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SOME RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper we obtain some retarded integral inequalities involving Stieltjes derivatives and we use our results in the study of various qualitative properties of a certain retarded impulsive differential equation.

1. Introduction

In this paper, we discuss various retarded integral inequalities of Stieltjes type and apply the inequalities to the study of various qualitative behaviors of a certain retarded differential equation involving impulses.

Differential equations with impulses arise in various real world phenomena in mathematical physics, mechanics, engineering, biology and so on. We refer to the monograph of Samoilenko and Perestyuk [8]. Also integral inequalities are very useful tools in global existence, uniqueness, stability and other properties of the solutions of various nonlinear differential equations, see, e.g., [5], and for retarded integral inequalities and their applications, see [6].

2. Preliminaries

In this section we state some materials that are needed in this paper.

Let $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ be the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integers, respectively, and let

$G(\mathbf{R}^+) = \{f : \mathbf{R}^+ \to \mathbf{R} |\ \forall t > 0$ both $f(t+)$ and $f(t-)$ exist, and $f(0+)$ exists $\}$.

For convenience we define

$$\Delta^+ f(t) = f(t+) - f(t), \Delta^- f(t) = f(t) - f(t-), \Delta f(t) = f(t+) - f(t-).$$

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Throughout this paper we use the Kurzweil-Stieltjes integrals and the Stieltjes derivatives. For the integrals and derivatives, and various notations that are used here, see, e.g., [3, 4, 10] and the references cited there.

We use the following results frequently.

Theorem 2.1 ([12, Theorem 2.15])**.** *Assume that $f \in G([a,b])$ and $g \in BV([a,b])$. Then both $f\,\mathrm{d}g$ and $g\,\mathrm{d}f$ are K-integrable on $[a,b]$.*

Theorem 2.2 ([3, 4])**.** *Assume that $f \in G([a,b])$ and a function $m : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and is not locally constant at $t \in [a,b]$. If f is continuous at t or m is not continuous at t, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_a^t f(s)\,\mathrm{d}m(s) = f(t).$$

Theorem 2.3 ([3, 4])**.** *Assume that $f \in G([a,b])$ and a function $m : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and that if m is constant on some neighborhood of t, then there exists a neighborhood of t such that both f and m are constant there. Suppose that $f'_m(t)$ exists at every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*) \int_a^b f'_m(s)\,\mathrm{d}m(s) = f(b) - f(a).$$

Theorem 2.4 ([11, p. 34, Corollary 4.13])**.** *Assume that $f \in G([a,b])$ and $m \in BV([a,b])$. Then for every $t \in [a,b]$ we have*

$$\lim_{\eta \to 0+} \int_a^{t \pm \eta} f(s)\,\mathrm{d}m(s) = \int_a^t f(s)\,\mathrm{d}m(s) \pm f(t)\Delta^{\pm} m(t).$$

Remark 2.5. In Theorem 2.3, we define, for every $i \in \mathbf{N}$, $f'_m(c_i) = 0$, and Theorem 2.4 implies that if m is continuous at t then $\int_a^{(\cdot)} f(s)\,\mathrm{d}m(s)$ is also continuous there.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots$. Then we define a function ϕ as

$$\phi(t) = \begin{cases} t, & \text{if } t \in [0, t_1] \\ t + k, & \text{if } t \in (t_k, t_{k+1}], \quad k \in \mathbf{N}. \end{cases} \tag{2.1}$$

For the function ϕ we have the following result.

Lemma 2.6 ([4]). *We have*

$$\int_0^t f(s)\,\mathrm{d}\phi(s) = \int_0^t f(s)\,\mathrm{d}s + \sum_{0<t_k<t} f(t_k).$$

We define $\mathbf{C}_m$ as the set of all points of continuity of a function m, and $\mathbf{D}_m$ as the set of all points of discontinuity of the function m, and we define

$$f \circ g(t) = f(g(t)),$$

and

$$A(t) = \begin{cases} 1, & t \in \mathbf{C}_m \\ 0, & t \notin \mathbf{C}_m \end{cases}, \quad B(t) = \begin{cases} 1, & t \in \mathbf{D}_m, \\ 0, & t \notin \mathbf{D}_m \end{cases}.$$

Throughout this paper, unless otherwise specified, we always assume the following conditions:

(H1) All functions in this paper are nonnegative regulated functions on $\mathbf{R}^+$.

