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HUGE COUPLED COINCIDENCE POINT THEOREM FOR GENERALIZED COMPATIBLE PAIR OF MAPPINGS WITH APPLICATIONS

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Abstract. We establish a coupled coincidence point theorem for generalized compatible pair of mappings under generalized nonlinear contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of $F : X \times X \to X$. An example supporting to our result has also been cited. As an application the solution of integral equations are obtained here to illustrate the usability of the obtained results. We improve, extend and generalize several known results.

1. Introduction and Preliminaries

Bhaskar and Lakshmikantham [2] introduced the notion of coupled fixed point, mixed monotone mappings in the setting of single-valued mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces.

In [2], Bhaskar and Lakshmikantham introduced the following:

Definition 1.1. *Let $(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:*

$$(u,\ v) \leq (x,\ y) \Leftrightarrow x \geq u \text{ and } y \leq v,\ \text{for all } (u,\ v),\ (x,\ y) \in X \times X.$$

Definition 1.2. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y.$$

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Definition 1.3. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all x, $y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \leq x_2 \implies F(x_1,\ y) \leq F(x_2,\ y)$$

and

$$y_1,\ y_2 \in X,\ y_1 \leq y_2 \implies F(x,\ y_1) \geq F(x,\ y_2).$$

Lakshmikantham and Ciric [12] extended the notion of mixed monotone property to mixed $g-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Bhaskar and Lakshmikantham [2].

In [12], Lakshmikantham and Ciric introduced the following:

Definition 1.4. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,\ y) = g(x) \text{ and } F(y,\ x) = g(y).$$

Definition 1.5. an element $(x, y) \in X \times X$ is called a *common coupled fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = F(x,\ y) = g(x) \text{ and } y = F(y,\ x) = g(y).$$

Definition 1.6. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *commutative* if

$$g(F(x,\ y)) = F(g(x),\ g(y)),\ \text{ for all } (x,\ y) \in X \times X.$$

Definition 1.7. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ and $g : X \to X$ are given mappings. We say that F has the *mixed $g-$monotone property* if for all x, $y \in X$, we have

$$x_1,\ x_2 \in X,\ g(x_1) \leq g(x_2) \implies F(x_1,\ y) \leq F(x_2,\ y)$$

and

$$y_1,\ y_2 \in X,\ g(y_1) \leq g(y_2) \implies F(x,\ y_1) \geq F(x,\ y_2).$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Later, Choudhury and Kundu [4] introduced the following notion of compatibility in the context of coupled coincidence point and used this notion to improve the results of Lakshmikantham and Ciric [12] :

Definition 1.8. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *compatible* if

$$\begin{aligned}
\lim_{n\to\infty} d(g(Fx_n,\ Fy_n),\ F(gx_n,\ gy_n)) &= 0,\\
\lim_{n\to\infty} d(g(Fy_n,\ Fx_n),\ F(gy_n,\ gx_n)) &= 0,
\end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned}
\lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x,\\
\lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y,\ \text{for some } x,\ y \in X.
\end{aligned}$$

These results used to study the existence and uniqueness of solution for periodic boundary value problems.

Hussain et al. [10] introduced a new concept of generalized compatibility of a pair of mappings $F, G : X \times X \to X$ defined on a product space and proved some coupled coincidence point results. Hussain et al. [10] also deduce some coupled fixed point results without mixed monotone property.

In [10], Hussain et al. introduced the following:

Definition 1.9. Suppose that $F, G : X \times X \to X$ are two mappings. F is said to be *$G-$increasing with respect to* $\leq$ if for all $x, y, u, v \in X$, with $G(x, y) \leq G(u, v)$ we have $F(x, y) \leq F(u, v)$.

Example 1.1. Let $X = (0, +\infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $F, G : X \times X \to X$ by $F(x, y) = \ln(x + y)$ and $G(x, y) = x + y$ for all $(x, y) \in X \times X$. Note that F is $G-$increasing with respect to $\leq$.

Example 1.2. Let $X = \mathbb{N}$ endowed with the partial order defined by $x, y \in X \times X$, $x \leq y$ if and only if y divides x. Define the mappings $F, G : X \times X \to X$ by $F(x, y) = x^2y^2$ and $G(x, y) = xy$ for all $(x, y) \in X \times X$. Then F is $G-$increasing with respect to $\leq$.

Definition 1.10. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F, G : X \times X \to X$ if $F(x, y) = G(x, y)$ and $F(y, x) = G(y, x)$.

Example 1.3. Let F, $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x,\ y) = xy$ and $G(x,\ y) = \frac{2}{3}(x + y)$ for all $(x,\ y) \in X \times X$. Note that (0, 0), (1, 2) and (2, 1) are coupled coincidence points of F and G.

