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HUGE CONTRACTION ON PARTIALLY ORDERED METRIC SPACES

Bhavana Deshpande [a,*], Amrish Handa [b] and Chetna Kothari [c]

Abstract. We establish coincidence point theorem for g-nondecreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. We also obtain the coupled coincidence point theorem for generalized compatible pair of mappings $F, G : X^2 \to X$ by using obtained coincidence point results. Furthermore, an example is also given to demonstrate the degree of validity of our hypothesis. Our results generalize, modify, improve and sharpen several well-known results.

1. Introduction and Preliminaries

In the sequel, we denote by X a non-empty set and $\preceq$ will represent a partial order on X. Given $n \in \mathbb{N}$ with $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). For simplicity, if $x \in X$, we denote $g(x)$ by gx.

The idea of the coupled fixed point was initiated by Guo and Lakshmikantham [9] in 1987.

Definition 1 ([9]). Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of F if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y. \tag{1}$$

Following this paper, Bhaskar and Lakshmikantham [2] where the authors introduced the notion of mixed monotone property for $F : X^2 \to X$ (wherein X is an ordered metric space) and utilized the same to prove some theorems on the existence and uniqueness of coupled fixed points.

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*Corresponding author.

Definition 2 ([2])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all x, $y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y), \tag{2}$$

and

$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \tag{3}$$

In 2009, Lakshmikantham and Ciric [15] generalized these results for nonlinear contraction mappings by introducing the notions of coupled coincidence point and mixed g-monotone property.

Definition 3 ([15])**.** Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of the mappings F and g if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy. \tag{4}$$

Definition 4 ([15])**.** Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *common coupled fixed point* of the mappings F and g if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy. \tag{5}$$

Definition 5 ([15])**.** The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be *commutative* if

$$gF(x,\ y) = F(gx,\ gy),\ \text{for all } (x,\ y) \in X^2. \tag{6}$$

Definition 6 ([15])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the *mixed g-monotone property* if for all x, $y \in X$, we have

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y), \tag{7}$$

and

$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \tag{8}$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Subsequently, Choudhury and Kundu [3] introduced the notion of compatibility and by using this notion to improve the results of Lakshmikantham and Ciric [15], thenafter several authors established coupled fixed/coincidence point theorems by using this notion.

Definition 7 ([3])**.** The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) = 0, \tag{9}$$
$$\lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n,\ y_n) = \lim_{n\to\infty} gx_n = x, \tag{10}$$
$$\lim_{n\to\infty} F(y_n,\ x_n) = \lim_{n\to\infty} gy_n = y,\ \text{for some } x,\ y \in X.$$

A great deal of these studies investigate contractions on partially ordered metric spaces because of their applicability to initial value problems defined by differential or integral equations.

Hussain et al. [11] introduced the notion of generalized compatibility of a pair $\{F, G\}$, of mappings $F, G : X \times X \to X$, then the authors employed this notion to obtained coupled coincidence point results for such a pair of mappings involving (φ, ψ)-contractive condition without mixed G-monotone property of F.

Definition 8 ([11])**.** Suppose that $F, G : X^2 \to X$ are two mappings. The mapping F is said to be $G-$*increasing* with respect to $\preceq$ if for all $x, y, u, v \in X$ with $G(x, y) \preceq G(u, v)$ we have $F(x, y) \preceq F(u, v)$.

Definition 9 ([11])**.** Let $F, G : X^2 \to X$ be two mappings. We say that the pair $\{F, G\}$ is *commuting* if

$$F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x)),\ \text{for all } x,\ y \in X. \tag{11}$$

Definition 10 ([11])**.** Suppose that $F, G : X^2 \to X$ are two mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of mappings F and G if

$$F(x,\ y) = G(x,\ y) \text{ and } F(y,\ x) = G(y,\ x). \tag{12}$$

Definition 11 ([11])**.** Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is *g-increasing with respect to* $\preceq$ if for any $x, y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \tag{13}$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \tag{14}$$

