J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2016.23.2.163
Volume 23, Number 2 (May 2016), Pages 163–179

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

Sung Jin Lee [a] and Jeong Pil Seo [b,*]

Abstract. Let

$$M_1f(x,y) := \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y),$$
$$M_2f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities

$$N\left(M_1f(x,y) - \rho M_2f(x,y), t\right) \ge \frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(M_2f(x,y) - \rho M_1f(x,y), t\right) \ge \frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy Banach spaces, where ρ is a fixed real number with $\rho \neq 1$.

1. Introduction and Preliminaries

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 16, 37]. In particular, Bag and Samanta [3], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 19, 20] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1 ([3, 19, 20, 21]). Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

---

Received by the editors March 31, 2016. Accepted April 19, 2016.
2010 *Mathematics Subject Classification.* Primary 46S40, 39B52, 39B62, 26E50, 47S40.
Key words and phrases. fuzzy Banach space, additive-quadratic ρ-functional inequality, Hyers-Ulam stability.
*Corresponding author.

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [19, 20].

Definition 1.2 ([3, 19, 20, 21])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3 ([3, 19, 20, 21])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [35] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 6, 7, 10, 17, 18, 22, 25, 26, 27, 29, 30, 31, 32, 33, 34, 38, 39]).

Park [23, 24] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space. Let ρ be a real number with $\rho \neq 1$.

2. Additive-quadratic ρ-functional Inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces.

We need the following lemma to prove the main results.

Lemma 2.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$M_1 f(x,y) = \rho M_2 f(x,y) \tag{2.1}$$

for all $x, y \in X$, then f is the Cauchy additive mapping.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (2.1), then f is the quadratic mapping.*

Proof. (i) Letting $y = x$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y)-f(x)-f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right) \\ &= \rho(f(x+y)-f(x)-f(y)) \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Letting $y=x$ in (2.1), we get $\frac{1}{2}f(2x)-2f(x)=0$ and so $f(2x)=4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y) \\ &=\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right) \\ &=\rho\left(\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right) \end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.4}$$

for all $x, y \in X$.
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_1 f(x,y) - \rho M_2 f(x,y), t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t+\frac{1}{2}\Psi(x,x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$, where $\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$.

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{t}{t + \frac{1}{2}\Phi(x, x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$, where $\Phi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$ for all $x, y \in X$.

Proof. (i) Letting $y = x$ in (2.5), we get

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x, x)} \tag{2.8}$$

and so

$$N\left(f\left(x\right) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$$

for all $x \in X$. Hence

(2.9)

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \\
&\quad\geq \min\Bigg\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, \\
&\qquad\qquad\qquad N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\Bigg\} \\
&\quad= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\} \\
&\quad\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\
&\quad= \min\left\{\frac{t}{t + 2^l\varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\
&\quad\geq \frac{t}{t + \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.4) and (2.9) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.6).

By (2.5),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right)$$
$$\geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t+2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = x$ in (2.5), we get

$$N\left(\frac{1}{2}f(2x) - 2f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.10}$$

and so

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{\frac{t}{2}}{\frac{t}{2}+\varphi\left(\frac{x}{2}, \frac{x}{2}\right)} = \frac{t}{t+2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$$

for all $x \in X$. Hence

$$\begin{aligned} &N\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right) \\ &\geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots,\right. \\ &\qquad\left. N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)\right\} \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right),\frac{t}{4^l}\right),\cdots,N\left(f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right),\frac{t}{4^{m-1}}\right)\right\}\\
&\ge \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l}+2\varphi\left(\frac{x}{2^{l+1}},\frac{x}{2^{l+1}}\right)},\cdots,\frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}}+2\varphi\left(\frac{x}{2^m},\frac{x}{2^m}\right)}\right\}\\
&= \min\left\{\frac{t}{t+2\cdot 4^l\varphi\left(\frac{x}{2^{l+1}},\frac{x}{2^{l+1}}\right)},\cdots,\frac{t}{t+2\cdot 4^{m-1}\varphi\left(\frac{x}{2^m},\frac{x}{2^m}\right)}\right\}\\
&\ge \frac{t}{t+\frac{1}{2}\sum_{j=l+1}^{m}4^j\varphi\left(\frac{x}{2^j},\frac{x}{2^j}\right)}
\end{aligned}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.4) and (2.11) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.7).

The rest of the proof is similar to the above additive case. □

Corollary 2.3. *Let $\theta \ge 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_1 f(x,y) - \rho M_2 f(x,y), t\right) \ge \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{2.12}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \ge \frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.12). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \ge \frac{(2^p-4)t}{(2^p-4)t+4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$.
(i) *Let $f : X \to Y$ be an odd mapping satisfying* (2.5). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \frac{1}{2}\Phi(x,x)}$$

for all $x \in X$ and all $t > 0$, where $\Phi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right)$ for all $x, y \in X$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.5). *Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{t}{t + \frac{1}{2}\Psi(x,x)}$$

for all $x \in X$ and all $t > 0$, where $\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi\left(2^j x, 2^j y\right)$ for all $x, y \in X$.