Definition 1.11. Let F, $G : X \times X \to X$ be two mappings. We say that the pair $\{F,\ G\}$ is *commuting* if

$$F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x)),\ \text{for all}\ x,\ y \in X.$$

Definition 1.12. Let $(X,\ \leq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that F is *$g-$increasing with respect to* $\leq$ if for any x, $y \in X$,

$$gx_1 \leq gx_2 \text{ implies } F(x_1,\ y) \leq F(x_2,\ y),$$

and

$$gy_1 \leq gy_2 \text{ implies } F(x,\ y_1) \leq F(x,\ y_2).$$

Definition 1.13. Let $(X,\ \leq)$ be a partially ordered set, $F : X \times X \to X$. We say that F is *increasing with respect to* $\leq$ if for any x, $y \in X$,

$$x_1 \leq x_2 \text{ implies } F(x_1,\ y) \leq F(x_2,\ y),$$

and

$$y_1 \leq y_2 \text{ implies } F(x,\ y_1) \leq F(x,\ y_2).$$

Definition 1.14. Let $F, G : X \times X \to X$. We say that the pair $\{F,\ G\}$ is *generalized compatible* if

$$\begin{aligned}
\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\
\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n, x_n),\ F(x_n,\ y_n))) &= 0,
\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}
\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x,\\
\lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y.
\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

In [7], Ding et al. proved coupled coincidence and common coupled fixed point theorems for generalized nonlinear contraction on partially ordered metric spaces which generalize the results of Lakshmikantham and Ciric [12]. The main result of Ding et al. [7] on complete partially ordered metric space X is as follows:

Theorem 1.1. *Assume that $g : X \to X$ is a continuous mapping, and $F : X \times X \to X$ is a continuous mapping with the mixed g-monotone property on X. Suppose that the following assumptions hold:*

($\mathbf{A_1}$) *there exists a non-decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, and*

$$
\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ d(gy,\ gv),\ D(gx,\ F(x,\ y)), \\ D(gu,\ F(u,\ v)),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right],
\end{aligned}
$$

for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$.

($\mathbf{A_2}$) *there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$.*

($\mathbf{A_3}$) *$F(X \times X) \subseteq g(X)$, g and F are commuting, that is, $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.*

Then F and g have a coupled coincidence point, that is, there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$.

Many authors focused on coupled fixed point theory including [1, 3, 5, 6, 8, 9, 11, 13, 14, 15, 17, 18, 19].

Recently Samet et al. [16] claimed that most of the coupled fixed point theorems on ordered metric spaces are consequences of well-known fixed point theorems.

In this paper, we establish a coupled coincidence point theorem for generalized compatible pair of mappings under generalized nonlinear contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of F. We also give an example and an application to integral equation to support our results presented here. We generalize the results of Bhaskar and Lakshmikantham [2], Ding et al. [7] and Lakshmikantham and Ciric [12].

2. Main Results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n\to\infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X \times X \to X$ be two generalized compatible mappings such that F is $G-$increasing with respect to $\leq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

Suppose that there exist $\varphi \in \Phi$ such that

$$\begin{aligned} &d(F(x,\ y),\ F(u,\ v)) \\ \leq\ \ &\varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)),\ d(G(x,\ y),\ F(x,\ y)), \\ d(G(u,\ v),\ F(u,\ v)),\ d(G(y,\ x),\ G(v,\ u)), \\ d(G(y,\ x),\ F(y,\ x)),\ d(G(v,\ u),\ F(v,\ u)), \\ \frac{d(G(x,\ y),\ F(u,\ v))+d(G(u,\ v),\ F(x,\ y))}{2}, \\ \frac{d(G(y,\ x),\ F(v,\ u))+d(G(v,\ u),\ F(y,\ x))}{2} \end{array}\right\}\right], \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$. Suppose that for any $x, y \in X$, there exist $u, v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{2.2}$$

Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

Then F and G have a coupled coincidence point.

Proof. By hypothesis, there exist $x_0, y_0 \in X$ such that

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

From (2.2), we can choose $x_1, y_1 \in X$ such that

$$G(x_1,\ y_1) = F(x_0,\ y_0) \text{ and } G(y_1,\ x_1) = F(y_0,\ x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$G(x_{n+1},\ y_{n+1}) = F(x_n,\ y_n) \text{ and } G(y_{n+1},\ x_{n+1}) = F(y_n,\ x_n), \text{ for all } n \geq 0. \tag{2.3}$$

We shall show that

$$G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \text{ and } G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}), \text{ for all } n \geq 0. \tag{2.4}$$

We shall use the mathematical induction. Let $n = 0$, since

$$\begin{aligned} G(x_0,\ y_0) &\leq F(x_0,\ y_0) = G(x_1,\ y_1), \\ G(y_0,\ x_0) &\geq F(y_0,\ x_0) = G(y_1,\ x_1), \end{aligned}$$

we have

$$G(x_0,\ y_0) \leq G(x_1,\ y_1) \text{ and } G(y_0,\ x_0) \geq G(y_1,\ x_1).$$

Thus (2.4) hold for $n = 0$. Suppose now that (2.4) hold for some fixed $n \in \mathbb{N}$. Then since

$$G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \text{ and } G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}),$$

and as F is $G-$increasing with respect to $\leq$, from (2.3), we have

$$\begin{aligned} G(x_{n+1},\ y_{n+1}) &= F(x_n,\ y_n) \leq F(x_{n+1},\ y_{n+1}) = G(x_{n+2},\ y_{n+2}), \\ G(y_{n+1},\ x_{n+1}) &= F(y_n,\ x_n) \geq F(y_{n+1},\ x_{n+1}) = G(y_{n+2},\ x_{n+2}). \end{aligned}$$

Thus by the mathematical induction we conclude that (2.4) hold for all $n \geq 0$. Therefore

$$G(x_0,\ y_0) \leq G(x_1,\ y_1) \leq ... \leq G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \leq ...$$

and

$$G(y_0,\ x_0) \geq G(y_1,\ x_1) \geq ... \geq G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}) \geq ...$$

