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WEIERSTRASS SEMIGROUPS OF PAIRS ON H-HYPERELLIPTIC CURVES

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Abstract. Kato[6] and Torres[9] characterized the Weierstrass semigroup of ramification points on h-hyperelliptic curves. Also they showed the converse results that if the Weierstrass semigroup of a point P on a curve C satisfies certain numerical condition then C can be a double cover of some curve and P is a ramification point of that double covering map. In this paper we expand their results on the Weierstrass semigroup of a ramification point of a double covering map to the Weierstrass semigroup of a pair (P, Q). We characterized the Weierstrass semigroup of a pair (P, Q) which lie on the same fiber of a double covering map to a curve with relatively small genus. Also we proved the converse: if the Weierstrass semigroup of a pair (P, Q) satisfies certain numerical condition then C can be a double cover of some curve and P, Q map to the same point under that double covering map.

1. Introduction and Preliminaries

Let C be a nonsingular complex projective curve of genus $g \geq 2$, $\mathcal{M}(C)$ denote the field of meromorphic functions on C and $\mathbb{N}_0$ be the set of all nonnegative integers. For two distinct points P, $Q \in C$, we define the Weierstrass semigroup $H(P) \subset \mathbb{N}_0$ of a point and the Weierstrass semigroup of a pair of points $H(P, Q) \subset \mathbb{N}_0^2$ by

$$\begin{aligned} H(P) &= \{\alpha \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}, \\ H(P, Q) &= \{(\alpha, \beta) \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}, \end{aligned}$$

where $(f)_\infty$ means the divisor of poles of f. Indeed, these sets form sub-semigroups of $\mathbb{N}_0$ and $\mathbb{N}_0^2$, respectively. The cardinality of the set $G(P) = \mathbb{N}_0 \setminus H(P)$ is exactly g. The set $G(P, Q) = \mathbb{N}_0^2 \setminus H(P, Q)$ is also finite, but its cardinality is dependent on the points P and Q. In [7], the upper and lower bound of such sets are given as

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$$\binom{g+2}{2} - 1 \leq \text{card } G(P,Q) \leq \binom{g+2}{2} - 1 - g + g^2.$$

We review some basic facts concerning the Weierstrass semigroups at a pair of points on a curve ([4], [7]).

Lemma 1.1. *For each $\alpha \in G(P)$, let $\beta_\alpha = \min\{\beta \mid (\alpha,\beta) \in H(P,Q)\}$. Then $\alpha = \min\{\gamma \mid (\gamma,\beta_\alpha) \in H(P,Q)\}$. Moreover, we have*

$$\{\beta_\alpha \mid \alpha \in G(P)\} = G(Q).$$

Proof. See [7]. □

Let $G(P) = \{p_1 < p_2 < \cdots < p_g\}$ and $G(Q) = \{q_1 < q_2 < \cdots < q_g\}$. Above lemma implies that the set $H(P,Q)$ defines a permutation $\sigma = \sigma(P,Q)$ satisfying that $(p_i, q_{\sigma(i)}) \in H(P,Q)$. Homma [4] obtained the formula for the cardinality of $G(P,Q)$ using the cardinality of the set of pairs (i,j) which are reversed by σ. Also we define $\tilde{\sigma} : G(P) \to G(Q)$ by $\tilde{\sigma}(p_i) = q_{\sigma(i)}$ which means nothing but $\tilde{\sigma}(\alpha) = \beta_\alpha$. Clearly $\tilde{\sigma}$ is a bijection. We use the following notations;

$$\begin{aligned} \Gamma = \Gamma(P,Q) &= \{(\alpha,\beta_\alpha) \mid \alpha \in G(P)\} \\ &= \{(p_i, q_{\sigma(i)}) \mid i = 1,2,\cdots,g\}, \\ \widetilde{\Gamma} = \widetilde{\Gamma}(P,Q) &= \Gamma(P,Q) \cup \big(H(P) \times \{0\}\big) \cup \big(\{0\} \times H(Q)\big). \end{aligned}$$

