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UNIFORMLY LIPSCHITZ STABILITY AND ASYMPTOTIC PROPERTY IN PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. This paper shows that the solutions to the perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds + h(t, y(t))$$

have asymptotic property and uniform Lipschitz stability. To show these properties, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), Ty(s))ds, h(t, y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$.

1. Introduction

Brauer[2] studied the asymptotic behavior of solutions of nonlinear systems and perturbations of nonlinear systems by means of analogue of the variation of constants formula for nonlinear systems due to V.M. Alekseev[1]. Elaydi and Farran[9] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS, which is introduced by Dannan and Elaydi[8]. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte[15,16] investigated the stability and asymptotic behavior of solutions of the functional differential equation. Gonzalez and Pinto[10] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al.[6,7] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Goo[11] and Choi et al.[3,5] investigated Lipschitz and asymptotic stability for perturbed differential systems. Also, Im and Goo[13] investigated asymptotic property for solutions of the perturbed functional differential systems.

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In this paper we will obtain some results on ULS and asymptotic property for perturbed nonlinear differential systems. We will employ the theory of integral inequalities to study ULS and asymptotic property for solutions of perturbed nonlinear differential systems.

2. Preliminaries

We consider the nonautonomous differential system

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds + h(t, y(t)),\ y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = 0$, $h(t, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel[9].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called (ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le$

$M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$,
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0) \le M$ for $|x_0| \le \delta$ and $t \ge t_0 \ge 0$.
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \le K\,|x_0|e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \le K\,e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \infty$

Remark 2.2 ([10]). The last definition implies that for $|x_0| \le \delta$

$$|x(t)| \le K\,|x_0|e^{-c(t-t_0)}, 0 \le t_0 \le t.$$

For the proof we prepare some related properties. We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ \ y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.3 ([2]). *Let x and y be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \ge t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.4. (Bihari−type Inequality) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ \ t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda(s) ds\right],$$

where $t_0 \le t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW^{-1}}\right\}.$$

Lemma 2.5 ([4])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \le c + &\int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^{t} \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \ \ 0 \le t_0 \le t. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + &\int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau \\ &+\lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

For the proof we need the following two corollaries from Lemma 2.5.

Corollary 2.6. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \mathrm{domW^{-1}}\right\}.$$

Corollary 2.7. *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)u(\tau)d\tau ds.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau) ds\right],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau) ds \in \mathrm{domW}^{-1}\right\}.$$

Lemma 2.8 ([12])**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) u(\tau) + \lambda_3(\tau) w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) u(r) dr) d\tau ds.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau ds\right],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \\ + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

For the proof we need the following corollary.

Corollary 2.9. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) w(u(\tau)) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) u(r) dr) d\tau ds.$$

Then

$$u(t) \quad \le W^{-1}\left[W(c) + \textstyle\int_{t_0}^{t} \lambda_1(s) \displaystyle\int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) dr) d\tau ds\right],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) dr) d\tau ds \in \mathrm{domW}^{-1}\right\}.$$

3. Main Results

In this section, we investigate ULS and asymptotic property for solutions of the perturbed nonlinear differential systems

To obtain ULS and asymptotic property, the following assumptions are needed:

(H1) The solution $x = 0$ of (1.1) is EASV.

(H2) $w(u)$ is nondecreasing in u, $u \le w(u)$.

Theorem 3.1. *Suppose that (H1), (H2), and the perturbing term $g(t, y, Ty)$ satisfies*

$$|g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t)w(|y(t)|) + |Ty(t)|\Big), \tag{3.1}$$

and

$$|Ty(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds, |h(t, y(t))| \le \int_{t_0}^{t} e^{-\alpha s}c(s)|y(s)|ds, \tag{3.2}$$

where $\alpha > 0$, $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $a, b, c, k \in L^1(\mathbb{R}^+)$. If

$$M(t_0) = W^{-1}\Bigg[W(c)+\int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}[a(\tau)+c(\tau)+b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds\Bigg] < \infty, \tag{3.3}$$

where $t \ge t_0$ and $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), the solution $x = 0$ of (2.1) is EASV. Therefore, it is EAS by remark 2.2. Using Lemma 2.3, together with (3.1) and (3.2), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big(\int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))|d\tau + |h(s, y(s))|\Big)ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s} e^{-\alpha\tau}\Big(a(\tau)w(|y(\tau)|) + c(\tau)|y(\tau)| \\
&\quad +b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr\Big)d\tau ds.
\end{aligned}$$

It follows from (H2) that

$$\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}\Big(a(\tau)w(|y(\tau)|e^{\alpha\tau}) + c(\tau)|y(\tau)|e^{\alpha\tau} \\
&+b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr\Big)d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Lemma 2.8 and (3.3) we obtain

$$\begin{aligned}|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + \int_{t_0}^{t} Me^{\alpha s}\int_{t_0}^{s}\Big(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr\Big)d\tau ds\Big]\\ &\le e^{-\alpha t}M(t_0),\end{aligned}$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. □

Remark 3.2. Letting $c(s) = 0$ for $t_0 \le s \le t$ in Theorem 3.1, we obtain the same result as that of Theorem 3.1 in [13].