Now by (2.1), we have

$$\begin{aligned} & d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})) \\ = \ & d(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)) \\ \leq \ & \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(x_{n-1},\ y_{n-1}),\ F(x_{n-1},\ y_{n-1})), \\ d(G(x_n,\ y_n),\ F(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ F(y_{n-1},\ x_{n-1})), \\ d(G(y_n,\ x_n),\ F(y_n,\ x_n)), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)) + d(G(x_n,\ y_n),\ F(x_{n-1},\ y_{n-1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ F(y_n,\ x_n)) + d(G(y_n,\ x_n),\ F(y_{n-1},\ x_{n-1}))}{2} \end{array} \right\} \right] \end{aligned}$$

$$\leq \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_{n+1},\ y_{n+1}))+d(G(x_n,\ y_n),\ G(x_n,\ y_n))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_{n+1},\ x_{n+1}))+d(G(y_n,\ x_n),\ G(y_n,\ x_n))}{2} \end{array} \right\} \right]$$

$$\leq \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1}))), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_{n+1},\ y_{n+1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_{n+1},\ x_{n+1}))}{2} \end{array} \right\} \right].$$

Thus

$$d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})) \leq \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1}))), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_{n+1},\ y_{n+1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_{n+1},\ x_{n+1}))}{2} \end{array} \right\} \right].$$

Similarly

$$d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \leq \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_{n+1},\ y_{n+1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_{n+1},\ x_{n+1}))}{2} \end{array} \right\} \right].$$

Combining them, we get

$$\max \left\{ \begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array} \right\}$$
$$\leq \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_{n+1},\ y_{n+1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_{n+1},\ x_{n+1}))}{2} \end{array} \right\} \right]$$

$$\leq \quad \varphi\left[\max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})), \\ \frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n))+d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1}))}{2}, \\ \frac{d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n))+d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1}))}{2} \end{array}\right\}\right]$$

$$\leq \quad \varphi\left[\max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1}))), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\}\right].$$

Thus

$$\max\left\{\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\} \leq \varphi\left[\max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\}\right]. \tag{2.5}$$

If we suppose that

$$\max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\} = \max\left\{\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\}.$$

Then by (2.5) and by the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$\begin{aligned} &\max\left\{\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\} \\ \leq \quad &\varphi\left[\max\left\{\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\}\right] \\ < \quad &\max\left\{\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\}, \end{aligned}$$

which is a contradiction. Thus, we must have

$$\max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)), \\ d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})), \\ d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \end{array}\right\} = \max\left\{\begin{array}{c} d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)), \\ d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n)) \end{array}\right\}.$$

Hence by (2.5), we have for all $n \in \mathbb{N}$,

$$\begin{aligned} & \max \left\{ \begin{array}{c} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), \\ d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) \end{array} \right\} \\ \leq\ & \varphi \left[\max \left\{ \begin{array}{c} d(G(x_{n-1}, y_{n-1}), G(x_n, y_n)), \\ d(G(y_{n-1}, x_{n-1}), G(y_n, x_n)) \end{array} \right\} \right] \\ \leq\ & \varphi^n \left[\max \left\{ \begin{array}{c} d(G(x_0, y_0), G(x_1, y_1)), \\ d(G(y_0, x_0), G(y_1, x_1)) \end{array} \right\} \right] \\ \leq\ & \varphi^n(\delta). \end{aligned}$$

Thus

$$\max \left\{ \begin{array}{c} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), \\ d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) \end{array} \right\} \leq \varphi^n(\delta), \tag{2.6}$$

where

$$\delta = \max \left\{ \begin{array}{c} d(G(x_0, y_0), G(x_1, y_1)), \\ d(G(y_0, x_0), G(y_1, x_1)) \end{array} \right\}.$$

Without loss of generality, one can assume that $\max\{d(G(x_0, y_0), G(x_1, y_1)), d(G(y_0, x_0), G(y_1, x_1))\} \neq 0$. In fact, if this is not true, then $G(x_0, y_0) = G(x_1, y_1) = F(x_0, y_0)$, $G(y_0, x_0) = G(y_1, x_1) = F(y_0, x_0)$, that is, (x_0, y_0) is a coupled coincidence point of F and G.

Thus, for $m, n \in \mathbb{N}$ with $m > n$, by triangle inequality and (2.6), we get

$$\begin{aligned} & d(G(x_n, y_n), G(x_{m+n}, y_{m+n})) \\ \leq\ & d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(x_{n+1}, y_{n+1}), G(x_{n+2}, y_{n+2})) \\ & + ... + d(G(x_{n+m-1}, y_{n+m-1}), G(x_{m+n}, y_{m+n})) \\ \leq\ & \max \left\{ \begin{array}{c} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})), \\ d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) \end{array} \right\} \\ & + \max \left\{ \begin{array}{c} d(G(x_{n+1}, y_{n+1}), G(x_{n+2}, y_{n+2})), \\ d(G(y_{n+1}, x_{n+1}), G(y_{n+2}, x_{n+2})) \end{array} \right\} \\ & + ... + \max \left\{ \begin{array}{c} d(G(x_{n+m-1}, y_{n+m-1}), G(x_{m+n}, y_{m+n})), \\ d(G(y_{n+m-1}, x_{n+m-1}), G(y_{m+n}, x_{m+n})) \end{array} \right\} \\ \leq\ & \varphi^n(\delta) + \varphi^{n+1}(\delta) + ... + \varphi^{n+m-1}(\delta) \\ \leq\ & \sum_{i=n}^{n+m-1} \varphi^i(\delta), \end{aligned}$$

which implies, by (ii_φ), that $\{G(x_n, y_n)\}$ is a Cauchy sequence in X. Similarly we obtain that $\{G(y_n, x_n)\}$ is also a Cauchy sequence in X. Since X is complete, there is some x, $y \in X$ such that

$$\lim_{n\to\infty} G(x_n,\ y_n) = \lim_{n\to\infty} F(x_n,\ y_n) = x, \tag{2.7}$$
$$\lim_{n\to\infty} G(y_n,\ x_n) = \lim_{n\to\infty} F(y_n,\ x_n) = y.$$

