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ON SOME UNBOUNDED DOMAINS FOR A MAXIMUM PRINCIPLE

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Abstract. In this paper, we study some characterizations of unbounded domains. Among these, so-called G-domain is introduced by Cabre for the Aleksandrov-Bakelman-Pucci maximum principle of second order linear elliptic operator in a non-divergence form. This domain is generalized to wG-domain by Vitolo for the maximum principle of an unbounded domain, which contains G-domain. We study the properties of these domains and compare some other characterizations. We prove that sA-domain is wG-domain, but using the Cantor set, we are able to construct a example which is wG-domain but not sA-domain.

1. Introduction

We consider the second order elliptic operator in the following non-divergence form

$$Lu(x) = a_{ij}(x)D_{ij}u(x) + b_i(x)D_iu(x) + c(x)u \tag{1.1}$$

in a given domain Ω in $\mathbb{R}^n$, where $D_i = \frac{\partial}{\partial x_i}$, $D_{ij} = D_iD_j$. The operator is called a uniformly elliptic if, for some positive constants λ, Λ,

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \Omega, \quad c(x) \leq 0. \tag{1.2}$$

For the elliptic operator, there is a well known property called maximum principles. For the bounded domain, it can be written as follows:

Theorem 1.1. *Let $Lu \geq 0$ for some bounded domain Ω, then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

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For an unbounded domain, one can consider the following simple example:

$$\Delta u = 0 \text{ in } \mathbb{R}^n_+, \qquad u = 0 \text{ on } \partial\mathbb{R}^n_+.$$

Here, Δ is the Laplace operator, $\mathbb{R}^n_+$ is an upper half plane. For the Dirichlet value problem, we have infinitely many solutions of the form $u(x) = u(x_1, x_2, ..., x_n) := kx_n$ for any $k \in \mathbb{R}$.

Thus, unlike bounded domains, the maximum principle is not easy to obtain for the unbounded domains, hence there are recent publications regarding the subject. For example, one may refer to [1, 2, 3, 5] and references therein.

Definition 1.1 ([1])**.** We say that the maximum principle holds for the operator L in Ω if

$$Lw \geq 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\limsup_{x \to \partial\Omega} w(x) \leq 0 \tag{1.4}$$

imply $w \leq 0$ in Ω.

Using an improved classical Alexandrov-Bakelman-Pucci maximum principle, Cabre [2] obtained the maximum principle above for the following type of domains, which will be denoted by G-domain hereafter.

Definition 1.2 ([2])**.** We say that Ω satisfies a *condition* G if there exist positive constants $\sigma < 1, \tau < 1$ and R_0 such that

$$\forall x \in \Omega \quad \exists B_{R_x} \quad \text{s.t.} \quad |B_{R_x} \setminus \Omega_{x,\tau}| \geq \sigma |B_{R_x}|, \tag{1.5}$$

where B_{R_x} is a ball containing x of radius $R_x \leq R_0$ and $\Omega_{x,\tau}$ is the component of $\Omega \cap B_{R_x/\tau}$ to which x belongs.

As noted by Vitolo [5], the G-domain contains connected open sets with finite measure, infinite cylinders, and strips. The explanations are presented in the next section.

By Cafagna and Vitolo [3], G-domain was generalized to wG-domain, and they obtained the maximum principle.

Definition 1.3 ([3])**.** We say that Ω satisfies a *condition* wG if there exist positive constants $\sigma < 1$ and $\tau < 1$ such that

$$\forall x \in \Omega \quad \exists B_{R_x} \quad \text{s.t.} \quad |B_{R_x} \setminus \Omega_{x,\tau}| \geq \sigma |B_{R_x}|, \tag{1.6}$$

where B_{R_x} is a ball containing x of radius R_x and $\Omega_{x,\tau}$ is the component of $\Omega \cap B_{R_x/\tau}$ to which x belongs.

Observe that in the definition, we do not impose any restriction on the boundedness of radius R unlike G-domain. It is immediate to see that G-domain is wG-domain. In the next section, we present examples of wG-domain, which is not G-domain.

The following A-domain appear in the book by O. A. Ladyzhenskaya and N. N. Uraltseva [4].

Definition 1.4. A domain Ω is called A-*domain* if there exists a constant $\sigma > 0$ and $R > 0$, such that for each $y \in \partial\Omega$ and $r \in (0, R)$, the Lebesgue measure

$$|B_r(y) \setminus \Omega| \geq \sigma |B_r|, \tag{1.7}$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at y.

Similar to wG-domain, we may also consider the following sA-domain. But in this case, the condition is stronger than A condition unlike G condition.

Definition 1.5. A domain Ω is called sA-*domain* if there exists a constant $\sigma > 0$, such that for each $y \in \partial\Omega$ and $r > 0$, the Lebesgue measure

$$|B_r(y) \setminus \Omega| \geq \sigma |B_r|, \tag{1.8}$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at y.