The above set $\Gamma(P,Q)$ is called the *generating subset* of the Weierstrass semigroup $H(P,Q)$. For given distinct two points P, Q, the set $\Gamma(P,Q)$ determines not only $\widetilde{\Gamma}(P,Q)$ but also the sets $H(P,Q)$ and $G(P,Q)$ completely, as described in the lemma below. To state the lemma we use the natural partial order on the set $\mathbb{N}_0^2$ defined as

$$(\alpha,\beta) \geq (\gamma,\delta) \text{ if and only if } \alpha \geq \gamma \text{ and } \beta \geq \delta,$$

and the least upper bound of two elements (α_1,β_1), (α_2,β_2) is defined as

$$\text{lub}\{(\alpha_1,\beta_1),(\alpha_2,\beta_2)\} = (\max\{\alpha_1,\alpha_2\},\ \max\{\beta_1,\beta_2\}).$$

Lemma 1.2. *(1) The subset $H(P,Q)$ of $\mathbb{N}_0^2$ is closed under the lub(least upper bound) operation. (2) Every element of $H(P,Q)$ is expressed as the lub of one or two elements of the set $\widetilde{\Gamma}(P,Q)$. (3) The set $G(P,Q) = \mathbb{N}_0^2 \setminus H(P,Q)$ is expressed as*

$$G(P,Q) = \bigcup_{l \in G(P)} \big(\{(l,\beta) | \beta = 0,1,\ldots,\tilde{\sigma}(l)-1\} \cup \{(\alpha,\tilde{\sigma}(l)) | \alpha = 0,1,\ldots,l-1\}\big).$$

Proof. See [7] and [8]. □

We say a pair $(\alpha,\beta) \in \mathbb{N}_0^2$ is *special* [resp. *nonspecial*, *canonical*] if the corresponding divisor $\alpha P + \beta Q$ is special [resp. nonspecial, canonical]. We denote $\dim(\alpha,\beta)$ the dimension of complete linear series $|\alpha P + \beta Q|$ and use the notations

$$\begin{aligned}
\Gamma_{\leq(\alpha,\beta)} &= \{(\gamma,\delta) \in \Gamma \mid (\gamma,\delta) \leq (\alpha,\beta)\} \\
\widetilde{\Gamma}_{\leq(\alpha,\beta)} &= \{(\gamma,\delta) \in \widetilde{\Gamma} \mid (\gamma,\delta) \leq (\alpha,\beta)\} \\
\mathbb{N}^2_{0\leq(\alpha,\beta)} &= \{(\gamma,\delta) \in \mathbb{N}_0^2 \mid (\gamma,\delta) \leq (\alpha,\beta)\} \\
H(P,Q)_{\leq(\alpha,\beta)} &= \{(\gamma,\delta) \in H(P,Q) \mid (\gamma,\delta) \leq (\alpha,\beta)\}.
\end{aligned}$$

We also need the following two theorems in [1].

Theorem 1.3 ([1, p.10] (Brill-Nöther Reciprocity)). *Let C be a curve of genus $g \geq 2$. If two linear series g_n^r and g_m^s on C are complete and residual to each other, i.e., $|g_n^r + g_m^s| = K$ where K is the canonical series, then $n - 2r = m - 2s$. This implies that if P is a base point of g_n^r then $|g_m^s + P|$ does not have P as a base point, this means that* $\dim|g_m^s + P| = s + 1$.

We use the following well-known lemmas to prove our theorems in this paper.

Lemma 1.4 ([1] (The Inequality of Castelnuovo-Severi)). *Let C, C_1 and C_2 be curves of respective genera g, g_1 and g_2. Assume that $\phi_i : C \to C_i$, $i = 1,2$ are d_i-sheeted coverings such that $\phi = \phi_1 \times \phi_2 : C \to C_1 \times C_2$ is birational onto its image. Then $g \leq (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2$.*

Lemma 1.5 ([2, p.116] (Castelnuovo's Bound)). *Let C be a smooth curve that admits a birational mapping onto a nondegenerate curve of degree d in $\mathbb{P}^r$. Then the genus of C satisfies the inequality*

$$g \leq \frac{m(m-1)}{2}(r-1) + m\epsilon,$$

where $m = \left[\frac{d-1}{r-1}\right]$ and $\epsilon = d - 1 - m(r-1)$.