Definition 12 ([11])**.** Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is *increasing with respect to* $\preceq$ if for any $x,\, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \tag{15}$$

and

$$y_1 \preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \tag{16}$$

Definition 13 ([11])**.** Let $F,\, G : X^2 \to X$ are two mappings. We say that the pair $\{F,\, G\}$ is *generalized compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0, \end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned} &\lim_{n\to\infty} G(x_n,\ y_n) = \lim_{n\to\infty} F(x_n,\ y_n) = x, \\ &\lim_{n\to\infty} G(y_n,\ x_n) = \lim_{n\to\infty} F(y_n,\ x_n) = y, \text{ for some } x,\ y \in X. \end{aligned} \tag{17}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Erhan et al. [7], announced that the results established in Hussain et al. [11] can be easily derived from the coincidence point results in the literature.

In [7], Erhan et al. recalled the following basic definitions:

Definition 14 ([1, 8])**.** A coincidence point of two mappings $T,\, g : X \to X$ is a point $x \in X$ such that $Tx = gx$.

Definition 15 ([7])**.** An ordered metric space $(X,\, d,\, \preceq)$ is a metric space $(X,\, d)$ provided with a partial order $\preceq$.

Definition 16 ([2, 11])**.** An ordered metric space $(X,\, d,\, \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all n, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all n. $(X,\, d,\, \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

Definition 17 ([7])**.** Let$(X,\ \preceq)$ be a partially ordered set and let $T,\ g : X \to X$ be two mappings. We say that T is $(g,\ \preceq)$-*non-decreasing* if $Tx \preceq Ty$ for all x, $y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$-*non-decreasing.*

Remark 18 ([7])**.** If T is $(g,\ \preceq)$-non-decreasing and $gx = gy$, then $Tx = Ty$. It follows that

$$gx = gy \Rightarrow \left\{ \begin{array}{c} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty. \tag{18}$$

Definition 19 ([18])**.** Let $(X,\ \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u,\ v),\ (x,\ y) \in X^2. \tag{19}$$

Definition 20 ([3, 10, 17, 18])**.** Let $(X,\ d,\ \preceq)$ be an ordered metric space. Two mappings $T,\ g : X \to X$ are said to be O-*compatible* if

$$\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0, \tag{20}$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Samet et al. [20] declared that most of the coupled fixed point theorems for single-valued mappings on ordered metric spaces can be derived from well-known fixed point theorems.

On the other hand, Ding et al. [6] proved coupled coincidence and common coupled fixed point theorems for generalized nonlinear contraction on partially ordered metric spaces which generalize the results of Lakshmikantham and Ciric [15]. Our fundamental sources are [4-7, 11-14, 16, 18-20].

In this paper, we obtain a coincidence point theorem for g-non-decreasing mappings satisfying generalized nonlinear contraction on partially ordered metric spaces. With the help of our result, we derive a coupled coincidence point theorem of generalized compatible pair of mappings $F,\ G : X^2 \to X$. We also give an example and an application to integral equation to support our results. Our results generalize, extend, modify, improve and sharpen the results of Bhaskar and Lakshmikantham [2], Ding et al. [6] and Lakshmikantham and Ciric [15].

2. Main Results

Lemma 21. *Let* $(X,\, d)$ *be a metric space. Suppose* $Y = X^2$ *and define* $\delta : Y \times Y \to [0,\, +\infty)$ *by*

$$\delta((x,\ y),\ (u,\ v)) = \max\left\{d(x,\ u),\ d(y,\ v)\right\},\ \text{for all } (x,\ y),\ (u,\ v) \in Y. \tag{21}$$

Then δ *is metric on* Y *and* $(X,\, d)$ *is complete if and only if* $(Y,\, \delta)$ *is complete.*

Let Φ denote the set of all functions $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_φ) φ is non-decreasing,
(ii_φ) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi^n(\varphi(t))$.
It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n\to\infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