Since the pair $\{F, G\}$ satisfies the generalized compatibility, from (2.7), we get

$$\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) = 0, \tag{2.8}$$

and

$$\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) = 0. \tag{2.9}$$

Now suppose that assumption (a) holds. Then

$$\begin{aligned} & d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\ \leq\ & d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) \\ & + d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(x,\ y)). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (2.7), (2.8) and the fact that F and G are continuous, we have

$$F(x,\ y) = G(x,\ y).$$

Similarly we can show that

$$F(y,\ x) = G(y,\ x).$$

Thus (x, y) is a coupled coincidence point of F and G.

Now, suppose that (b) holds. By (2.4) and (2.7), we have $\{G(x_n, y_n)\}$ is a non-decreasing sequence, $G(x_n, y_n) \to x$ and $\{G(y_n, x_n)\}$ is a non-increasing sequence, $G(y_n, x_n) \to y$ as $n \to \infty$. Thus for all n, we have

$$G(x_n,\ y_n) \leq x \text{ and } G(y_n,\ x_n) \geq y. \tag{2.10}$$

Since G is continuous, by (2.7), (2.8) and (2.9), we have

$$\begin{aligned} & \lim_{n\to\infty} G(G(x_n,\ y_n),\ G(y_n,\ x_n)) \\ & = G(x,\ y) \\ & = \lim_{n\to\infty} G(F(x_n,\ y_n),\ F(y_n,\ x_n)) \\ & = \lim_{n\to\infty} F(G(x_n,\ y_n),\ G(y_n,\ x_n)) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} G(G(y_n,\ x_n),\ G(x_n,\ y_n)) \\
&= G(y,\ x) \\
&= \lim_{n\to\infty} G(F(y_n,\ x_n),\ F(x_n,\ y_n)) \\
&= \lim_{n\to\infty} F(G(y_n,\ x_n),\ G(x_n,\ y_n)). \qquad (2.12)
\end{aligned}$$

Since G has the mixed monotone property, it follows from (2.10) that $G(G(x_n, y_n), G(y_n, x_n)) \leq G(x, y)$ and $G(G(y_n, x_n), G(x_n, y_n)) \geq G(y, x)$. Now using (2.1) and (2.11), we get

$$\begin{aligned}
& d(G(x,\ y),\ F(x,\ y)) \\
= & \lim_{n\to\infty} d\left(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(x,\ y)\right) \\
= & \lim_{n\to\infty} d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) \\
\leq & \lim_{n\to\infty} \varphi\left[\Delta_n\right],
\end{aligned}$$

where

$$\Delta_n = \max\left\{\begin{array}{c}
d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)), \\
d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(G(x_n,\ y_n), G(y_n,\ x_n))), \\
d(G(x,\ y),\ F(x,\ y)),\ d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(y,\ x)), \\
d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))), \\
d(G(y,\ x),\ F(y,\ x)), \\
\frac{d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y))+d(G(x,\ y),\ F(G(x_n,\ y_n),G(y_n,\ x_n)))}{2}, \\
\frac{d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ F(y,\ x))+d(G(y,\ x),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n)))}{2}
\end{array}\right\}.$$

Similarly

$$d(G(y,\ x),\ F(y,\ x)) \leq \lim_{n\to\infty} \varphi\left[\Delta_n\right].$$

Combining them, we get

$$\max\left\{d(G(x,\ y),\ F(x,\ y)),\ d(G(y,\ x),\ F(y,\ x))\right\} \leq \lim_{n\to\infty} \varphi\left[\Delta_n\right], \qquad (2.13)$$

which, by (2.7), implies

$$\begin{aligned}
& \max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)), \\ d(G(y,\ x),\ F(y,\ x)) \end{array}\right\} \qquad (2.14) \\
& \leq \varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)), \\ d(G(y,\ x),\ F(y,\ x)) \end{array}\right\}\right].
\end{aligned}$$

Now, we claim that

$$\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)), \\ d(G(y,\ x),\ F(y,\ x)) \end{array}\right\} = 0. \qquad (2.15)$$

If this is not true, then

$$\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)),\\ d(G(y,\ x),\ F(y,\ x))\end{array}\right\} > 0.$$

Thus, by (2.14) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get

$$\begin{aligned} &\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)),\\ d(G(y,\ x),\ F(y,\ x))\end{array}\right\}\\ \leq\ &\varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)),\\ d(G(y,\ x),\ F(y,\ x))\end{array}\right\}\right]\\ <\ &\max\left\{\begin{array}{c} d(G(x,\ y),\ F(x,\ y)),\\ d(G(y,\ x),\ F(y,\ x))\end{array}\right\},\end{aligned}$$

which is a contradiction. So (2.15) holds. Thus, it follows that

$$G(x,\ y) = F(x,\ y) \text{ and } G(y,\ x) = F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and G. □

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X \times X \to X$ be two commuting mappings such that F is $G-$increasing with respect to $\leq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

Suppose that the inequalities (2.1) and (2.2) hold and either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

Then F and G have a coupled coincidence point.