Proof. (i) It follows from (2.8) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq \frac{2t}{2t + \varphi(x,x)} = \frac{t}{t + \frac{1}{2}\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (2.10) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), t\right) \geq \frac{2t}{2t + \varphi(x,x)} = \frac{t}{t + \frac{1}{2}\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying* (2.12). *Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.12). *Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

3. Additive-quadratic ρ-functional Inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$M_2 f(x, y) = \rho M_1 f(x, y) \tag{3.1}$$

for all $x, y \in X$, then f is the Cauchy additive mapping.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and* (3.1), *then f is the quadratic mapping.*

Proof. (i) Letting $y = 0$ in (3.1), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Letting $y = 0$ in (3.1), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}
&\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y) \\
&\quad = 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\
&\quad = \rho\left(\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y)\right)
\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. $\square$

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.4}$$

for all $x, y \in X$.

(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y) - \rho M_1 f(x,y), t\right) \geq \frac{t}{t + \varphi(x,y)} \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)} \tag{3.6}$$

for all $x \in X$ and all $t > 0$, where $\Phi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$ for all $x, y \in X$.

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{t}{t + \Psi(x, 0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$, where $\Psi(x,y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$ for all $x, y \in X$.

Proof. (i) Letting $y = 0$ in (3.5), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.8}$$

for all $x \in X$. Hence

(3.9)

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\\
&\quad \geq \min\Bigg\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots,\\
&\qquad\qquad\qquad N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\Bigg\}\\
&\quad = \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\quad \geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&\quad = \min\left\{\frac{t}{t + 2^l \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}\\
&\quad \geq \frac{t}{t + \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.4) and (3.9) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.6).

By (3.5),

$$\begin{aligned}
&N\Bigg(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\\
&\qquad\qquad - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\Bigg)\\
&\quad \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right.$$
$$\left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right)$$
$$\geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t + 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = \rho\left(A(x+y) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = 0$ in (3.5), we get

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) = N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.10}$$

for all $x \in X$. Hence

(3.11)

$$N\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)$$
$$\geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, \right.$$
$$\left. N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)\right\}$$
$$= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right), \frac{t}{4^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right), \frac{t}{4^{m-1}}\right)\right\}$$
$$\geq \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}$$
$$= \min\left\{\frac{t}{t + 4^l \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 4^{m-1}\varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\}$$
$$\geq \frac{t}{t + \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right)}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.4) and (3.11) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.7).

The rest of the prrof is similar to the above additive case. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x, y) - \rho M_1 f(x, y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.12}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.12). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$.
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.5). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)}$$

for all $x \in X$ and all $t > 0$, where $\Phi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right)$ for all $x, y \in X$.

(ii) *Let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and* (3.5). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{t}{t + \Psi(x, 0)}$$

for all $x \in X$ *and all* $t > 0$*, where* $\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi\left(2^j x, 2^j y\right)$ *for all* $x, y \in X$.

Proof. (i) It follows from (3.8) that

$$N\left(f(x) - \frac{1}{2} f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2} f(2x), t\right) \geq \frac{2t}{2t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{2}\varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (3.10) that

$$N\left(f(x) - \frac{1}{4} f(2x), \frac{t}{4}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4} f(2x), t\right) \geq \frac{4t}{4t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{4}\varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 1$. *Let* X *be a normed vector space with norm* $\|\cdot\|$.

(i) *Let* $f : X \to Y$ *be an odd mapping satisfying* (3.12). *Then* $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

(ii) *Let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and* (3.12). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

References

1. M. Adam: On the stability of some quadratic functional equation. *J. Nonlinear Sci. Appl.* **4** (2011), 50-59.
2. T. Aoki: On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2** (1950), 64-66.
3. T. Bag & S.K. Samanta: Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.* **11** (2003), 687-705.
4. ______: Fuzzy bounded linear operators. *Fuzzy Sets and Systems* **151** (2005), 513-547.
5. L. Cădariu, L. Găvruta & P. Găvruta: On the stability of an affine functional equation. *J. Nonlinear Sci. Appl.* **6** (2013), 60-67.
6. A. Chahbi & N. Bounader: On the generalized stability of d'Alembert functional equation *J. Nonlinear Sci. Appl.* **6** (2013), 198-204.
7. I. Chang & Y. Lee: Additive and quadratic type functional equation and its fuzzy stability. *Results Math.* **63** (2013), 717-730.
8. S.C. Cheng & J.M. Mordeson: Fuzzy linear operators and fuzzy normed linear spaces. *Bull. Calcutta Math. Soc.* **86** (1994), 429-436.
9. P.W. Cholewa: Remarks on the stability of functional equations. *Aequationes Math.* **27** (1984), 76-86.
10. G.Z. Eskandani & P. Găvruta: Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces. *J. Nonlinear Sci. Appl.* **5** (2012), 459-465.
11. C. Felbin: Finite dimensional fuzzy normed linear spaces. *Fuzzy Sets and Systems* **48** (1992), 239-248.
12. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), 431-436.
13. D.H. Hyers: On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.
14. A.K. Katsaras: Fuzzy topological vector spaces II. *Fuzzy Sets and Systems* **12** (1984), 143-154.
15. I. Kramosil & J. Michalek: Fuzzy metric and statistical metric spaces. *Kybernetica* **11** (1975), 326-334.
16. S.V. Krishna & K.K.M. Sarma: Separation of fuzzy normed linear spaces. *Fuzzy Sets and Systems* **63** (1994), 207-217.
17. G. Lu, Y. Wang & P. Ye: n-Jordan $*$-derivations on induced fuzzy C^*-algebras. *J. Comput. Anal. Appl.* **20** (2016), 266-276.