Lemma 1.6 ([2, p.251] (Clifford's Theorem)). *For any two effective divisors on a smooth curve C,*

$$\dim|D| + \dim|D'| \leq \dim|D + D'|$$

and for $|D|$ special

$$\dim|D| \leq d/2$$

with equality holding only if $D = 0$, $D = K$, or C is hyperelliptic.

In Section 2, we study the Weierstrass semigroups of pairs on h-hyperelliptic curves.

2. Semigroups on h-hyperelliptic Curves

Recall that a curve C is called h-hyperelliptic if it admits a double covering map $\pi : C \to C_h$ where C_h is a curve of genus h, or equivalently, if there is an automorphism of order two on C which is defined by interchanging of the two sheets of this covering. Such π is unique if $g > 4h + 1$ [3], which we can prove easily using above Lemma 1.4. Usually, 0-hyperelliptic curves and 1-hyperelliptic curves are said to be hyperelliptic and bi-elliptic, respectively. The results in this section was motivated by [6] and [9], where the authors studied ordinary Weierstrass semigroups of points on h-hyperelliptic curves.

Lemma 2.1. *Let C be a curve of genus g. Suppose that C is an h-hyperelliptic curve for some $h \geq 0$ with a double covering map $\pi : C \to C_h$. If a linear series g^1_k is base point free and not compounded of π, then $k > g - 2h$.*

Proof. The k-sheeted map $\phi_{g^1_k} : C \to \mathbb{P}^1$ and 2-sheeted map $\pi : C \to C_h$ induce a birational map

$$\phi_{g^1_k} \times \pi : C \to \mathbb{P}^1 \times C_h$$

onto its image. By Lemma 1.4, $g \leq (k-1)(2-1)+k\cdot 0+2\cdot h$ so we get $k > g-2h$. $\square$

Theorem 2.2. *Let C be an h-hyperelliptic curve of genus $g \geq 6h + 2$ with a double covering map $\pi : C \to C_h$. Let $P, Q \in C$ be distinct points and $\pi(P) = \pi(Q) = P'$. Then*

$$H(P,Q)_{\leq(2h+1,2h+1)} = \{(k,k) \mid k \in H(P'),\ k \leq 2h+1\}.$$

Proof. Suppose that there exists an element $(\alpha, \beta) \in H(P,Q)_{\leq(2h+1,2h+1)}$ not contained in $\{(k,k) \mid k \in H(P'),\ k \leq 2h+1\}$. Let $g^1_{\alpha+\beta}$ be a linear subseries of $|\alpha P + \beta Q|$ which is base-point-free and not necessarily complete. If $\alpha \neq \beta$, $g^1_{\alpha+\beta}$ is not compounded of π. If $\alpha = \beta$ and $\alpha \notin H(P')$, let $H(P')_{\leq 2h} = \{n_0 = 0, n_1, \cdots, n_h = 2h\}$. For some i, $n_i < \alpha < n_{i+1}$ and $\dim|n_i(P+Q)| < \dim|\alpha(P+Q)|$ by the assumption on α. Also $\dim|n_i(P+Q)| \geq \dim|\alpha P'| = i$ so we have $\dim|\alpha P'| < \dim|\alpha(P+Q)|$. Thus $|\alpha(P+Q)|$ and $g^1_{\alpha+\beta}$ is not compounded of π again. Now by Lemma 2.1,

$$\alpha + \beta > g - 2h \geq (6h+2) - 2h \geq 4h + 2$$

which contradicts the choice of $(\alpha, \beta) \in H(P, Q)_{\leq(2h+1,2h+1)}$. □

Each of the following two theorems is a converse of Theorem 2.2 in a different view point. For the next theorem, we need two lemmas.