Now, we deduce results which are analogous to Theorem 1.1 without $g-$mixed monotone property of F.

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F is $g-$increasing with respect to $\leq$, and there exist $\varphi \in \Phi$ such that*

$$\begin{aligned} &d(F(x,\ y),\ F(u,\ v))\\ \leq\ &\varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ d(gy,\ gv),\ D(gx,\ F(x,\ y)),\\ D(gu,\ F(u,\ v)),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)),\\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2}\end{array}\right\}\right],\end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, *where* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$. *Suppose that* $F(X \times X) \subseteq g(X)$, g *is continuous and monotone increasing with respect to* $\leq$ *and the pair* $\{F, g\}$ *is compatible. Also suppose that either*

(a) F *is continuous or*

(b) X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$ *in* X *then* $x_n \leq x$, *for all* n,

(ii) *if a non-increasing sequence* $\{x_n\} \to x$ *in* X *then* $x \leq x_n$, *for all* n.

If there exist two elements $x_0,\ y_0 \in X$ *with*

$$gx_0 \leq F(x_0,\ y_0) \text{ and } gy_0 \geq F(y_0,\ x_0).$$

Then F *and* g *have a coupled coincidence point.*

Corollary 2.4. *Let* $(X, \leq)$ *be a partially ordered set such that there exists a complete metric* d *on* X. *Assume* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that* F *is* $g-$*increasing with respect to* $\leq$, *and there exist* $\varphi \in \Phi$ *such that*

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ d(gy,\ gv),\ D(gx,\ F(x,\ y)), \\ D(gu,\ F(u,\ v)),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right], \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, *where* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$. *Suppose that* $F(X \times X) \subseteq g(X)$, g *is continuous and monotone increasing with respect to* $\leq$ *and the pair* $\{F, g\}$ *is commuting. Also suppose that either*

(a) F *is continuous or*

(b) X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$ *in* X *then* $x_n \leq x$, *for all* n,

(ii) *if a non-increasing sequence* $\{x_n\} \to x$ *in* X *then* $x \leq x_n$, *for all* n.

If there exist two elements $x_0,\ y_0 \in X$ *with*

$$gx_0 \leq F(x_0,\ y_0) \text{ and } gy_0 \geq F(y_0,\ x_0).$$

Then F *and* g *have a coupled coincidence point.*

Corollary 2.5. *Let* $(X, \leq)$ *be a partially ordered set such that there exists a complete metric* d *on* X. *Assume* $F : X \times X \to X$ *be an increasing mapping with respect to* $\leq$ *and there exist* $\varphi \in \Phi$ *such that*

$$d(F(x,\ y),\ F(u,\ v))$$

$$\leq \varphi\left[\max\left\{\begin{array}{c} d(x, u), d(y, v), D(x, F(x, y)), \\ D(u, F(u, v)), D(y, F(y, x)), D(v, F(v, u)), \\ \frac{D(x, F(u, v))+D(u, F(x, y))}{2}, \frac{D(y, F(v, u))+D(v, F(y, x))}{2} \end{array}\right\}\right],$$

for all $x, y, u, v \in X$, where $x \leq u$ and $y \geq v$. Also suppose that either

(*a*) *F is continuous or*

(*b*) *X has the following properties:*

(*i*) *if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,*

(*ii*) *if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.*

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0).$$

Then F has a coupled fixed point.

Example 2.1. Suppose that $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$. Let $F, G : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2-y^2}{3} & \text{if } x \geq y \\ 0 & \text{if } x < y, \end{cases}$$

and

$$G(x, y) = \begin{cases} x^2 - y^2 & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases} .$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{3}, & \text{for } t \neq 1 \\ 1, & \text{for } t = 1. \end{cases}$$

First, we shall show that the mappings F and G satisfy the condition (2.1). Let x, $y \in X$ such that $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$, we have

$$\begin{aligned} & d(F(x, y), F(u, v)) \\ = & \left|\frac{x^2 - y^2}{3} - \frac{u^2 - v^2}{3}\right| \\ = & \frac{1}{3}\left|G(x, y) - G(u, v)\right| \\ = & \frac{1}{3} d(G(x, y), G(u, v)) \\ \leq & \frac{1}{3}\max\left\{\begin{array}{c} d(G(x, y), G(u, v)), d(G(x, y), F(x, y)), \\ d(G(u, v), F(u, v)), d(G(y, x), G(v, u)), \\ d(G(y, x), F(y, x)), d(G(v, u), F(v, u)), \\ \frac{d(G(x, y), F(u, v))+d(G(u, v), F(x, y))}{2}, \\ \frac{d(G(y, x), F(v, u))+d(G(v, u), F(y, x))}{2} \end{array}\right\} \end{aligned}$$

$$\leq \quad \varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)),\ d(G(x,\ y),\ F(x,\ y)), \\ d(G(u,\ v),\ F(u,\ v)),\ d(G(y,\ x),\ G(v,\ u)), \\ d(G(y,\ x),\ F(y,\ x)),\ d(G(v,\ u),\ F(v,\ u)), \\ \frac{d(G(x,\ y),\ F(u,\ v))+d(G(u,\ v),\ F(x,\ y))}{2}, \\ \frac{d(G(y,\ x),\ F(v,\ u))+d(G(v,\ u),\ F(y,\ x))}{2} \end{array}\right\}\right].$$

Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$. In addition, like in [10], all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0)$ is a coincidence point of F and G.

