(s) \int_{t_0}^s q(\tau) d\tau) ds \right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + c(s) \int_{t_0}^s q(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying the nonlinear variation of constants formula, Lemma 2.4, together with (3.5), and (3.6), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left((a(s) + b(s)) w(|y(s)|) \right. \\ &\quad \left. + c(s) \int_{t_0}^s q(\tau) w(|y(\tau)|) d\tau \right) ds. \end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left((a(s) + b(s)) w\left(\frac{|y(s)|}{h(s)}\right) \right. \\ &\quad \left. + c(s) \int_{t_0}^s q(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 2.10, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + c(s) \int_{t_0}^s q(\tau) d\tau) ds \right],$$

where $c = c_1|y_0| h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$. This completes the proof. \square

Remark 3.5. Letting $b(t) = c(t) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.2 in [9].

We need the following lemma for the proof of Theorem 3.7.

Lemma 3.6. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that, for some $c \geq 0$, we have*

$$(3.7) \quad u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \left(\int_{t_0}^s (\lambda_3(\tau)w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(s)u(r)dr) d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau \right) ds, \quad t \geq t_0.$$

Then

$$(3.8) \quad u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t [\lambda_1(s) + \lambda_2(s) \left(\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr) d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau \right)] ds \right], \quad t \geq t_0.$$

Proof. Define a function $v(t)$ by the right member of (3.7). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t) \left(\int_{t_0}^t (\lambda_3(s)w(u(s)) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)u(\tau)d\tau) ds \right. \\ &\quad \left. + \lambda_6(t) \int_{t_0}^t \lambda_7(s)w(u(s))ds \right) \\ &\leq \left[\lambda_1(t) + \lambda_2(t) \left(\int_{t_0}^t (\lambda_3(s) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau) ds \right. \right. \\ &\quad \left. \left. + \lambda_6(t) \int_{t_0}^t \lambda_7(s)ds \right) \right] w(v(t)), \end{aligned}$$

$t \geq t_0$, since $v(t)$ is nondecreasing, $u \leq w(u)$, and $u(t) \leq v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$(3.9) \quad v(t) \leq c + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau \right. \\ \left. + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau) d\tau \right) w(v(s)) ds.$$

Thus, by Lemma 2.8, (3.9) yields the estimate (3.8). \square

Theorem 3.7. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies*

$$(3.10) \quad |g(t, y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)|y(s)| ds$$

and

$$(3.11) \quad |h(t, y(t), Ty(t))| \leq c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^t q(s)w(|y(s)|) ds,$$

where $t \geq t_0 \geq 0$ and $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau \right. \\ \left. + c(s) \int_{t_0}^s q(\tau) d\tau] ds \right],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau \right. \\ \left. + c(s) \int_{t_0}^s q(\tau) d\tau] ds \in \text{dom} W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying the nonlinear variation of constants formula, Lemma 2.4, together with (3.10), and (3.11), we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ \leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(\int_{t_0}^s (a(\tau)w(|y(\tau)|) \right. \\ \left. + b(\tau) \int_{t_0}^{\tau} k(r)|y(r)| dr) d\tau + c(s)(w(|y(s)|) + \int_{t_0}^s q(\tau)w(|y(\tau)|) d\tau) \right) ds.$$

Using the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)\left(c(s)w\left(\frac{|y(s)|}{h(s)}\right) + \int_{t_0}^s (a(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right) + b(\tau) \int_{t_0}^\tau k(r)\frac{|y(r)|}{h(r)}dr)d\tau + c(s) \int_{t_0}^s q(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 3.6, we have

$$|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2 \int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^\tau k(r)dr)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau]ds\right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. \square

Remark 3.8. Letting $c(t) = 0$ in Theorem 3.7, we obtain the similar result as that of Theorem 3.7 in [8].

Theorem 3.9. Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.2) satisfies

$$(3.12) \quad \int_{t_0}^s |g(\tau, y(\tau))|d\tau \leq a(s)w(|y(s)|) + b(s) \int_{t_0}^s k(\tau)|y(\tau)|d\tau$$

and

$$(3.13) \quad |h(s, y(s), Ty(s))| \leq c(s)(w(|y(s)|) + |Ty(s)|),$$

$$|Ty(s)| \leq \int_{t_0}^s q(\tau)w(|y(\tau)|)d\tau,$$

where $s \geq t_0 \geq 0$ and $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau)ds\right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau)d\tau + c(s) \int_{t_0}^s q(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t, t_0, y_0)$ of (2.2) passing through (t_0, y_0) is given by

(3.14)

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) \left(\int_{t_0}^s g(\tau, y(\tau)) d\tau + h(s, y(s), Ty(s)) \right) ds.$$

By the same argument as in the proof in Theorem 3.2, the solution $z = 0$ of (2.4) is hS. Applying Lemma 2.4, together with (3.12), (3.13), and (3.14), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left((a(s)w(|y(s)|)) \right. \\ &\quad \left. + b(s) \int_{t_0}^s k(\tau) |y(\tau)| d\tau + c(s)(w(|y(s)|) + \int_{t_0}^s q(\tau)w(|y(\tau)|) d\tau) \right) ds. \end{aligned}$$

It follows from (H2) and (H3) that

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left((a(s) + c(s))w\left(\frac{|y(s)|}{h(s)}\right) \right. \\ &\quad \left. + b(s) \int_{t_0}^s k(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau + c(s) \int_{t_0}^s q(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau \right) ds. \end{aligned}$$

Defining $u(t) = |y(t)||h(t)|^{-1}$, then, by Lemma 2.9, we have

$$\begin{aligned} |y(t)| &\leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau) d\tau \right. \\ &\quad \left. + c(s) \int_{t_0}^s q(\tau) d\tau \right], \end{aligned}$$

where $t_0 \leq t < b_1$ and $c = c_1 |y_0| h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. Hence, the proof is complete. \square

Remark 3.10. Letting $c(s) = b(s) = 0$ in Theorem 3.9, we obtain the same result as that of Theorem 3.2 in [9].

Remark 3.11. Letting $c(s) = 0$ in Theorem 3.9, we obtain the same result as that of Theorem 3.4 in [8].

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