Theorem 22. *Let* $(X,\, d,\, \preceq)$ *be a partially ordered metric space and let* $T,\, g : X \to X$ *be two mappings such that the following properties are fulfilled:*

(i) $T(X) \subseteq g(X)$,

(ii) *T is $(g,\, \preceq)$-non-decreasing,*

(iii) *there exists* $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$,

(iv) *there exists* $\varphi \in \Phi$ *such that*

$$d(Tx,\ Ty) \leq \varphi\left(M(x,\ y)\right),$$

where

$$M(x,\ y) = \max\left\{\begin{array}{c} d(gx,\ gy),\ d(gx,\ Tx),\ d(gy,\ Ty), \\ \frac{d(gx,\ Ty)+d(gy,\ Tx)}{2} \end{array}\right\},$$

for all $x,\, y \in X$ *such that* $gx \preceq gy$. *Also assume that, at least, one of the following conditions holds:*

(a) *$(X,\, d)$ is complete, T and g are continuous and the pair $(T,\, g)$ is O-compatible,*

(b) *$(X,\, d)$ is complete, T and g are continuous and commuting,*

(c) *$(g(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular,*

(d) *$(X,\, d)$ is complete, $g(X)$ is closed and $(X,\, d,\, \preceq)$ is non-decreasing-regular,*

(e) *$(X,\, d)$ is complete, g is continuous, the pair $(T,\, g)$ is O-compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular.*

Then T and g have, at least, a coincidence point.

Proof. We divide the proof into four steps.

Step 1. We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $gx_{n+1} = Tx_n$, for all $n \geq 0$. Let $x_0 \in X$ be arbitrary. Since

$Tx_0 \in T(X) \subseteq g(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g, \preceq)$-non-decreasing, therefore $Tx_0 \preceq Tx_1$. Again, since $Tx_1 \in T(X) \subseteq g(X)$, therefore there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g, \preceq)$-non-decreasing, therefore $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that $\{gx_n\}$ is $\preceq$-non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{22}$$

Step 2. We claim that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Now, by contractive condition (iv), we have

$$d(gx_{n+1},\ gx_{n+2}) = d(Tx_n,\ Tx_{n+1}) \leq \varphi\left(M(x_n,\ x_{n+1})\right), \tag{23}$$

where

$$\begin{aligned} & M(x_n,\ x_{n+1}) \\ = \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ Tx_n),\ d(gx_{n+1},\ Tx_{n+1}), \\ \frac{d(gx_n,\ Tx_{n+1})+d(gx_{n+1},\ Tx_n)}{2} \end{array} \right\} \\ = \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2}), \\ \frac{d(gx_n,\ gx_{n+2})+d(gx_{n+1},\ gx_{n+1})}{2} \end{array} \right\} \\ \leq \ & \max\left\{d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2})\right\}. \end{aligned}$$

If $d(gx_{n+1}, gx_{n+2}) \geq d(gx_n, gx_{n+1})$. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_{n+1},\ gx_{n+2}). \tag{24}$$

From (23), (24) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\left(d(gx_{n+1},\ gx_{n+2})\right) < d(gx_{n+1},\ gx_{n+2}),$$

which is a contradiction. Hence, $d(gx_n, gx_{n+1}) \geq d(gx_{n+1}, gx_{n+2})$. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_n,\ gx_{n+1}). \tag{25}$$

Thus, by (23) and (25), we have for all $n \in \mathbb{N}$,

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\left(d(gx_n,\ gx_{n+1})\right) \leq \varphi^n\left(d(gx_0,\ gx_1)\right) \leq \varphi^n(\delta), \tag{26}$$

where

$$\delta = d(gx_0,\ gx_1).$$

Without loss of generality, we can assume that $d(gx_0, gx_1) \neq 0$. In fact, if this is not true, then $gx_0 = gx_1 = Tx_0$, that is, x_0 is a coincidence point of g and T.