Now we prove the uniqueness of the coupled coincidence point. Note that if $(X, \leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x,\ y),\ (u,\ v) \in X \times X$:

$$(x,\ y) \leq (u,\ v) \Longleftrightarrow G(x,\ y) \leq G(u,\ v) \text{ and } G(y,\ x) \geq G(v,\ u),$$

where $G : X \times X \to X$ is one-one.

Theorem 2.6. *In addition to the hypotheses of Theorem 2.1, suppose that for every $(x,\ y),\ (x^*,\ y^*)$ in $X \times X$, there exists another $(u,\ v)$ in $X \times X$ which is comparable to $(x,\ y)$ and $(x^*,\ y^*)$, then F and G have a unique coupled coincidence point.*

Proof. From Theorem 2.1, the set of coupled coincidence points of F and G is non-empty. Assume that $(x,\ y),\ (x^*,\ y^*) \in X \times X$ are two coupled coincidence points of F and G, that is,

$$\begin{aligned} F(x,\ y) &= G(x,\ y) \text{ and } F(y,\ x) = G(y,\ x), \\ F(x^*,\ y^*) &= G(x^*,\ y^*) \text{ and } F(y^*,\ x^*) = G(y^*,\ x^*). \end{aligned}$$

We shall prove that $G(x,\ y) = G(x^*,\ y^*)$ and $G(y,\ x) = G(y^*,\ x^*)$. By assumption, there exists $(u,\ v) \in X \times X$, that is, comparable to $(x,\ y)$ and $(x^*,\ y^*)$. We define the sequences $\{G(u_n,\ v_n)\}$ and $\{G(v_n,\ u_n)\}$ as follows, with $u_0 = u,\ v_0 = v$:

$$G(u_{n+1},\ v_{n+1}) = F(u_n,\ v_n),\ G(v_{n+1},\ u_{n+1}) = F(v_n,\ u_n),\ n \geq 0.$$

Since $(u,\ v)$ is comparable to $(x,\ y)$, we may assume that $(x,\ y) \leq (u,\ v) = (u_0,\ v_0)$, which implies that $G(x,\ y) \leq G(u_0,\ v_0)$ and $G(y,\ x) \geq G(v_0,\ u_0)$. We suppose that $(x,\ y) \leq (u_n,\ v_n)$ for some n. We prove that $(x,\ y) \leq (u_{n+1},\ v_{n+1})$. Since F is $G-$increasing, we have $G(x,\ y) \leq G(u_n,\ v_n)$ implies $F(x,\ y) \leq F(u_n,\ v_n)$ and $G(y,\ x) \geq G(v_n,\ u_n)$ implies $F(y,\ x) \geq F(v_n,\ u_n)$. Therefore

$$G(x,\ y) = F(x,\ y) \leq F(u_n,\ v_n) = G(u_{n+1},\ v_{n+1})$$

and

$$G(y,\ x) = F(y,\ x) \geq F(v_n,\ u_n) = G(v_{n+1},\ u_{n+1}).$$

Thus, we have

$$(x,\ y) \leq (u_{n+1},\ v_{n+1}),\ \text{for all } n.$$

Now, by (2.1), we have

$$\begin{aligned} & d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) = d(F(x,\ y),\ F(u_n,\ v_n)) \leq \varphi[\nabla_n], \\ & d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) = d(F(y,\ x),\ F(v_n,\ u_n)) \leq \varphi[\nabla_n], \end{aligned} \tag{2.16}$$

where

$$\nabla_n = \max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_n,\ v_n)),\ d(G(u_n,\ v_n),\ G(u_{n+1},\ v_{n+1})), \\ d(G(y,\ x),\ G(v_n,\ u_n)),\ d(G(v_n,\ u_n),\ G(v_{n+1},\ u_{n+1})), \\ \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1}))+d(G(u_n,\ v_n),\ G(x,\ y))}{2}, \\ \frac{d(G(y,\ x),\ G(v_{n+1},\ u_{n+1}))+d(G(v_n,\ u_n),\ G(y,\ x))}{2} \end{array} \right\}.$$

Since, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\nabla_n = \max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_n,\ v_n)),\ d(G(y,\ x),\ G(v_n,\ u_n)), \\ \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1}))+d(G(u_n,\ v_n),\ G(x,\ y))}{2}, \\ \frac{d(G(y,\ x),\ G(v_{n+1},\ u_{n+1}))+d(G(v_n,\ u_n),\ G(y,\ x))}{2} \end{array} \right\}. \tag{2.17}$$

Combining (2.16) and (2.17), we get

$$\max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})), \\ d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) \end{array} \right\} \leq \varphi[\nabla_n]. \tag{2.18}$$

We claim that

$$\nabla_n = \max\left\{d(G(x,\ y),\ G(u_n,\ v_n)),\ d(G(y,\ x),\ G(v_n,\ u_n))\right\}. \tag{2.19}$$