(H2) A left-continuous function m is strictly increasing and differentiable, and $m' > 0$, except for a countable set of $\mathbf{R}^+$. And a function w is nondecreasing, continuous and positive on $(0, \infty)$. We define

$$E(t) = \int_1^t \frac{\mathrm{d}s}{w(s)},$$

and E^{-1} represents the inverse of the function E, and $\mathrm{Dom}(E^{-1})$ represents the domain of the function E^{-1}.

(H3) A left-continuous function α is nondecreasing, and $0 \leq \alpha(t) \leq t$ for every $t \in \mathbf{R}^+$, and $\alpha'(t)$ exists whenever $m'(t)$ exists, and α is continuous at t whenever m is continuous there.

3. Some Retarded Integral Inequalities

In order to obtain some integral inequalities we need following results.

Lemma 3.1. *Assume that a positive left-continuous function z is nondecreasing on $\mathbf{R}^+$. If z is continuous at t and $z'_m(t)$ exists, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} = \frac{z'_m(t)}{w(z(t))}.$$

If $t \in \mathbf{D}_m$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} \le \frac{z'_m(t)}{w(z(t))}.$$

Proof. First assume that the function z is constant on some open neighborhood of t, then $z'_m(t) = 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} = 0 = \frac{z'_m(t)}{w(z(t))},$$

so in this case the lemma is true.

If z is not locally constant at t and continuous there, then by definition we have

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} = \lim_{\eta,\delta\to 0+} \frac{\int_{z(t-\delta)}^{z(t+\eta)} \frac{\mathrm{d}s}{w(s)}}{z(t+\eta) - z(t-\delta)} \cdot \frac{z(t+\eta) - z(t-\delta)}{m(t+\eta) - m(t-\delta)} = \frac{z'_m(t)}{w(z(t))}.$$

Now assume that $t \in \mathbf{D}_m$. Then, since both m and z are left-continuous, and both w and z are nondecreasing, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} &= \lim_{\eta,\delta\to 0+} \frac{\int_{z(t-\delta)}^{z(t+\eta)} \frac{\mathrm{d}s}{w(s)}}{m(t+\eta) - m(t-\delta)} = \frac{\int_{z(t)}^{z(t+)} \frac{\mathrm{d}s}{w(s)}}{m(t+) - m(t)} \\ &\le \frac{1}{w(z(t))} \cdot \frac{z(t+) - z(t)}{m(t+) - m(t)} = \frac{z'_m(t)}{w(z(t))}.\end{aligned}$$

The proof is complete. □

From now on we define

$$M_f(t) = \sup_{\alpha(t) \le s \le \alpha(t+)} f(s).$$

Then we have the following result.

Lemma 3.2. *If α'_m exists except for a countable subset of $\mathbf{C}_m$, then there is a countable set $\mathbf{D}$ of $\mathbf{C}_m$ such that for every $t \in \mathbf{C}_m - \mathbf{D}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^t f(s)\,\mathrm{d}m(s) = f(t), \quad \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^{\alpha(t)} f(s)\,\mathrm{d}s = f(\alpha(t))\,\alpha'_m(t), \tag{3.1}$$

and if $t \in \mathbf{D}_m$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^t f(s)\,\mathrm{d}m(s) = f(t), \quad \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^{\alpha(t)} f(s)\,\mathrm{d}s \le M_f(t)\,\alpha'_m(t). \tag{3.2}$$

Proof. Since every function in $G(\mathbf{R}^+)$ has at most a countable number of discontinuities([2, p.17, Corollary 3.2.]), by Theorem 2.2, there is a countable set $\mathbf{D}_1 \subset \mathbf{C}_m$ such that for every $t \in \mathbf{C}_m - \mathbf{D}_1$, the first equality of (3.1) is true, and again by Theorem 2.2, the first equality of (3.2) is also true.