Lemma 2.3. *Let* (α, β) *be an element in* $\mathbb{N}_0^2$ *with* $\beta \geq 1$ [resp. $\alpha \geq 1$]. Then

$$\dim(\alpha, \beta) = \dim(\alpha, \beta - 1) + 1 \,[\text{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1]$$

if and only if there exists $(\gamma, \beta) \in \widetilde{\Gamma}$ [resp. $(\alpha, \delta) \in \widetilde{\Gamma}$] with $0 \leq \gamma \leq \alpha$ [resp. $0 \leq \delta \leq \beta$].

Proof. See [7].

Lemma 2.4. *Let* $H \subset \mathbb{N}$ *be a semigroup. Assume that* H *contains* h *terms in* $\{1, 2, \cdots, 2h\}$ *and* $2h, 2h + 1 \in H$. *Then* H *contains any integers* $k \geq 2h$.

Proof. First, we show that $2h + 2 \in H$. The set $I_{2h+1} = \{1, 2, \cdots, 2h, 2h + 1\}$ has $h + 1$ elements of H. Consider a partition of I_{2h+1}

$$\{1, 2h + 1\}, \{2, 2h\}, \{3, 2h - 1\}, \cdots, \{h + 1\}.$$

If $h + 1 \in H$, then $2h + 2 \in H$ since H is a semigroup. If $h + 1 \notin H$, then at least one of the sets other than $\{h+1\}$ is contained in H, and hence we have $2h+2 \in H$.

Next, we show that $2h + 3 \in H$. The set $I_{2h+2} = \{1, 2, \cdots, 2h, 2h + 1, 2h + 2\}$ has $h + 2$ elements of H. Consider a partition of I_{2h+2}

$$\{1, 2h + 2\}, \{2, 2h + 1\}, \{3, 2h\}, \cdots, \{h + 1, h + 2\}.$$

Then at least of one is contained in H and hence $2h + 3 \in H$.

Repeating this process, we conclude that $k \in H$ for all $k \geq 2h$. □

Theorem 2.5. *Let* C *be a curve of genus* $g \geq 6h + 4$ *and* $P, Q \in C$. *Assume that* $H(P, Q)$ *contains exactly* h *terms in* $\{(1, 1), (2, 2), \cdots, (2h, 2h)\}$ *and that*

$$(2h, 2h), (2h + 1, 2h + 1) \in H(P, Q).$$

Then C *is* h*-hyperelliptic with the double covering map* $\phi : C \to C_h$ *for some* C_h. *Moreover* $\phi(P) = \phi(Q)$ *and* $H(\phi(P)) = \{k \mid (k, k) \in H(P, Q)\}$.

Proof. By Lemma 2.4, $(k, k) \in H(P, Q)$ for all $k \geq 2h$. By Lemma 2.3,

$$\dim |(3h + 1)(P + Q)| \geq 2h + 1.$$

Let $s + 1 = \dim |(3h + 1)(P + Q)|$ and let's denote $|(3h + 1)(P + Q)|$ by g^{s+1}_{6h+2}. Consider a rational map $\phi : C \to \mathbb{P}^{s+1}$ defined by g^{s+1}_{6h+2}.

Claim: $s = 2h$.

Suppose that $s \geq 2h+1$. If ϕ is birational, then

$$m = \left[\frac{(6h+2)-1}{(s+1)-1}\right] = 2,\ \epsilon = (6h+1) - 2s.$$

So by Lemma 1.5, we get

$$g \leq 12h + 2 - 3s \leq 6h - 1$$

which contradicts our bound of genus. Let t be the degree of ϕ and C' be a normalization of $\phi(C)$. Then C' admits a complete base-point-free linear series $g^{s+1}_{\frac{6h+2}{t}}$. Since $s+1 < \frac{6h+2}{t}$, we have $t = 2$. Thus C is a double covering of the curve C' and we have a complete linear series $g^{s+1}_{3h+1}(C')$. By Clifford's theorem, it is a complete nonspecial linear series on C', hence the genus of C' is $h' = 3h - s < h$. Here we have two possibilities

$$\phi(P) = \phi(Q) \text{ or } \phi(P) \neq \phi(Q).$$

Subclaim: $\phi(P) = \phi(Q)$.