Thus, for $m,\ n \in \mathbb{N}$ with $m > n$, by triangle inequality and (26), we get

$$\begin{aligned} & d(gx_n,\ gx_{m+n}) \\ \leq\ & d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2}) + ... + d(gx_{n+m-1},\ gx_{m+n}) \\ \leq\ & \varphi^n(\delta) + \varphi^{n+1}(\delta) + ... + \varphi^{n+m-1}(\delta) \\ \leq\ & \sum_{i=n}^{n+m-1} \varphi^i(\delta), \end{aligned}$$

which implies, by (ii_φ), that $\{gx_n\}$ is a Cauchy sequence in X.

Step 3. We claim that T and g have a coincidence point distinguishing between cases $(a) - (e)$.

Suppose now that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible. Since $(X,\ d)$ is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$ and $\{Tx_n\} \to z$. Since T and g are continuous, therefore $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Since the pair $(T,\ g)$ is O-compatible, therefore $\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$. Thus, we conclude that

$$d(gz,\ Tz) = \lim_{n\to\infty} d(ggx_{n+1},\ Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0,$$

that is, z is a coincidence point of T and g.

Suppose now that (b) holds, that is, $(X,\ d)$ is complete, T and g are continuous and commuting. It is evident that (b) implies (a).

Suppose now that (c) holds, that is, $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X),\ d)$, so there exists $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$, then $\{gx_n\} \to gz$. Indeed, as $(X,\ d,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$d(gx_{n+1},\ Tz) = d(Tx_n,\ Tz) \leq \varphi\left(M(x_n,\ z)\right), \tag{27}$$

where

$$\begin{aligned} M(x_n,\ z) &= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ Tx_n),\ d(gz,\ Tz), \\ \frac{d(gx_n,\ Tz)+d(gz,\ Tx_n)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ gx_{n+1}),\ d(gz,\ Tz), \\ \frac{d(gx_n,\ Tz)+d(gz,\ gx_{n+1})}{2} \end{array} \right\}. \end{aligned}$$

Since $\{gx_n\} \to gz$, therefore there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$M(x_n,\ z) = d(gz,\ Tz). \tag{28}$$

By (27) and (28), we get

$$d(gx_{n+1},\ Tz) \leq \varphi\left(d(gz,\ Tz)\right.$$

Now, we claim that $d(gz, Tz) = 0$. If this is not true, then $d(gz, Tz) > 0$, which, by the fact that $\varphi(t) < t$ for all $t > 0$, implies

$$d(gx_{n+1},\ Tz) < d(gz,\ Tz).$$

Letting $n \to \infty$ in the above inequality and using $\lim_{n\to\infty} gx_n = gz$, we get

$$d(gz,\ Tz) < d(gz,\ Tz),$$

which is a contradiction. Hence we must have $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g.

Suppose now that (d) holds, that is, (X, d) is complete, $g(X)$ is closed and $(X, d, \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. Thus (d) implies (c).

Suppose now that (e) holds, that is, (X, d) is complete, g is continuous, the pair (T, g) is O-compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair (T, g) is O-compatible, we have $\lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. As $\{ggx_n\} \to gz$ the previous property means that $\{Tgx_n\} \to gz$.

Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$d(Tgx_n,\ Tz) \leq \varphi\left(M(gx_n,\ z)\right), \tag{29}$$

where

$$M(gx_n,\ z) = \max\left\{ \begin{array}{c} d(ggx_n,\ gz),\ d(ggx_n,\ Tgx_n),\ d(gz,\ Tz), \\ \frac{d(ggx_n,\ Tz)+d(gz,\ Tgx_n)}{2} \end{array} \right\}.$$

Since $\{ggx_n\} \to gz$, therefore there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$M(gx_n,\ z) = d(gz,\ Tz). \tag{30}$$