Let $\mathbf{K} = \{t \in \mathbf{C}_m : \alpha \text{ is constant on some open neighborhood of } t\}$. Then, since in this case $\alpha'_m(t) = 0$, we have

$$f(\alpha(t))\,\alpha'_m(t) = 0 = \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^{\alpha(t)} f(s)\,\mathrm{d}s,$$

and so in this case the lemma is also true.

Note that since $f \in G(\mathbf{R}^+)$ and α is nondecreasing on $\mathbf{R}^+$ there is a countable set $\mathbf{D}_2$ such that, for every $t \in (\mathbf{C}_m - \mathbf{K}) - \mathbf{D}_2$, α is not locally constant at t, and f is continuous at $\alpha(t)$. By (H3), $t \in \mathbf{C}_m$ implies $t \in \mathbf{C}_\alpha$. So by the definition of Stieltjes derivatives, for every $t \in (\mathbf{C}_m - \mathbf{K}) - \mathbf{D}_2$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^{\alpha(t)} f(s)\,\mathrm{d}s &= \lim_{\eta,\delta\to 0+} \frac{\int_{\alpha(t-\delta)}^{\alpha(t+\eta)} f(s)\,\mathrm{d}s}{m(t+\eta) - m(t-\delta)} \\ &= \lim_{\eta,\delta\to 0+} \frac{\int_{\alpha(t-\delta)}^{\alpha(t+\eta)} f(s)\,\mathrm{d}s}{\alpha(t+\eta) - \alpha(t-\delta)} \cdot \frac{\alpha(t+\eta) - \alpha(t-\delta)}{m(t+\eta) - m(t-\delta)} \\ &= f(\alpha(t))\,\alpha'_m(t).\end{aligned}$$

Now we put $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2$. Then for every $t \in \mathbf{C}_m - \mathbf{D}$ the lemma is true.

And if $t \in \mathbf{D}_m$ then we have

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} \int_0^{\alpha(t)} f(s)\,\mathrm{d}s = \lim_{\eta,\delta\to 0+} \frac{\int_{\alpha(t-\delta)}^{\alpha(t+\eta)} f(s)\,\mathrm{d}s}{m(t+\eta) - m(t-\delta)}$$

$$= M_f(t) \lim_{\eta,\delta\to 0+} \frac{\int\limits_{\alpha(t-\delta)}^{\alpha(t+\eta)} \mathrm{d}s}{m(t+\eta)-m(t-\delta)} = M_f(t)\,\alpha'_m(t).$$

Thus the proof is complete. □

The following is a Gronwall-Bellman type integral inequality.

Theorem 3.3. *Assume that functions n and m are nondecreasing on $\mathbf{R}^+$.*
If a function u satisfies

$$u(t) \le n(t) + q(t) \int\limits_0^t a(s)u(s)\,\mathrm{d}m(s), \tag{3.3}$$

then we have

$$u(t) \le n(t)\left[1 + q(t)\int\limits_0^t a(s)\exp\left(\int\limits_s^t aq\,\mathrm{d}m\right)\mathrm{d}m(s)\right]. \tag{3.4}$$

Proof. Without loss of generality we may assume that $n(t)$ is positive. If $n(t)$ is nonnegative we use $n(t)+\varepsilon(\varepsilon>0)$ and let $\varepsilon\to 0+$.

From (3.3), since $n(t)$ is positive, we have

$$\frac{u(t)}{n(t)} \le 1 + q(t)\int\limits_0^t a(s)\frac{u(s)}{n(s)}\,\mathrm{d}m(s).$$

By [3, Theorem 4.4.] we get

$$\frac{u(t)}{n(t)} \le 1 + q(t)\int\limits_0^t a(s)\exp\left(\int\limits_s^t aq\,\mathrm{d}m\right)\mathrm{d}m(s).$$

This inequality yields (3.4). The proof is complete. □

From now on we assume that $L \ge 0$ is a constant.