If $\phi(P) \neq \phi(Q)$, then $\phi^*(\phi(P)) = 2P$ and $\phi^*(\phi(Q)) = 2Q$, since the divisor $(3h+1)(P+Q)$ is the pull-back of some divisor on C' via ϕ. In this case, $3h+1$ must be even and hence h is odd. Consider a linear series $|(3h+2)(P+Q)|$ and let its dimension be $u+1$. Then $s+2 \geq u \geq s+1 \geq 2h+2$. Through the similar steps as above, we conclude that C is a double covering of another curve C'' of genus $h'' \leq h-1$, and the series $|(3h+2)(P+Q)|$ is compounded of the latter map ϕ'. Since h is odd, $3h+2$ is also odd. Hence $\phi'^*(\phi'(P)) = P + Q$. Now $\phi \times \phi'$ is birational, and by Lemma 1.4, we have $g \leq 1 + 4h$ contrary to our assumption. Therefore we proved the Subclaim $\phi(P) = \phi(Q)$.

Since $k(P+Q) = \phi^*(k\phi(P))$ for any integer k, we have $(k,k) \in H(P,Q)$ for $k \in H(\phi(P))$. Then the cardinality of the set $\{(k,k) \mid (k,k) \notin H(P,Q),\ k \geq 1\}$ is less than h, which is a contradiction to our assumption. Thus we proved the Claim $s = 2h$.

Now we have a complete linear series $g^{2h+1}_{6h+2} = |(3h+1)(P+Q)|$ and a rational map $\phi : C \to \mathbb{P}^{2h+1}$ induced from g^{2h+1}_{6h+2}. Suppose ϕ is birational. Then by Lemma 1.5, we get $g(C) \leq 6h+3$ which contradicts the assumption $g \geq 6h+4$.

Thus ϕ is a double covering map from C to $\phi(C)$ with $g(\phi(C)) = h$. Therefore C is h-hyperelliptic. Since $|(2h+1)(P+Q)|$ and $|2h(P+Q)|$ is also compounded of ϕ, we conclude that $\phi(P) = \phi(Q)$. □

Remark 2.6. The above theorem is a modification of Theorem A in [9].

Theorem 2.7. *Let C be a curve of genus $g \geq 6h+5$. Suppose that* $(2h, 2h)$, $(2h+1, 2h+1) \in H(P,Q)$ *and* $\dim(2h,2h) = h$, $\dim(2h+1,2h+1) = h+1$. *Then C is an h-hyperelliptic curve. Moreover, P and Q have same image under the double covering map.*

Proof. Consider the rational map $\phi : C \to \mathbb{P}^{h+1}$ defined by the linear series

$$g^{h+1}_{4h+2} = |(2h+1)(P+Q)|\,.$$

If ϕ is birational, then $g \leq 6h+4$ by Lemma 1.5. Thus ϕ is not birational. Let t be the degree of ϕ and C' be a normalization of $\phi(C)$. Thus C' admits a complete base-point-free linear series $g^{h+1}_{\frac{4h+2}{t}}(C')$. Since $h+1 \leq \frac{4h+2}{t}$, we have $t=2$ or $t=3$.

If $t=2$, then we have $g^{h+1}_{\frac{4h+2}{2}}(C') = g^{h+1}_{2h+1}(C')$ on C'. Since $h+1 > \frac{2h+1}{2}$, this series is nonspecial by Lemma 1.6 and the genus of C' is exactly h. Since $2h+1$ is odd and the divisor $(2h+1)(P+Q)$ is also a pull-back of some divisor via a double covering map ϕ, we conclude that $\phi(P) = \phi(Q)$.