By (29) and (30), we get

$$d(Tgx_n,\ Tz) \leq \varphi\left(d(gz,\ Tz)\right),$$

Now, we claim that $d(gz, Tz) = 0$. If this is not true, then $d(gz, Tz) > 0$, which, by the fact that $\varphi(t) < t$ for all $t > 0$, implies

$$d(Tgx_n,\ Tz) < d(gz,\ Tz).$$

Letting $n \to \infty$ in the above inequality and using $\{Tgx_n\} \to gz$, we get

$$d(gz,\ Tz) < d(gz,\ Tz),$$

which is a contradiction. Hence we must have $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g. □

Next, we derive the two dimensional version of Theorem 22. For the ordered metric space $(X, d, \preceq)$, let us consider the ordered metric space $(X^2, \delta, \sqsubseteq)$, where δ was defined in Lemma 21 and $\sqsubseteq$ was introduced in (19). Define the mappings T_F, $T_G : X^2 \to X^2$, for all $(x, y) \in X^2$, by,

$$T_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } T_G(x,\ y) = (G(x,\ y),\ G(y,\ x)). \tag{31}$$

Under these conditions, the following properties hold:

Lemma 23. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F, G : X^2 \to X$ be two mappings. Then*

(1) (X, d) is complete if and only if (X^2, δ) is complete.

(2) If $(X, d, \preceq)$ is regular, then $(X^2, \delta, \sqsubseteq)$ is also regular.

(3) If F is d-continuous, then T_F is δ-continuous.

(4) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G, \sqsubseteq)$-nondecreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(6) For any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) Assume there exists $\varphi \in \Phi$ such that

$$d(F(x,\ y),\ F(u,\ v)) \le \varphi\left(M(x,\ y,\ u,\ v)\right), \tag{32}$$

where

$$\begin{aligned} &M(x,\ y,\ u,\ v) \\ = \ &\max \left\{ \begin{array}{c} d(G(x,\ y),\ G(u,\ v)),\ d(G(x,\ y),\ F(x,\ y)), \\ d(G(u,\ v),\ F(u,\ v)),\ \frac{d(G(x,\ y),\ F(u,\ v)) + d(G(u,\ v),\ F(x,\ y))}{2}, \\ d(G(y,\ x),\ G(v,\ u)),\ d(G(y,\ x),\ F(y,\ x)), \\ d(G(v,\ u),\ F(v,\ u)),\ \frac{d(G(y,\ x),\ F(v,\ u)) + d(G(v,\ u),\ F(y,\ x))}{2} \end{array} \right\}, \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$, then

$$\delta(T_F(x,\ y),\ T_F(u,\ v)) \leq \varphi\left(M_\delta((x,\ y),\ (u,\ v))\right),$$

where

$$M_\delta((x,\ y),\ (u,\ v)) = \max\left\{\begin{array}{c} \delta(T_G(x,\ y),\ T_G(u,\ v)), \\ \delta(T_G(x,\ y),\ T_F(x,\ y)), \\ \delta(T_G(u,\ v),\ T_F(u,\ v)), \\ \frac{\delta(T_G(x,\ y),\ T_F(u,\ v))+\delta(T_G(u,\ v),\ T_F(x,\ y))}{2} \end{array}\right\},$$

for all $(x,\ y),\ (u,\ v) \in X^2$, where $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$.

(8) If the pair $\{F,\ G\}$ is generalized compatible, then the mappings T_F and T_G are O-compatible in $(X^2,\ \delta,\ \sqsubseteq)$.

(9) A point $(x,\ y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

Proof. Statement (1) follows from Lemma 21 and (2), (3), (5), (6) and (9) are obvious.

(4) Assume that F is G-increasing with respect to $\preceq$ and let $(x,\ y),\ (u,\ v) \in X^2$ be such that $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$. Then $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Since F is G-increasing with respect to $\preceq$, we have that $F(x,\ y) \preceq F(u,\ v)$ and $F(y,\ x) \succeq F(v,\ u)$. Therefore $T_F(x,\ y) \sqsubseteq T_F(u,\ v)$ which shows that T_F is $(T_G,\ \sqsubseteq)$-non-decreasing.