In fact, if

$$\nabla_n = \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(u_n,\ v_n),\ G(x,\ y))}{2},$$

then, by (2.16) and the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$\begin{aligned} & d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) \\ \leq\ & \varphi\left[\frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(u_n,\ v_n),\ G(x,\ y))}{2}\right] \\ <\ & \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(u_n,\ v_n),\ G(x,\ y))}{2}, \end{aligned}$$

which implies that

$$d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) < d(G(u_n,\ v_n),\ G(x,\ y)),$$

and so

$$\begin{aligned} \nabla_n &= \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(u_n,\ v_n),\ G(x,\ y))}{2} \\ &< d(G(u_n,\ v_n),\ G(x,\ y)), \end{aligned}$$

which is a contradiction. In addition, if

$$\nabla_n = \frac{d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) + d(G(v_n,\ u_n),\ G(y,\ x))}{2},$$

then there is again a contradiction. So (2.19) holds. Thus, by (2.18), we have

$$\begin{aligned} &\max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})), \\ d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) \end{array} \right\} \\ &\leq \varphi\left[\max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_n,\ v_n)), \\ d(G(y,\ x),\ G(v_n,\ u_n)) \end{array} \right\}\right]. \end{aligned}$$

Then it follows that

$$\begin{aligned} &\max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})), \\ d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) \end{array} \right\} \\ &\leq \varphi^{n+1}\left[\max\left\{ \begin{array}{c} d(G(x,\ y),\ G(u_0,\ v_0)), \\ d(G(y,\ x),\ G(v_0,\ u_0)) \end{array} \right\}\right]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (ii_φ), we get

$$G(x,\ y) = \lim_{n\to\infty} G(u_{n+1},\ v_{n+1}) \text{ and } G(y,\ x) = \lim_{n\to\infty} G(v_{n+1},\ u_{n+1}).$$

Similarly, we can show that

$$G(x^*,\ y^*) = \lim_{n\to\infty} G(u_{n+1},\ v_{n+1}) \text{ and } G(y^*,\ x^*) = \lim_{n\to\infty} G(v_{n+1},\ u_{n+1})).$$

Thus $G(x,\, y) = G(x^*,\, y^*)$ and $G(y,\, x) = G(y^*,\, x^*)$. □

3. Application to Integral Equations

As an application of the results established in section 2 of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))\,[f(q,\ x(q)) + g(q,\ x(q))]\,dq + h(p), \tag{3.1}$$

for all $p \in I = [a,\, b]$.

Let Θ denote the set of all functions $\theta : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_θ) θ is non-decreasing,
(ii_θ) $\theta(p) \leq p$.

Assumption 3.1. We assume that the functions K_1, K_2, f, g fulfill the following conditions:

(i) $K_1(p, q) \geq 0$ and $K_2(p, q) \leq 0$ for all p, $q \in I$,

(ii) There exists λ, μ and $\theta \in \Theta$ such that for all x, $y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\theta(x - y) \tag{3.2}$$

and

$$-\mu\theta(x - y) \leq g(q,\ x) - g(q,\ y) \leq 0, \tag{3.3}$$

(iii)

$$\max\{\lambda,\ \mu\} \sup_{p \in I} \int_a^b [K_1(p,\ q) - K_2(p,\ q)]dq \leq \frac{1}{4}. \tag{3.4}$$

Definition 3.1 ([13])**.** A pair $(\alpha, \beta) \in X^2$ with $X = C(I, \mathbb{R})$, where $C(I, \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a *coupled lower-upper solution* of (3.1) if, for all $p \in I$,

$$\begin{aligned} \alpha(p) \ \leq \ & \int_a^b K_1(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq + h(p), \end{aligned}$$

and

$$\begin{aligned} \beta(p) \ \geq \ & \int_a^b K_1(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq + h(p). \end{aligned}$$

Theorem 3.1. *Consider the integral equation* (3.1) *with* K_1, $K_2 \in C(I \times I, \mathbb{R})$, f, $g \in C(I \times \mathbb{R}, \mathbb{R})$ *and* $h \in C(I, \mathbb{R})$. *Suppose that there exists a coupled lower-upper solution* (α, β) *of* (3.1) *and Assumption 3.1 is satisfied. Then the integral equation* (3.1) *has a solution in* $C(I, \mathbb{R})$.

Proof. Consider $X = C(I, \mathbb{R})$, the natural partial order relation, that is, for x, $y \in C(I, \mathbb{R})$,

$$x \leq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} |x(p) - y(p)|\,.$$

Now define on $X \times X$ the following partial order: for $(x, y), (u, v) \in X^2$,

$$(x,\ y) \leq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \text{ for all } p \in I.$$

Obviously, for any $(x, y) \in X \times X$, the functions $\max\{x, y\}$ and $\min\{x, y\}$ are the upper and lower bounds of x and y respectively. Therefore for every (x, y), $(u, v) \in X \times X$, there exists the element $(\max\{x,\ u\},\ \min\{y,\ v\})$ which is comparable to (x, y) and (u, v). Define $\varphi : [0, +\infty) \to [0, +\infty)$ as follows

$$\varphi(t) = \begin{cases} \frac{t}{2}, \text{ for } t \neq 1 \\ 1, \text{ for } t = 1. \end{cases}$$

