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HYERS-ULAM STABILITY OF TERNARY (σ, τ, ξ)-DERIVATIONS ON C^*-TERNARY ALGEBRAS: REVISITED

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Abstract. In [1], the definition of C^*-Lie ternary (σ, τ, ξ)-derivation is not well-defined and so the results of [1, Section 4] on C^*-Lie ternary (σ, τ, ξ)-derivations should be corrected.

1. Hyers-Ulam Stability of C^*-Lie Ternary (σ, τ, ξ)-derivations

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \to [xyz]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[xy[zwv]] = [x[wzy]v] = [[xyz]wv]$, and satisfies $\|[xyz]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[xxx]\| = \|x\|^3$.

Definition 1.1 ([1])**.** Let A be a C^*-ternary algebra and let $\sigma, \tau, \xi : A \to A$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $L : A \to A$ is called a *C^*-Lie ternary (σ, τ, ξ)-derivation* if

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)xz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$, where $[xyz]_{(\sigma,\tau,\xi)} = x\tau(y)\xi(z) - \sigma(z)\tau(y)x$.

The x- and z-variables of the left side are $\mathbb{C}$-linear and the y-variable of the left side is conjugate $\mathbb{C}$-linear. But the x-variable of the right side is not $\mathbb{C}$-linear and the y-variable of the right side is not conjugate $\mathbb{C}$-linear. Furthermore, the y-variable of the right side in the definition of $[xyz]$ is $\mathbb{C}$-linear. But the y-variable of the left side is conjugate $\mathbb{C}$-linear. Thus we correct the definition of C^*-Lie ternary (σ, τ, ξ)-derivation as follows.

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Definition 1.2. Let A be a C^*-ternary algebra and let $\sigma, \tau, \xi : A \to A$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $L : A \to A$ is called a *C^*-Lie ternary (σ, τ, ξ)-derivation* if

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)^*x^*z]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$, where $[xyz]_{(\sigma,\tau,\xi)} = x\tau(y)^*\xi(z) - \sigma(z)\tau(y)^*x$.

Throughout this paper, assume that A is a C^*-ternary with norm $\|\cdot\|$, and that $\sigma, \tau, \xi : A \to A$ are $\mathbb{C}$-linear mappings. Let q be a positive rational number.

We prove the Hyers-Ulam stability of C^*-Lie ternary (σ, τ, ξ)-derivations on C^*-ternary algebras, associated with the Euler-Lagrange type additive mapping.

Theorem 1.3. *Let $n \in \mathbb{N}$. Assume that $r > 3$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1), (2.3)–(2.5) *of* [1] *and*

$$\begin{aligned} &\|f([xyz]) - [f(x)yz]_{(g,h,k)} - [f(y)^*x^*z]_{(g,h,k)} - [f(z)yx]_{(g,h,k)}\| \\ &\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{1.1}$$

for all $x, y, z \in A$. Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying (2.6)-(2.8) *of* [1] *and*

$$\|f(x) - L(x)\| \leq \frac{\theta}{(nq)^r - nq}\|x\|^r \tag{1.2}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of [1, Theorem 2.1], one can show that there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique $\mathbb{C}$-linear mapping $L : A \to A$ satisfying (2.6)–(2.8) of [1] and (1.2). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$.

It follows from (1.1) that

$$\begin{aligned} &\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)^*x^*z]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\ &= \lim_{m\to\infty} (nq)^{3m} \left(\left\| f\left(\frac{[xyz]}{(nq)^{3m}}\right) - \left[f\left(\frac{x}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{z}{(nq)^m} \right]_{(g,h,k)} \right. \right. \end{aligned}$$

$$- \left[f\left(\frac{y}{(nq)^m}\right)^* \frac{x^*}{(nq)^m} \frac{z}{(nq)^m} \right]_{(g,h,k)} - \left[f\left(\frac{z}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{x}{(nq)^m} \right]_{(g,h,k)} \Bigg\| \Bigg)$$
$$\leq \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{mr}} (\|x\|^r + \|y\|^r + \|z\|^r) = 0$$

for all $x, y, z \in A$. So

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)^* x^* z]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of [1, Theorem 2.1]. □