(7) Let $(x,\ y),\ (u,\ v) \in X^2$ be such that $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$. Therefore $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. From (32), we have

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{33}$$

Furthermore $G(y,\ x) \succeq G(v,\ u)$ and $G(x,\ y) \preceq G(u,\ v)$, the contractive condition (32) implies that

$$d(F(y,\ x),\ F(v,\ u)) \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{34}$$

Combining (33) and (34), we get

$$\max\left\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\right\} \leq \varphi\left(M(x,\ y,\ u,\ v)\right). \tag{35}$$

It follows from (35) that

$$\begin{aligned} & \delta(T_F(x,\ y),\ T_F(u,\ v)) \\ = & \ \delta\left((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u))\right) \\ = & \ \max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\} \\ \leq & \ \varphi\left(M(x,\ y,\ u,\ v)\right) \\ \leq & \ \varphi\left(M_\delta((x,\ y),\ (u,\ v))\right). \end{aligned}$$

(8) Let $\{(x_n,\ y_n)\} \subseteq X^2$ be any sequence such that $T_F(x_n,\ y_n) \xrightarrow{\delta} (x,\ y)$ and $T_G(x_n,\ y_n) \xrightarrow{\delta} (x,\ y)$ (Note that it is not require to suppose that $\{T_G(x_n,\ y_n)\}$ is $\sqsubseteq$-monotone). Thus

$$\begin{aligned} & (F(x_n,\ y_n),\ F(y_n,\ x_n)) \xrightarrow{\delta} (x,\ y) \\ \Rightarrow & \ F(x_n,\ y_n) \xrightarrow{d} x \text{ and } F(y_n,\ x_n) \xrightarrow{d} y, \end{aligned}$$

and

$$\begin{aligned} & (G(x_n,\ y_n),\ G(y_n,\ x_n)) \xrightarrow{\delta} (x,\ y) \\ \Rightarrow & \ G(x_n,\ y_n) \xrightarrow{d} x \text{ and } G(y_n,\ x_n) \xrightarrow{d} y. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} G(x_n,\ y_n) = x \in X, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} G(y_n,\ x_n) = y \in X. \end{aligned}$$

Since the pair $\{F, G\}$ is generalized compatible, therefore

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0. \end{aligned}$$

In particular,

$$\begin{aligned} & \lim_{n\to\infty} \delta(T_G T_F(x_n,\ y_n),\ T_F T_G(x_n,\ y_n)) \\ = & \lim_{n\to\infty} \delta(T_G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ T_F(G(x_n,\ y_n),\ G(y_n,\ x_n))) \\ = & \lim_{n\to\infty} \delta\begin{pmatrix} (G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))), \\ (F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{pmatrix} \\ = & \lim_{n\to\infty} \max\left\{ \begin{array}{c} d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(G(x_n,\ y_n),\ G(y_n,\ x_n))), \\ d(G(F(y_n,\ x_n),\ F(x_n,\ y_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{array} \right\} \\ = & \ 0. \end{aligned}$$

Hence, the mappings T_F and T_G are O-compatible in $(X^2, \delta, \sqsubseteq)$. □

Theorem 24. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exists $\varphi \in \Phi$ satisfying (32) and for any $x, y \in X$, there exist u, $v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{36}$$

Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Proof. It is only require to use Theorem 22 to the mappings $T = T_F$ and $g = T_G$ in the ordered metric space $(X^2, \delta, \sqsubseteq)$ with Lemma 23. □

Corollary 25. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two commuting mappings satisfying (32) and (36) such that F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Next, we deduce results without g-mixed monotone property of F.