Theorem 3.4. *If a function u satisfies*

$$u(t) \le L + \int\limits_0^t c(s)\,\mathrm{d}s + \int\limits_0^t a(s)w(u(s))\,\mathrm{d}m(s) + \int\limits_0^{\alpha(t)} b(s)w(u(s))\,\mathrm{d}s, \tag{3.5}$$

then for some $T \ge 0$ we have

$$u(t) \le E^{-1}[\gamma(t)],$$

where

$$\gamma(t) = \int_0^t \Big(A[a + (b \circ \alpha)\,\alpha'_m] + B[a + M_b\,\alpha'_m] \Big)\,\mathrm{d}m + E\left(L + \int_0^t c(s)\,\mathrm{d}s \right),$$

and the number T is chosen so that

$$\gamma(t) \in \mathrm{Dom}(E^{-1}),$$

for every $t \in [0,T]$.

Proof. We define

$$x(t) = L + \int_0^t c(s)\,\mathrm{d}s$$

and

$$y(t) = \int_0^t a(s)w(u(s))\,\mathrm{d}m(s) + \int_0^{\alpha(t)} b(s)w(u(s))\,\mathrm{d}s.$$

Then $u(t) \le x(t) + y(t)$.

By Lemma 3.1 since x is continuous on $\mathbf{R}^+$ there is a countable set $\mathbf{D}_1 \subset \mathbf{C}_m$ such that for every $t \in \mathbf{R}^+ - \mathbf{D}_1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}m(t)} E(x(t)) = \frac{\mathrm{d}}{\mathrm{d}m(t)} \int_1^{x(t)} \frac{\mathrm{d}s}{w(s)} = \frac{x'_m(t)}{w(x(t))}. \tag{3.6}$$

By Lemma 3.2 there is a countable set $\mathbf{D}_2 \subset \mathbf{C}_m$ such that for every $t \in \mathbf{C}_m - \mathbf{D}_2$ we have

$$\begin{aligned} y'_m(t) &\le a(t)w(u(t)) + b(\,\alpha(t))w(u(\,\alpha(t)))\,\alpha'_m(t) \\ &\le a(t)w((x+y)(t)) + b(\,\alpha(t))w((x+y)(\,\alpha(t)))\,\alpha'_m(t) \\ &\le [a(t) + b(\,\alpha(t))\,\alpha'_m(t)]w((x+y)(t)), \end{aligned}$$

and for every $t \in \mathbf{D}_m$ we get

$$\begin{aligned} y'_m(t) &\le a(t)w((x+y)(t)) + M_b(t)w((x+y)(t))\,\alpha'_m(t) \\ &\le [a(t) + M_b(t)\,\alpha'_m]w((x+y)(t)). \end{aligned}$$

Now, let $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2$, and considering Remark 2.5, for every $t \in \mathbf{D}$, we define $0 = x'_m(t) = y'_m(t) = \alpha'_m(t) = (E \circ x)'_m(t)$. Then for every $t \in \mathbf{R}^+$, by the previous inequalities and (3.6), for every $t \in \mathbf{R}^+$, we have

$$\begin{aligned}\frac{(x+y)'_m(t)}{w((x+y)(t))} &\le A(t)[a(t)+b(\alpha(t))\,\alpha'_m(t)] + B(t)[a(t)+M_b(t)\,\alpha'_m(t)] + \frac{x'_m(t)}{w((x+y)(t))}\\ &\le A(t)[a(t)+b(\alpha(t))\,\alpha'_m(t)] + B(t)[a(t)+M_b(t)\,\alpha'_m(t)] + \frac{x'_m(t)}{w(x(t))}\\ &= A(t)[a(t)+b(\alpha(t))\,\alpha'_m(t)] + B(t)[a(t)+M_b(t)\,\alpha'_m(t)] + (E\circ x)'_m(t).\end{aligned}$$

Thus we get

$$\begin{aligned}(K^*)\int_0^t \frac{(x+y)'_m}{w\circ(x+y)}\,\mathrm{d}m &\le \int_0^t \Big(A[a+(b\circ\,\alpha)\,\alpha'_m] + B[a+M_b\,\alpha'_m]\Big)\,\mathrm{d}m\\ &\quad + (K^*)\int_0^t (E\circ x)'_m(t)\,\mathrm{d}m(s).\end{aligned}$$