Define now the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x,\ y)(p) \quad &= \quad \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq + h(p), \end{aligned}$$

for all $p \in I$. We can prove, like in [13], that F is increasing. Now for $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$, by using (3.2) and (3.3), we have

$$\begin{aligned} & F(x_1,\ y)(p) - F(x_2,\ y)(p) \\ = \; & \int_a^b K_1(p,\ q)\,[f(q,\ x_1(q)) + g(q,\ y(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x_1(q))]\,dq \\ & - \int_a^b K_1(p,\ q)\,[f(q,\ x_2(q)) + g(q,\ y(q))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x_2(q))]\,dq \\ = \; & \int_a^b K_1(p,\ q)\,[f(q,\ x_1(q)) - f(q,\ x_2(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[g(q,\ x_1(q)) - g(q,\ x_2(q))]\,dq \end{aligned}$$

$$
\begin{aligned}
\leq \quad & \int_a^b K_1(p,\ q)\left[\lambda\theta\left(x(q)-u(q)\right)+\mu\theta\left(v(q)-y(q)\right)\right]dq \\
& -\int_a^b K_2(p,\ q)\left[\lambda\theta\left(v(q)-y(q)\right)+\mu\theta\left(x(q)-u(q)\right)\right]dq.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& F(x,\ y)(p)-F(u,\ v)(p) \\
& \leq \int_a^b K_1(p,\ q)\left[\lambda\theta\left(x(q)-u(q)\right)+\mu\theta\left(v(q)-y(q)\right)\right]dq \\
& -\int_a^b K_2(p,\ q)\left[\lambda\theta\left(v(q)-y(q)\right)+\mu\theta\left(x(q)-u(q)\right)\right]dq.
\end{aligned} \tag{3.5}
$$

Since the function θ is non-decreasing and $x \geq u$ and $y \leq v$, we have

$$
\begin{aligned}
\theta\left(x(q)-u(q)\right) \quad & \leq \quad \theta\left(\sup_{p\in I}|x(q)-u(q)|\right)=\theta(d(x,\ u)), \\
\theta\left(v(q)-y(q)\right) \quad & \leq \quad \theta\left(\sup_{p\in I}|v(q)-y(q)|\right)=\theta(d(y,\ v)).
\end{aligned}
$$

Hence by (3.5), in view of the fact that $K_2(p, q) \leq 0$, we obtain

$$
\begin{aligned}
& |F(x,\ y)(p)-F(u,\ v)(p)| \\
\leq \quad & \int_a^b K_1(p,\ q)\left[\lambda\theta(d(x,\ u))+\mu\theta(d(y,\ v))\right]dq \\
& -\int_a^b K_2(p,\ q)\left[\lambda\theta(d(y,\ v))+\mu\theta(d(x,\ u))\right]dq, \\
\leq \quad & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(x,\ u))+\max\{\lambda,\ \mu\}\theta(d(y,\ v))\right]dq \\
& -\int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(y,\ v))+\max\{\lambda,\ \mu\}\theta(d(x,\ u))\right]dq,
\end{aligned}
$$

as all the quantities on the right hand side of (3.5) are non-negative. Now, taking the supremum with respect to p we get, by using (3.4),

$$
\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq \quad & \max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b \left(K_1(p,\ q)-K_2(p,\ q)\right)dq.\left[\theta(d(x,\ u))+\theta(d(y,\ v))\right] \\
\leq \quad & \frac{\theta(d(x,\ u))+\theta(d(y,\ v))}{4}.
\end{aligned}
$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{4}. \tag{3.6}$$

Now, since θ is non-decreasing, we have

$$\begin{aligned} \theta(d(x,\ u)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \\ \theta(d(y,\ v)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \end{aligned}$$

which implies, by (ii_θ), that

$$\begin{aligned} \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) \\ &\leq \max\left\{d(x,\ u),\ d(y,\ v)\right\}. \end{aligned}$$

Hence

$$\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{4} \leq \frac{1}{2}\max\left\{d(x,\ u),\ d(y,\ v)\right\}. \tag{3.7}$$

Thus by (3.6) and (3.7), we have

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \frac{1}{2}\max\left\{d(x,\ u),\ d(y,\ v)\right\} \\ \leq\ & \frac{1}{2}\max\left\{\begin{array}{c} d(x,\ u),\ d(y,\ v),\ D(x,\ F(x,\ y)), \\ D(u,\ F(u,\ v)),\ D(y,\ F(y,\ x)),\ D(v,\ F(v,\ u)), \\ \frac{D(x,\ F(u,\ v)) + D(u,\ F(x,\ y))}{2},\ \frac{D(y,\ F(v,\ u)) + D(v,\ F(y,\ x))}{2} \end{array}\right\} \\ \leq\ & \varphi\left[\max\left\{\begin{array}{c} d(x,\ u),\ d(y,\ v),\ D(x,\ F(x,\ y)), \\ D(u,\ F(u,\ v)),\ D(y,\ F(y,\ x)),\ D(v,\ F(v,\ u)), \\ \frac{D(x,\ F(u,\ v)) + D(u,\ F(x,\ y))}{2},\ \frac{D(y,\ F(v,\ u)) + D(v,\ F(y,\ x))}{2} \end{array}\right\}\right], \end{aligned}$$

which is the contractive condition in Corollary 2.5. Now, let $(\alpha,\ \beta) \in X \times X$ be a coupled upper-lower solution of (3.1), then we have $\alpha(p) \leq F(\alpha,\ \beta)(p)$ and $\beta(p) \geq F(\beta,\ \alpha)(p)$, for all $p \in I$, which shows that all hypothesis of Corollary 2.5 are satisfied. This proves that F has a coupled fixed point $(x,\ y) \in X \times X$ which is the solution in $X = C(I,\ \mathbb{R})$ of the integral equation (3.1). □

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