There are examples which are sA-domain, but not A-domain, which is also presented in the next section.

So far, we introduce 4 types of condition, G, wG, A, sA conditions. By its definition, it is rather easy to tell the inclusion of between G and wG, A and sA. In the paper, we show that the sA condition imply the wG condition, but the converse does not hold.

Theorem 1.2. *Any sA domain is wG domain, but the converse does not hold.*

It is proved in Theorem 2.2 and Theorem 2.3. The main idea for a counter example is to use Cantor set for its construction, such that, we are able to construct locally A-domain, but not sA-domain for big r for any σ.

2. Main Results

In this section, we prove main results of the paper, and some known and unknown but simple related facts. Firstly, we enlist some known examples of G-domain.

Example 2.1. Any connected open set Ω with finite measure is a G-domain. Namely Ω does satisfy Definition 1.2. Let the Lebesgue measure of Ω, $|\Omega| = m$, choose sufficiently large R such that $|B_R| \geq 2m$. Then Ω satisfies Definition with $\sigma = \frac{1}{2}$, for any $\tau > 1$, $R_0 = R$. Note that for any $x \in \Omega$, there exists $B_{R_x} = B_R(x)$,

$$|B_{R_x} \setminus \Omega_{x,\tau}| \geq |B_R| - |\Omega| \geq \frac{1}{2}|B_R|.$$

Example 2.2. Any infinite cylinder and strips are G-domain. Let $\Omega = \{x \in \mathbb{R}^n \,|\, |x'| \leq r, r > 0, x = (x', x_n)\}$. Then for each $x \in \Omega$, there exists $B_{R_x} = B_{2r}(x)$ such that, for any $\tau > 1$,

$$|B_{R_x} \setminus \Omega_{x,\tau}| \geq |B_{r/2}(y)| \geq \frac{1}{4^n}|B_{2r}|$$

for some $y \in \mathbb{R}^n \setminus \Omega$. Domains of strips case is similar.

Example 2.3. A checked domain is also G-domain. Let

$$\Omega_1 := \{x \in \mathbb{R} \,|\, x \in (2i, 2i+1) \text{ for some integer } i\}, \qquad \Omega := \Omega_1 \times \Omega_1.$$

Note that for any $x \in \Omega$, $B_{10}(x)$ contains a unit square in $\mathbb{R}^n \setminus \Omega$. Similarly, one can prove n-dimensional case.

The G-domain (Definition 1.2) is generalized to wG-domain (Definition 1.3). The following examples show that the converse does not hold.

Example 2.4. Any open connected cone whose closure is different from the whole space is wG-domain, but not G-domain. For example we consider 2-dimensional case. Let $\Omega := \{x \in \mathbb{R}^2 \,|\, x_2 > x_1 \wedge x_2 > -x_1, x = (x_1, x_2)\}$. For any $x \in \Omega$, choose $B_{2|x|}(0)$. With this ball, it is easy to that it satisfies the definition 1.3. But, note that for any positive y, the point $(0, \sqrt{2}y)$ in Ω has a distance of y to its boundary. Thus at least $B_{y/2}$ is needed to touch outside of Ω containing the point. This means that one can not impose the boundedness of R in the definition 1.2. Thus in all Ω is not G-domain.

Example 2.5. Let

$$\Omega_1 := \{x \in \mathbb{R} \,|\, x \in (2^{2i}, 2^{2i+1}) \text{ for some natural number } i\}, \qquad \Omega := \Omega_1 \times \Omega_1.$$

Similar to the previous example, Ω is wG-domain, but not G-domain.

As discussed in the introduction, there are examples which are sA-domain, but not A-domain.

Theorem 2.1. *There exist a domain which does satisfies A condition, but not sA condition.*

Proof. Consider the following domain in $\mathbb{R}^2$

$$\Omega := \{x \in \mathbb{R}^2 \,|\, x_2 < x_1^2, x = (x_1, x_2)\}.$$

It is immediate to see that Ω is A-domain considering the unit ball centering on its boundary. Considering the unit ball centered at the origin,

$$|B_r(0) \setminus \Omega| = 2\int_0^{\sqrt{\frac{-1+\sqrt{1+4r^2}}{2}}} \sqrt{r^2 - x^2} - x^2 dx \leq 2 \cdot r \cdot \sqrt{r}.$$

Thus,

$$\frac{|B_r(0) \setminus \Omega|}{|B_r|} \to 0 \quad \text{as } r \nearrow \infty.$$

□

The next theorem implies that the sA condition imply the wG condition.