So by Theorem 2.3 and Lemma 3.1 we get

$$\begin{aligned}&E((x+y)(t)) - E((x+y)(0))\\ &\le \int_0^t \Big(A[a+(b\circ\,\alpha)\,\alpha'_m] + B[a+M_b\,\alpha'_m]\Big)\,\mathrm{d}m + E(x(t)) - E(x(0)).\end{aligned}$$

Since $x(0)=L, y(0)=0$ we have

$$\begin{aligned}&E((x+y)(t)) - E(L)\\ &\le \int_0^t \Big(A[a+(b\circ\,\alpha)\,\alpha'_m] + B[a+M_b\,\alpha'_m]\Big)\,\mathrm{d}m + E(x(t)) - E(L).\end{aligned}$$

This implies that for every $t \in [0,T]$,

$$u(t) \le (x+y)(t) \le E^{-1}\left[\gamma(t)\right].$$

The proof is complete. □

From now on we define $\alpha_+(t) = \alpha(t+)$. Then we have the following result.

Theorem 3.5. *Assume that a function h is nondecreasing on $\mathbf{R}^+$ and for every $u, v \ge 0$, $w(uv) \le w(u)w(v)$. If a function u satisfies*

(3.7)

$$u(t) \le L + q(t)\int_0^t c(s)u(s)\,\mathrm{d}h(s) + \int_0^t a(s)w(u(s))\,\mathrm{d}m(s) + \int_0^{\alpha(t)} b(s)w(u(s))\,\mathrm{d}s,$$

then we have for some $T \geq 0$

$$u(t) \leq k(t)E^{-1}[\gamma(t)], \tag{3.8}$$

where

$$\gamma(t) = E(L) + \int_0^t \Big(A[a \cdot (w \circ k) + (b \circ \alpha) \cdot (w \circ k \circ \alpha)\, \alpha'_m] + B[a \cdot (w \circ k) + M_b \cdot (w \circ k \circ \alpha_+)\, \alpha'_m] \Big)\, \mathrm{d}m,$$

and $k(t) = 1 + q(t) \int_0^t c(s) \exp\left(\int_s^t cq\, \mathrm{d}h \right) \mathrm{d}h(s)$, *and the number* T *is chosen so that*

$$\gamma(t) \in \mathrm{Dom}(E^{-1}),$$

for all $t \in [0, T]$.

Proof. Define a function $n(t)$ by

$$n(t) = L + \int_0^t a(s) w(u(s))\, \mathrm{d}m(s) + \int_0^{\alpha(t)} b(s) w(u(s))\, \mathrm{d}s,$$

then (3.7) can be written as

$$u(t) \leq n(t) + q(t) \int_0^t c(s) u(s)\, \mathrm{d}h(s).$$

Since $n(t)$ is nondecreasing on $\mathbf{R}^+$, by applying Theorem 3.3, we have

$$u(t) \leq k(t)n(t). \tag{3.9}$$

By Lemma 3.2, there is a countable set $\mathbf{D} \subset \mathbf{C}_m$ such that for every $t \in \mathbf{C}_m - \mathbf{D}$, we have

$$\begin{aligned} n'_m(t) &\leq [a \cdot (w \circ u) + (b \circ \alpha) \cdot (w \circ u \circ \alpha) \cdot \alpha'_m](t) \\ &\leq [a \cdot (w \circ (k \cdot n)) + (b \circ \alpha) \cdot (w \circ (k \cdot n) \circ \alpha) \cdot \alpha'_m](t) \\ &\leq [a \cdot (w \circ k) \cdot (w \circ n) + (b \circ \alpha) \cdot (w \circ k \circ \alpha) \cdot (w \circ n \circ \alpha) \cdot \alpha'_m](t) \\ &\leq [a \cdot (w \circ k) + (b \circ \alpha) \cdot (w \circ k \circ \alpha) \cdot \alpha'_m](t) \cdot (w \circ n)(t) \end{aligned}$$

and for every $t \in \mathbf{D}_m$ we get similarly

$$\begin{aligned} n_m'(t) &\le \left[a \cdot (w \circ u) + \left(\sup_{\alpha(t) \le s \le \alpha(t+)} b(s)(w \circ u)(s)\right) \cdot \alpha_m'\right](t) \\ &\le [a \cdot (w \circ k) \cdot (w \circ n) + M_b \cdot (w \circ k \circ \alpha_+) \cdot (w \circ n \circ \alpha_+) \cdot \alpha_m'](t) \\ &\le [a \cdot (w \circ k) + M_b \cdot (w \circ k \circ \alpha_+) \cdot \alpha_m'](t) \cdot (w \circ n)(t). \end{aligned}$$