Corollary 26. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X, $F : X \times X \to X$ and $g : X \to X$ be two compatible mappings such that F is g-increasing with respect to $\preceq$. Assume there exists $\varphi \in \Phi$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \le \varphi\left(M_g(x,\ y,\ u,\ v)\right), \tag{37}$$

where

$$M_g(x,\ y,\ u,\ v) = \max\left\{\begin{array}{c} d(gx,\ gu),\ d(gx,\ F(x,\ y)),\ d(gu,\ F(u,\ v)), \\ \frac{d(gx,\ F(u,\ v))+d(gu,\ F(x,\ y))}{2}, \\ d(gy,\ gv),\ d(gy,\ F(y,\ x)),\ d(gv,\ F(v,\ u)), \\ \frac{d(gy,\ F(v,\ u))+d(gv,\ F(y,\ x))}{2} \end{array}\right\},$$

for all $x,\ y,\ u,\ v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Furthermore $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 27. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ and $g : X \to X$ be two commuting mappings satisfying (37) such that F is g-increasing with respect to $\preceq$. Furthermore $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$. Also suppose that either*

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Now, we deduce result without mixed monotone property of F.

Corollary 28. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ be an increasing mapping with respect to $\preceq$ and there exists $\varphi \in \Phi$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi\left(m(x,\ y,\ u,\ v)\right),$$

where

$$m(x,\ y,\ u,\ v) = \max\left\{\begin{array}{c} d(x,\ u),\ d(x,\ F(x,\ y)),\ d(u,\ F(u,\ v)), \\ \frac{d(x,\ F(u,\ v))+d(u,\ F(x,\ y))}{2}, \\ d(y,\ v),\ d(y,\ F(y,\ x)),\ d(v,\ F(v,\ u)), \\ \frac{d(y,\ F(v,\ u))+d(v,\ F(y,\ x))}{2} \end{array}\right\},$$

for all $x,\ y,\ u,\ v \in X$, *where* $x \preceq u$ *and* $y \succeq v$. *Also suppose that either*

(a) F *is continuous or*

(b) $(X,\ d,\ \preceq)$ *is regular.*

If there exist two elements $x_0,\ y_0 \in X$ *with*

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F *has a coupled fixed point.*

Example 29. Suppose that $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$. Let $F,\ G : X \times X \to X$ be defined as

$$\begin{aligned} F(x,\ y) &= \begin{cases} \frac{x^2-y^2}{3}, \text{ if } x \geq y, \\ 0, \text{ if } x < y, \end{cases} \\ \text{and } G(x,\ y) &= \begin{cases} x^2 - y^2, \text{ if } x \geq y, \\ 0, \text{ if } x < y. \end{cases} \end{aligned}$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{3}, \text{ for } t \neq 1, \\ 1, \text{ for } t = 1. \end{cases}$$

First, we shall show that the contractive condition (32) holds for the mappings F and G. Let $x,\ y,\ u,\ v \in X$ such that $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$, we have

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &= \left| \frac{x^2-y^2}{3} - \frac{u^2-v^2}{3} \right| \\ &= \frac{1}{3} \left| G(x,\ y) - G(u,\ v) \right| \\ &= \frac{1}{3} d(G(x,\ y),\ G(u,\ v)) \\ &\leq \frac{1}{3} M(x,\ y,\ u,\ v) \\ &\leq \varphi\left(M(x,\ y,\ u,\ v)\right). \end{aligned}$$

Thus the contractive condition (32) holds for all $x,\ y,\ u,\ v \in X$. In addition, like in [11], all the other conditions of Theorem 24 are satisfied and $z = (0,\ 0)$ is a coincidence point of F and G.

Remark 30. Using the same technique that can be used in [12 – 14, 18, 19, 20] it is possible to derive tripled, quadruple and in general, multidimensional coincidence point theorems from Theorem 22.

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[a]Department of Mathematics, Govt. Arts & Science P. G. College, Ratlam (M.P.), India
Email address: bhavnadeshpande@yahoo.com

[b]Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M.P.), India
Email address: amrishhanda83@gmail.com

[c]Department of Mathematics, Govt. P. G. College Jaora (M.P.), India
Email address: chetnakothari123@gmail.com