Now it remains to show that the case $t=3$ can not occur. If $t=3$, then $(4h+2)$ is a multiple of 3 and we have a complete $g^{h+1}_{\frac{4h+2}{3}}(C')$ on C'. By Lemma 1.6 again, this linear series is nonspecial, and the genus of C' is $\frac{h-1}{3}$. If $\phi(P) = \phi(Q)$, then $\phi^*(\phi(P)) = 2P+Q$ or $P+2Q$. Then $(2h+1)(P+Q)$ can not be a pull-back of any divisor on C'. Thus we have

$$\phi^*(\phi(P)) = 3P \text{ and } \phi^*(\phi(Q)) = 3Q.$$

Now $V = \left|\frac{2h+1}{3}\phi(P) + \frac{2h-2}{3}\phi(Q)\right|$ is a complete linear series on C' of degree $\frac{4h-1}{3}$. Since $\frac{4h-1}{3} \geq 2 \cdot g(C')$ so V is base point free. Then

$$|(2h+1)P + (2h-2)Q| = |\phi^*(V)|$$

which is obtained from the pullback of V is also base point free and we have

$$(2h+1, 2h-2) \in H(P,Q).$$

Since $(2h,2h) \in H(P,Q)$ by assumption, we have $(2h+1,2h) \in H(P,Q)$ by Lemma 1.2. Thus

$$\dim(2h+1,2h+1) > \dim(2h+1,2h) > \dim(2h,2h) = h$$

which contradicts the assumption $\dim(2h+1,2h+1) = h+1$. Hence the case $t=3$ can not occur. □

Remark 2.8. In Theorem 2.7, we assume the existence of only two elements in $H(P, Q)$ and their dimensions without assuming the sequence of elements in $H(P, Q)$.

We state a generalized version of Theorem 2.7.

Theorem 2.9. *Let C be a curve of genus $g \geq 6h + a$, $a \geq 5$. Suppose that there exists an integer n satisfying that* (i) $2h+1 \leq n \leq \frac{g+a-3}{2}$, (ii) $\dim|n(P+Q)| = n-h$ *and* $(n, n) \in H(P, Q)$ *and* (iii) $\dim|(n-1)(P+Q)| = (n-1)-h$ *and* $(n-1, n-1) \in H(P, Q)$*. Then C is h-hyperelliptic with double covering map $\pi : C \to C_h$ with*

$$\pi(P) = \pi(Q) = P' \in C_h \textit{ and } \{k \mid (k, k) \in H(P, Q)\} = H(P').$$

Proof. If $n = 2h+1$, we already proved in Theorem 2.7. Now we assume $n \geq 2h+2$.

Let n be a number such that $2h+1 \leq n \leq \frac{g+a-3}{2}$, $(n, n) \in H(P, Q)$ and $\dim|n(P+Q)| = n - h$. Let $|n(P + Q)| = g_{2n}^{n-h}$ and $\phi_n : C \to \mathbb{P}^{n-h}$ be a rational map defined by g_{2n}^{n-h}.

Claim 1: ϕ_n is not birational if $n \geq 2h + 2$.

Suppose that $\phi_n : C \to \mathbb{P}^{n-h}$ is birational. Then using the Castelnuovo bound, the genus of C satisfies the inequality $g \leq \frac{m(m-1)}{2}(r - 1) + m\epsilon$, where $m = \left[\frac{d-1}{r-1}\right]$ and $\epsilon = d - 1 - m(r - 1)$. In this theorem, m satisfies $m = \left[\frac{2n-1}{n-h-1}\right]$=2 or 3. If $m = 2$ and $\epsilon = 2h+1$ then $g \leq n+3h+1 \leq g-\frac{1}{2}$ which is a contradiction. If $m = 3$ and $\epsilon = -n + 3h + 2$ then $g \leq 6h + 3 < g$ which is a contradiction again. Thus ϕ_n is not birational if $n \geq 2h + 2$.

Let $\deg \phi_n = t \geq 2$. Since ϕ_n is nondegenerate, $n - h \leq \frac{2n}{t}$ so $\deg \phi_n = 2$ or $\deg \phi_n = 3$.

Claim 2: If $(n, n), (n-1, n-1) \in H(P, Q)$, $\dim|n(P+Q)| = n-h$ and $\dim|(n-1)(P + Q)| = (n - 1) - h$, then $\deg \phi_n = 2$ and $g(\phi_n(C)) = h$.