Considering Remark 2.5, for every $t \in \mathbf{D}$, we define $n_m'(t) = 0 = \alpha_m'(t)$. Then, by the previous inequalities, for every $t \in \mathbf{R}^+$, we have

$$\begin{aligned} n_m'(t) \le \Big(&A[a \cdot (w \circ k) + (b \circ \alpha) \cdot (w \circ k \circ \alpha) \cdot \alpha_m'] \\ &+ B[a \cdot (w \circ k) + M_b \cdot (w \circ k \circ \alpha_+) \cdot \alpha_m']\Big)(t) \cdot (w \circ n)(t). \end{aligned}$$

This implies

$$\begin{aligned} \frac{n_m'(t)}{w \circ n(t)} \le \Big(&A[a \cdot (w \circ k) + (b \circ \alpha) \cdot (w \circ k \circ \alpha) \cdot \alpha_m'] \\ &+ B[a \cdot (w \circ k) + M_b \cdot (w \circ k \circ \alpha_+) \cdot \alpha_m']\Big)(t). \end{aligned}$$

By Theorem 2.3 and Lemma 3.1, integrating both sides of the above inequality gives

$$\begin{aligned} E(n(t)) - E(n(0)) \le \int_0^t \Big(&A[a \cdot (w \circ k) + (b \circ \alpha) \cdot (w \circ k \circ \alpha) \cdot \alpha_m'] \\ &+ B[a \cdot (w \circ k) + M_b \cdot (w \circ k \circ \alpha_+) \cdot \alpha_m']\Big)\, \mathrm{d}m. \end{aligned}$$

Since $n(0) = L$ we have

$$n(t) \le E^{-1}[\gamma(t)]. \tag{3.10}$$

Hence (3.9) and (3.10) gives (3.8). The proof is complete. □

4. Some Applications

There are many applications of the inequalities obtained in the previous section. Here we shall give some examples that are sufficient to show the usefulness of our results.

We consider the following retarded impulsive differential equation:

$$x'(t) + Ax(t) = F(t, x(t)) + G(s, x(\alpha(t))),\ t \ne t_k \tag{4.1}$$

$$x(0) = x_0, \Delta x(t_k) = I_k(x(t_k)), A > 0, k \in \mathbf{N},$$

where $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < \cdots$.

In this section, for every $k \in \mathbf{N}$, we assume the following conditions:

(C1)A strictly increasing left-continuous function $\alpha \in G(\mathbf{R}^+)$ is continuous at every $t \neq t_k$, and differentiable at every $t \in \mathbf{R}^+$, where $\alpha'(t_k)$ implies the left hand derivative at t_k, and $0 \leq \alpha(t) \leq t$.

(C2)A left-continuous function $x \in G(\mathbf{R}^+)$ is continuous at every $t \neq t_k$, and $x'(t)$ exists for every t, where for $t = t_k$ or $\alpha(t) = t_k$ we define $x'(t)$ as the left hand derivative at t.

(C3)A continuous function $I_k : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies

$$|I_k(x)| \leq a_k w(|x|),\ a_k \geq 0.$$

(C4)A function $G : \mathbf{R}^+ \times \mathbf{R} \longrightarrow \mathbf{R}$ is continuous, and a function $F : \mathbf{R}^+ \times \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at every $(t, x), t \neq t_k$, and for every t_k, $F(t_k, x) = I_k(x)$ and,

$$|F(t, x)| \leq a(t)w(|x|) + c(t),\ |G(t, x)| \leq j(t)w(|x|),$$

for some nonnegative functions $a, c \in G(\mathbf{R}^+)$, where $a(t_k) = a_k, c(t_k) = 0$, and a function j that is continuous on $\mathbf{R}^+$.