Theorem 3.3. *Suppose that (H1), (H2), and the perturbing term $g(t, y, Ty)$ satisfies*

$$|g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t)w(|y(t)|) + |Ty(t)|\Big), \tag{3.4}$$

and

$$|Ty(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds, |h(t, y(t))| \le \int_{t_0}^{t} e^{-\alpha s}c(s)w(|y(s)|)ds, \tag{3.5}$$

where $\alpha > 0$, $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $a, b, c, k \in L^1(\mathbb{R}^+)$. If

$$M(t_0) = W^{-1}\Big[W(c)+\int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}[a(\tau)+c(\tau)+b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds\Big] < \infty,\ t \ge t_0, \tag{3.6}$$

where $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), the solution $x = 0$ of (2.1) is EASV, and so it is EAS by remark 2.2. Applying Lemma 2.3, together with (3.4) and (3.5), we have

$$\begin{aligned}|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s} e^{-\alpha\tau}\big((a(\tau) + c(\tau))w(|y(\tau)|)\\ +b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr\big)d\tau ds.\end{aligned}$$

From (H2) , we obtain

$$\begin{aligned}|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}\big((a(\tau) + c(\tau))w(|y(\tau)|e^{\alpha\tau})\\ +b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr\big)d\tau ds.\end{aligned}$$

Defining $u(t) = |y(t)|e^{\alpha t}$, then by corollary 2.9 and (3.6) we obtain

$$
\begin{aligned}
|y(t)| &\leq e^{-\alpha t} W^{-1}\Big[W(c) + \int_{t_0}^{t} M e^{\alpha s} \int_{t_0}^{s} \Big(a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr\Big) d\tau ds\Big] \\
&\leq e^{-\alpha t} M(t_0),
\end{aligned}
$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Remark 3.4. Letting $c(s) = 0$ for $t_0 \leq s \leq t$ in Theorem 3.3, we obtain the same result as that of Theorem 3.1 in [13].

Theorem 3.5. *Suppose that (H1), (H2), and the perturbed term $g(t, y, Ty)$ satisfies*

$$
\int_{t_0}^{t} |g(s, y(s), Ty(s))| ds \leq e^{-\alpha t}\Big(a(t)|y(t)| + |Ty(t)|\Big), \tag{3.7}
$$

and

$$
|Ty(t)| \leq b(t) \int_{t_0}^{t} k(s)|y(s)|ds, |h(t, y(t))| \leq e^{-\alpha t} c(t) w(|y(t)|), \tag{3.8}
$$

where $\alpha > 0$, $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $a, b, c, k \in L^1(\mathbb{R}^+)$. If

$$
M(t_0) = W^{-1}\Big[W(c) + M \int_{t_0}^{\infty} \Big(a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau\Big) ds\Big] < \infty, \tag{3.9}
$$

where $b_1 = \infty$ and $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. From (H1), the solution $x = 0$ of (2.1) is EASV. Therefore, it is EAS. Using Lemma 2.3, together with (3.7) and (3.8), we have

$$
\begin{aligned}
|y(t)| \leq M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} M e^{-\alpha(t-s)}\Big(e^{-\alpha s} a(s)|y(s)| \\
&+ e^{-\alpha s} b(s) \int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + e^{-\alpha s} c(s) w(|y(s)|)\Big) ds.
\end{aligned}
$$

Applying (H2) , we obtain

$$
\begin{aligned}
|y(t)| \leq M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} M e^{-\alpha t}\Big(a(s)|y(s)|e^{\alpha s} + c(s) w(|y(s)|e^{\alpha s})\Big) ds \\
&+ \int_{t_0}^{t} M e^{-\alpha t} b(s) \int_{t_0}^{s} k(\tau)|y(\tau)|e^{\alpha \tau} d\tau ds.
\end{aligned}
$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, it follows from Corollary 2.6 and (3.9) that

$$
\begin{aligned}
|y(t)| &\le e^{-\alpha t} W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau\Big)ds\Big] \\
&\le e^{-\alpha t} M(t_0),
\end{aligned}
$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. □

Remark 3.6. Letting $b(s) = c(s) = 0$ for $t_0 \le s \le t$ in Theorem 3.5, we obtain the same result as that of Corollary 3.8 in [5].

Theorem 3.7. *Suppose that (H1), (H2), and the perturbed term $g(t, y, Ty)$ satisfies*

$$
\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le e^{-\alpha t}\Big(a(t)w(|y(t)|) + |Ty(t)|\Big), \tag{3.10}
$$

and

$$
|Ty(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds, \; |h(t, y(t))| \le e^{-\alpha t}c(t)w(|y(t)|), \tag{3.11}
$$

where $\alpha > 0$, $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $a, b, c, k \in L^1(\mathbb{R}^+)$. If

$$
M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau\Big)ds\Big] < \infty, \tag{3.12}
$$

where $b_1 = \infty$ and $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. From the assumption (H1), the solution $x = 0$ of (2.1) is EASV, and so it is EAS. Using Lemma 2.3, together with (3.10) and (3.11), we have