If $t = 3$, then $2n$ is a multiple of 3 and there is a complete and nonspecial $g_{\frac{2n}{3}}^{n-h}(C')$ on $C' = \phi_n(C)$. Hence the genus of C' is $\frac{3h-n}{3}$. If $\phi_n(P) = \phi_n(Q)$, then $\phi_n^*(\phi_n(P)) = 2P + Q$ or $P + 2Q$ and the pullback of a multiple of $\phi(P)$ can not be $n(P + Q)$. Thus we have $\phi_n(P) \neq \phi_n(Q)$ and hence

$$\phi_n^*(\phi_n(P)) = 3P,\ \phi_n^*(\phi_n(Q)) = 3Q.$$

Since $|nP + (n - 3)Q| = |\phi_n^*(\frac{n}{3}\phi_n(P) + \frac{n-3}{3}\phi_n(Q))|$ is base point free, $(n, n - 3) \in H(P, Q)$. Then $\dim|nP+nQ| = \dim|(n-1)P+(n-1)Q|+2$ which is a contradiction to our assumption.

Therefore we conclude $\deg\phi_n = t = 2$ and there is a complete, nonspecial $g^{n-h}_{\frac{2n}{2}}(C')$ on $C' = \phi_n(C)$. Hence the genus of C' is h and C is h-hyperelliptic with double covering map $\pi = \phi_n : C \to C' = C_h$.

Claim 3: $\pi(P) = \pi(Q) = P'$ and $\{k \mid (k,k) \in H(P,Q)\} = H(P')$

Case 1: n is odd.

Since $\pi = \phi_n$ is a double covering map by Claim 2, there is a complete, nonspecial $g^{n-h}_{\frac{2n}{2}}(C') = g^{n-h}_n(C')$ on C'. By Riemann-Roch Theorem, $g(C') = k - (k-h) = h$. Since $n(P+Q)$ is a pullback of some divisor D on $C' = C_h$, i.e., $n(P+Q) = \pi^*(D)$ and n is odd, we get $\pi(P) = \pi(Q)$.

Case 2: n is even.

Suppose that $\phi_n(P) \neq \phi_n(Q)$. Since $n \geq 2h+1$ and n is even, $n \geq 2h+2$ and $\dim|(n-1)(P+Q)| = (n-1)-h$ and $(n-1,n-1) \in H(P,Q)$ by the assumption on n. Consider ϕ_{n-1} which is defined by $g^{(n-1)-h}_{2(n-1)} = |(n-1)(P+Q)|$. By Castelnuovo's bound, ϕ_{n-1} is not birational and $\deg\phi_{n-1} = 2$ or 3. If $\deg\phi_{n-1} = 3$, there is a complete, nonspecial $g^{(n-1)-h}_{\frac{2(n-1)}{3}}$ on $C'' = \phi_{n-1}(C)$. So $g(C'') = h - \frac{(n-1)}{3}$. Then the 3:1 map $\phi_{n-1} : C \to C_{h-\frac{n-1}{3}}$ and the 2:1 map $\phi_n : C \to C_h$ induce a map $\phi_{n-1}\times\phi_n : C \to C_{h-\frac{n-1}{3}} \times C_h$ which is birational onto its image. By Lemma 1.4, $g(C) \leq (3-1)(2-1) + 3(h - \frac{n-1}{3}) + 2h = 2 + 5h - (n-1) \leq 2 + 3h < g$ which is a contradiction. Thus $\deg\phi_{n-1} = 2$ and there is a complete, nonspecial $g^{(n-1)-h}_{\frac{2(n-1)}{2}}$ on $\phi_{n-1}(C)$. In this case $g(\phi_{n-1}(C)) = h$. Let $\phi_{n-1}(C) = C'_h$. Since $\phi_n(P) \neq \phi_n(Q)$ and $\phi_{n-1}(P) = \phi_{n-1}(Q)$, $\phi_{n-1}\times\phi_n : C \to C'_h \times C_h$ is birational onto its image. Again by Lemma 1.4, $g(C) \leq (2-1)(2-1) + 2h + 2h = 4h+1 < g$ which is a contradiction.

Thus we have $\pi(P) = \pi(Q)$ and the last assertion follows from Theorem 2.2. □

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