18. D. Mihet & R. Saadati: On the stability of the additive Cauchy functional equation in random normed spaces. *Appl. Math. Lett.* **24** (2011), 2005-2009.
19. A.K. Mirmostafaee, M. Mirzavaziri & M.S. Moslehian: Fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems* **159** (2008), 730-738.
20. A.K. Mirmostafaee & M.S. Moslehian: Fuzzy versions of Hyers-Ulam-Rassias theorem. *Fuzzy Sets and Systems* **159** (2008), 720-729.
21. ______: Fuzzy approximately cubic mappings, *Inform. Sci.* **178** (2008), 3791-3798.
22. E. Movahednia, S.M. Mosadegh, C. Park & D. Shin: Stability of a lattice preserving functional equation on Riesz space: fixed point alternative. *J. Comput. Anal. Appl.* **21** (2016), 83-89.
23. C. Park: Additive ρ-functional inequalities and equations. *J. Math. Inequal.* **9** (2015), 17-26.
24. ______: Additive ρ-functional inequalities in non-Archimedean normed spaces. *J. Math. Inequal.* **9** (2015), 397-407.
25. ______: Stability of ternary quadratic derivation on ternary Banach algebras: revisited. *J. Comput. Anal. Appl.* **20** (2016), 21-23.
26. C. Park, K. Ghasemi, S.G. Ghaleh & S. Jang: Approximate n-Jordan $*$-homomorphisms in C^*-algebras. *J. Comput. Anal. Appl.* **15** (2013), 365-368.
27. C. Park, A. Najati & S. Jang: Fixed points and fuzzy stability of an additive-quadratic functional equation. *J. Comput. Anal. Appl.* **15** (2013), 452-462.
28. Th. M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
29. K. Ravi, E. Thandapani & B.V. Senthil Kumar: Solution and stability of a reciprocal type functional equation in several variables. *J. Nonlinear Sci. Appl.* **7** (2014), 18-27.
30. S. Schin, D. Ki, J. Chang & M. Kim: Random stability of quadratic functional equations: a fixed point approach. *J. Nonlinear Sci. Appl.* **4** (2011), 37-49.
31. S. Shagholi, M. Bavand Savadkouhi & M. Eshaghi Gordji: Nearly ternary cubic homomorphism in ternary Fréchet algebras. *J. Comput. Anal. Appl.* **13** (2011), 1106-1114.
32. S. Shagholi, M. Eshaghi Gordji & M. Bavand Savadkouhi: Stability of ternary quadratic derivation on ternary Banach algebras. *J. Comput. Anal. Appl.* **13** (2011), 1097-1105.
33. D. Shin, C. Park & Sh. Farhadabadi: On the superstability of ternary Jordan C^*-homomorphisms. *J. Comput. Anal. Appl.* **16** (2014), 964-973.
34. ______: Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation. *J. Comput. Anal. Appl.* **17** (2014), 125-134.
35. F. Skof: Propriet locali e approssimazione di operatori. *Rend. Sem. Mat. Fis. Milano* **53** (1983), 113-129.
36. S.M. Ulam: *A Collection of the Mathematical Problems.* Interscience Publ. New York, 1960.

37. J.Z. Xiao & X.H. Zhu: Fuzzy normed spaces of operators and its completeness. *Fuzzy Sets and Systems* **133** (2003), 389-399.
38. C. Zaharia: On the probabilistic stability of the monomial functional equation. *J. Nonlinear Sci. Appl.* **6** (2013), 51-59.
39. S. Zolfaghari: Approximation of mixed type functional equations in p-Banach spaces. *J. Nonlinear Sci. Appl.* **3** (2010), 110-122.

[a]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
Email address: hyper@daejin.ac.kr

[b]Ohsang High School, Gumi, Kyeongsangbuk-do 730-842, Korea
Email address: sjp4829@hanmail.net