Theorem 2.2. *Any domain Ω which satisfies Definition 1.5 does satisfy Definition 1.3.*

Proof. Let Ω be a sA-domain, x be an arbitrary point in Ω, $d(x)$ be a distance of x to $\partial\Omega$, and $|x - y| = d(x)$ for some $y \in \partial\Omega$. Choose $R = 2d(x)$, then $x \in B_{2d(x)}(y)$ and

$$|B_{2d(x)}(y) \setminus \Omega_{x,\tau}| \geq |B_{2d(x)}(y) \setminus \Omega| \geq \sigma|B_{2d(x)}| \tag{2.1}$$

by Definition 1.5. Thus in all, Ω is wG domain with the same σ and for any $\tau < 1$. □

For the next, we will present an example which is wG-domain, but not sA-domain. Thus, the converse of the previous theorem does not hold.

First consider the domain in $\mathbb{R}^2$ using the Cantor set. The Cantor set C is defined by

$$C = [0, 1] \setminus \cup_{m=1}^{\infty} \cup_{k=0}^{3^{m-1}-1} (\frac{3k+1}{3^m}, \frac{3k+2}{3^m}).$$

For each positive integer m, let

$$D_m := \cup_{k=0}^{3^{m-1}-1}(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}), \quad D := \cup_{m=1}^{\infty} D_m.$$

Note that $|C| = 0$, $|D| = 1$. Now we define a set operations as follows: for any set S, we set $-S = \{-x \,|\, x \in S\}$, $m + S = S + m := \{m + x \,|\, x \in S\}$.

We define an open set in $\mathbb{R}^2$.

$$\Omega_1^+ := \cup_{m=1}^{\infty}((m-1)+D_m)\times(-\infty,+\infty), \quad \Omega_1^- := -\Omega_1^+, \quad \Omega_1 := \Omega_1^+ \cup \Omega_1^-.$$

Note that Ω_1 satisfies Definition 1.3, but is not connected. To connect these components, we add the following sets: for any $m \in \mathbb{N}$,

$$B_m := ([m-1,m)\cup(-m,1-m])\times(-\frac{1}{2\cdot 3^m},\frac{1}{2\cdot 3^m}).$$

Now we take Ω_1' to be $\cup_{m=1}^{\infty}B_m\cup\Omega_1$. Then Ω_1' is wG-domain. For Ω_1', it is not easy to check that it satisfies Definition 1.5 due to the irregularity of the domain. It is difficult to estimate $\frac{|B_r(0)\setminus\Omega_1'|}{|B_r|}$.

We will modify the above idea to obtain the following theorem.

Theorem 2.3. *There is wG-domain, which is not sA-domain.*

Proof. First recall that the Cantor set C is defined by

$$C = [0,1]\setminus\cup_{m=1}^{\infty}\cup_{k=0}^{3^{m-1}-1}(\frac{3k+1}{3^m},\frac{3k+2}{3^m}).$$

For each positive integer m, let

$$D_m := \cup_{k=0}^{3^{m-1}-1}(\frac{3k+1}{3^m},\frac{3k+2}{3^m}), \quad D := \cup_{m=1}^{\infty}D_m.$$

Note that $|C| = 0$, $|D| = 1$.

Now we define a set operations as follows: for any set S, we set $m+S = S+m := \{m+x\,|\,x\in S\}$. Also we define

$$E := \cup_{m=1}^{\infty}(m-1)+D_m.$$

Let

$$\Omega := \{x\in\mathbb{R}^n\,|\,|x|\in E\}\cup\{x\in\mathbb{R}^n\,|\,x=(x_1,x'),|x_1|\in[m-1,m),|x'|\le\frac{1}{3^m}\text{ for some } m\in\mathbb{N}\}.$$

It is easy to see that Ω is connected since $x'=0$ is contained in Ω. Observe that

$$\frac{|\Omega\cap B_r|}{|B_r|}\nearrow 1 \quad \text{as } r\nearrow\infty. \tag{2.2}$$

For any $x\in\Omega$ and $|x|\le m$, then $x\in B_{\frac{2}{3^m}}(y)$ for some $y\in\partial\Omega$, and $B_{\frac{1}{2\cdot 3^m}}(z)\subset\mathbb{R}^n\setminus\Omega$, $B_{\frac{1}{2\cdot 3^m}}(z)\subset B_{\frac{2}{3^m}}(y)$. This is due to the fact that if $|x|\le m$, then x belongs to a locally connected component of width $\frac{1}{3^m}$. Thus in all, Ω is wG-domain.

But $0\in\Omega$ and $B_r(0)$ is a disjoint union of $B_r\setminus\Omega$ and $B_r\cap\Omega$, we have that

$$\frac{|B_r(0)\setminus\Omega|}{|B_r|}\searrow 0 \quad \text{as } r\nearrow\infty$$

due to (2.2). Thus Ω is not sA-domain. □

Remark 2.6. In the definition of sA-domain, one may replace Ω by $\Omega_{x,\tau}$ as in the definition of G or wG-domain. But the above example in the proof still works as a counterexample.

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