Now to accomplish our purpose we need some preliminaries.

Let X be a linear space, recall that a semi-norm on X, is a mapping $|\cdot| : X \longrightarrow \mathbf{R}^+$ having all the properties of a norm except that $|x| = 0$ does not allways imply that $x = 0$.

Suppose that we have a countable family of semi-norms on X, $|\cdot|_n$; we say that this family is *sufficient* if and only if for every $x \in X, x \neq 0$ there exists a positive integer n such that $|x|_n \neq 0$.

Every space $(X, |\cdot|_n)$, endowed with a countable and sufficient family of semi-norms can be organized as a metric space by setting the metric

$$\mathrm{d}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x - y|_n}{1 + |x - y|_n}.$$

It is well-known fact that (X, d) forms a locally convex space (see, e.g.,[7]).

Recall that the convergence determined by the metric d can be characterized as follows:

$x_k \to x$ if and only if for every positive integer n, $\lim_{k\to\infty} |x_k - x|_n = 0$.

To achieve our purpose we need the following result.

Theorem 4.1 ([9, Schaefer's fixed point theorem])**.** *Assume that X is a linear space with a countable and sufficient family of semi-norms $|\cdot|_n, n \in \mathbf{N}$.*

Let $T : X \longrightarrow X$ be a completely continuous map. If the set

$$\Lambda = \{x \in X : x = \lambda T x \ for\ some\ \lambda \in (0,1)\}$$

is bounded, then T has a fixed point.

A set $\mathcal{A} \subset G([a,b])$ has *uniform one-sided limits at* $t_0 \in [a,b]$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathcal{A}$ we have: if $t_0 < t < t_0 + \delta$ then $|x(t) - x(t_0+)| < \varepsilon$; if $t_0 - \delta < t < t_0$ then $|x(t) - x(t_0-)| < \varepsilon$.

A set $\mathcal{A} \subset G([a,b])$ is called *equi-regulated on* $[a,b]$ if it has uniform one-sided limits at every point $t_0 \in [a,b]$.

For compactness of a set $\mathcal{A} \in G([a,b])$, we have the following result.

Theorem 4.2. ([1, Corollary 2.4]) *A set $\mathcal{A} \in G([a,b])$ is relatively compact if and only if it is equi-regulated on $[a,b]$ and for every $t \in [a,b]$ the set $\{x(t) : x \in \mathcal{A}\}$ is bounded in $\mathbf{R}$.*

Lemma 4.3. *If a function x is a solution of*

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)}F(s,x(s))\,\mathrm{d}s + \int_0^t e^{-A(t-s)}G(s,x(\alpha(s)))\,\mathrm{d}s + \sum_{0<t_k<t} e^{-A(t-t_k)}I_k(x(t_k)),$$

then the function x is a solution of the equation (4.1).

Proof. First assume that $t \neq t_k$ and $\alpha(t) \neq t_k, \forall k \in \mathbf{N}$. Then, since both x and α are continuous at t, by the conditions (C1)-(C4) we have

$$\begin{aligned}
x'(t) &= -Ax_0e^{-At} - A\int_0^t e^{-A(t-s)}F(s,x(s))\,\mathrm{d}s + e^{-At}e^{At}F(t,x(t)) \\
&\quad - A\int_0^t e^{-A(t-s)}G(s,x(\alpha(s)))\,\mathrm{d}s + e^{-At}e^{At}G(t,x(\alpha(t))) \\
&= -A\Big[x_0e^{-At} + \int_0^t e^{-A(t-s)}F(s,x(s))\,\mathrm{d}s
\end{aligned} \tag{4.2}$$

$$+ \int_0^t e^{-A(t-s)} G(s, x(\alpha(s)))\,\mathrm{d}s \Big] + F(t, x(t)) + G(t, x(\alpha(t)))$$
$$= -Ax(t) + F(t, x(t)) + G(t, x(\alpha(t))).$$

This implies x satisfies (4.1) at t.