Theorem 1.4. *Let $n \in \mathbb{N}$. Assume that $0 < r < 1$ if $nq > 1$ and that $r > 3$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1), (2.3)–(2.5) *of* [1] *and* (1.1). *Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying* (2.12)–(2.14) *of* [1] *and*

$$\|f(x) - L(x)\| \leq \frac{\theta}{nq - (nq)^r} \|x\|^r \tag{1.3}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of [1, Theorem 2.2], there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique $\mathbb{C}$-linear mapping $L : A \to A$ satisfying (2.1), (2.3)–(2.5) of [1] and (1.3). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} \frac{1}{(nq)^m} f((nq)^m x)$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorem 1.3 and [1, Theorem 2.1]. □

Theorem 1.5. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq > 1$ and that $0 < nr < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5), (2.17) *of* [1] *and*

$$\begin{aligned} &\|f([xyz]) - [f(x)yz]_{(g,h,k)} - [f(y)^* x^* z]_{(g,h,k)} - [f(z)yx]_{(g,h,k)}\| \\ &\leq \theta \|x\|^r \|y\|^r \|z\|^r \end{aligned} \tag{1.4}$$

for all $x, y, z \in A$. Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying (2.6)–(2.8) *of* [1] *and*

$$\|f(x) - L(x)\| \leq \frac{\theta}{n((nq)^{nr} - nq)} \|x\|^{nr} \tag{1.5}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of [1, Theorem 2.3], there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ and a unique $\mathbb{C}$-linear mapping $L : A \to A$ satisfying (2.6)–(2.8) of [1] and (1.5). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} (nq)^m f\left(\frac{x}{(nq)^m}\right)$$

for all $x \in A$.

It follows from (1.4) that

$$\begin{aligned}
&\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)^* x^* z]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} (nq)^{3m} \left(\left\| f\left(\frac{[xyz]}{(nq)^{3m}}\right) - \left[f\left(\frac{x}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{z}{(nq)^m} \right]_{(g,h,k)} \right.\right. \\
&\quad \left.\left. - \left[f\left(\frac{y}{(nq)^m}\right)^* \frac{x^*}{(nq)^m} \frac{z}{(nq)^m} \right]_{(g,h,k)} - \left[f\left(\frac{z}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{x}{(nq)^m} \right]_{(g,h,k)} \right\| \right) \\
&\leq \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{3mr}} (\|x\|^r \|y\|^r \|z\|^r) = 0
\end{aligned}$$

for all $x \in A$. Hence

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)^* x^* z]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$ and the proof of the theorem is complete. □

Theorem 1.6. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq < 1$ and that $0 < nr < 1$ if $nq > 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5), (2.17) *of* [1] *and* (1.4). *Then there exist unique $\mathbb{C}$-linear mappings $\sigma, \tau, \xi : A \to A$ to A and a unique C^*-ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying* (2.12)–(2.14) *of* [1] *and*

$$\|f(x) - L(x)\| \leq \frac{\theta}{n(nq - (nq)^{nr})} \|x\|^{nr} \tag{1.6}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of [1, Theorem 2.4], there exist unique $\mathbb{C}$-linear mappings $\sigma,\tau,\xi : A \to A$ and a unique $\mathbb{C}$-linear mapping $L : A \to A$ satisfying (2.12)–(2.14) of [1] and (1.6). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} \frac{1}{(nq)^m} f((nq)^m x)$$

for all $x \in A$.

It follows from (1.4) that

$$\begin{aligned}
&\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)^*x^*z]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} \frac{1}{(nq)^{3m}} (\|f((nq)^{3m}[xyz]) - [f((nq)^m x)(nq)^m y(nq)^m z]_{(g,h,k)} \\
&\quad - [f((nq)^m y)^*(nq)^m x^*(nq)^m z)]_{(g,h,k)} - [f((nq)^m z)(nq)^m y(nq)^m x]_{(g,h,k)}\|) \\
&\le \lim_{m\to\infty} \frac{(nq)^{3mr}\theta}{(nq)^{3m}} (\|x\|^r\|y\|^r\|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)^*x^*z]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x \in A$ and the proof of the theorem is complete. □

References

1. M. Eshaghi Gordji, R. Farrokhzad & S.A.R. Hosseinioun: Hyers-Ulam stability of ternary (σ,τ,ζ)-derivations on C^*-ternary algebras. *J. Math. Phys. Anal. Geom.* **8** (2012), no. 1, 3-20.

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