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(e^{-\alpha s}a(s)w(|y(s)|) \\
&+ e^{-\alpha s}b(s)\int_{t_0}^{s}k(\tau)|y(\tau)|d\tau + e^{-\alpha s}c(s)w(|y(s)|)\Big)ds.
\end{aligned}
$$

By the assumption (H2) , we obtain

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} Me^{-\alpha t}(a(s)+c(s))w(|y(s)|e^{\alpha s})ds \\
&+ \int_{t_0}^{t} Me^{-\alpha t}b(s)\int_{t_0}^{s}k(\tau)|y(\tau)|e^{\alpha\tau}d\tau ds.
\end{aligned}
$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, it follows from Corollary 2.7 and (3.12) that

$$
\begin{aligned}
|y(t)| &\le e^{-\alpha t} W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau\Big)ds\Big] \\
&\le e^{-\alpha t} M(t_0),
\end{aligned}
$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Remark 3.8. Letting $c(s) = 0$ for $t_0 \leq s \leq t$ in Theorem 3.7, we obtain the same result as that of Theorem 3.3 in [13].

Theorem 3.9. *For the perturbed (2.2), we suppose that (H2),*

$$\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \leq a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)|y(s)|ds + |Ty(t)|, \tag{3.13}$$

and

$$|Ty(t)| \leq c(t)w(|y(t)|), |h(t, y(t))| \leq d(t)\int_{t_0}^{t} q(s)w(|y(s)|)ds \tag{3.14}$$

where $a, b, c, d, k, q, u \in C(\mathbb{R}^+)$, $a, b, c, d, k, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty)$, $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$, *and*

(3.15)

$$M(t_0) = W^{-1}\Big[W(K) + K\int_{t_0}^{\infty}\Big(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big],$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *If the zero solution of (2.1) is ULSV, then the zero solution of (2.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS by Theorem 3.3[8]. In view of Lemma 2.3, together with ULSV condition of $x = 0$ of (2.1), (3.13) and (3.14), we obtain

$$\begin{aligned}
|y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big(\int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))|d\tau + |h(s, y(s))|\Big)ds \\
&\leq K|y_0| + \int_{t_0}^{t} K|y_0|\Big(a(s)\frac{|y(s)|}{|y_0|} + c(s)w\Big(\frac{|y(s)|}{|y_0|}\Big)\Big)ds \\
&\quad + \int_{t_0}^{t} K|y_0|(b(s)\Big(\int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{|y_0|}d\tau + d(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{|y_0|})d\tau\Big)ds.
\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, by Lemma 2.5, we have

$$|y(t)| \leq |y_0|W^{-1}\Big[W(K) + K\int_{t_0}^{t}\Big(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big].$$

The above estimation and (3.15) yield the desired result. Hence the proof is complete. □

Remark 3.10. Letting $b(s) = c(s) = d(s) = 0$ for $t_0 \leq s \leq t$ in Theorem 3.9, we obtain the same result as that of Corollary 3.4 in [5].

Theorem 3.11. *For the perturbed (2.2), we suppose that (H2),*

$$|g(t, y(t), Ty(t))| \leq a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s)|y(s)|ds + |Ty(t)|, \tag{3.16}$$

and

$$|Ty(t)| \leq c(t)w(|y(t)|), |h(t, y(t))| \leq \int_{t_0}^{t} q(s)w(|y(s)|)ds \tag{3.17}$$

where $a, b, c, k, q, u \in C(\mathbb{R}^+)$*,* $a, b, c, k, q \in L^1(\mathbb{R}^+)$ *,* $w \in C((0, \infty))$*,* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$*, and*

$$M(t_0) = W^{-1}\Big[W(K) + K \int_{t_0}^{\infty} \int_{t_0}^{s} \Big(a(\tau) + c(\tau) + q(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr\Big)d\tau ds\Big], \tag{3.18}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$*. If the zero solution of (2.1) is ULSV, then the zero solution of (2.2) ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Using Lemma 2.3, together with ULSV condition of $x = 0$ of (2.1), (3.16), and (3.17) , we obtain

$$\begin{aligned} |y(t)| \leq K|y_0| &+ \int_{t_0}^{t} K|y_0| \int_{t_0}^{s} \Big(a(\tau)\frac{|y(\tau)|}{|y_0|} + (c(\tau) + q(\tau))w\Big(\frac{|y(\tau)|}{|y_0|}\Big)\Big)d\tau ds \\ &+ \int_{t_0}^{t} K|y_0| \int_{t_0}^{s} b(\tau) \int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|}drd\tau ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Lemma 2.8 yields

$$|y(t)| \leq |y_0|W^{-1}\Big[W(K) + K \int_{t_0}^{t} \int_{t_0}^{s} \Big(a(\tau) + c(\tau) + q(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr\Big)d\tau ds\Big].$$

Thus, by (3.18), we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 3.12. Letting $b(s) = c(s) = q(s) = 0$ for $t_0 \leq s \leq t$ in Theorem 3.11, we obtain the same result as that of Corollary 3.2 in [5].

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