Assume that $t \neq t_k$ for every k and $\alpha(t) = t_j$ for some j. Then by the conditions (C1) and (C2) x is continuous at t and left-continuous at t_j, and α is left-continuous at t. So we have the left hand derivative of x at t. As in (4.2) we can show that

$$x'(t) \equiv x'_-(t) = -Ax(t) + F(t, x(t)) + G(t, x(\alpha(t))).$$

Finally assume that $t = t_k$ for some k. Then we can easily verify that

$$\Delta x(t_k) = I_k(x(t_k)).$$

The proof is complete. □

From now on we let $m = \phi$ (see (2.1)). Then we have the following result.

Lemma 4.4. *If a function x is a solution of*

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)} F(s, x(s))\,\mathrm{d}m(s) + \int_0^t e^{-A(t-s)} G(s, x(\alpha(s)))\,\mathrm{d}s, \tag{4.3}$$

then the function x is a solution of the equation (4.1).

Proof. By Lemma 2.6 and the condition (C4), we have

$$e^{-At}x_0 + \int_0^t e^{-A(t-s)} F(s, x(s))\,\mathrm{d}s + \sum_{0<t_k<t} e^{-A(t-t_k)} I_k(x(t_k))$$
$$+ \int_0^t e^{-A(t-s)} G(s, x(\alpha(s)))\,\mathrm{d}s$$
$$= e^{-At}x_0 + \int_0^t e^{-A(t-s)} F(s, x(s))\,\mathrm{d}s + \sum_{0<t_k<t} e^{-A(t-t_k)} F(t_k, x(t_k))$$
$$+ \int_0^t e^{-A(t-s)} G(s, x(\alpha(s)))\,\mathrm{d}s$$

$$= e^{-At}x_0 + \int_0^t e^{-A(t-s)}F(s, x(s))\,\mathrm{d}m(s) + \int_0^t e^{-A(t-s)}G(s, x(\alpha(s)))\,\mathrm{d}s.$$

By the above equality and Lemma 4.3 we see that the lemma is true. □

From now on we let for $k \in \mathbf{N}$

$$b(s) = \begin{cases} \frac{j\circ\alpha^{-1}(s)}{\alpha'\circ\alpha^{-1}(s)}, & s \in [0, \alpha(t_1)) \cup \left(\cup_{k=1}^{\infty} (\alpha(t_k+), \alpha(t_{k+1})) \right) \\ 0, & s \in \cup_{k=1}^{\infty}[\alpha(t_k), \alpha(t_k+)]. \end{cases}$$

Then we have the following result.

Lemma 4.5. *Assume that the function $b \in G(\mathbf{R}^+)$. Then for every*

$$t \in [t_k, t_{k+1}), k \in \mathbf{N} \cup \{0\},$$

we have

$$\int_0^t j(s)w(|x(\alpha(s))|)\,\mathrm{d}s \le \int_0^{\alpha(t)} b(s)w(|x(s)|)\,\mathrm{d}s.$$

Proof. We have, for every $t \in [t_k, t_{k+1}), k \in \mathbf{N} \cup \{0\}$,

$$\int_0^t j(s)w(|x(\alpha(s))|)\,\mathrm{d}s = \int_0^{t_1} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s$$
$$+ \sum_{i=2}^{k} \int_{t_{i-1}}^{t_i} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s + \int_{t_k}^{t} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s.$$

Here by change of variables we get

$$\int_0^{t_1} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s = \int_0^{\alpha(t_1)} j \circ \alpha^{-1}(v)w(|x(v)|)\frac{1}{\alpha' \circ \alpha^{-1}(v)}\,\mathrm{d}v$$
$$= \int_0^{\alpha(t_1)} b(v)w(|x(v)|)\,\mathrm{d}v,$$

and

$$\int_{t_{i-1}}^{t_i} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s = \lim_{\eta\to 0+} \int_{t_{i-1}+\eta}^{t_i} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s$$

$$= \lim_{\eta\to 0+} \int_{\alpha(t_{i-1}+\eta)}^{\alpha(t_i)} j\circ\alpha^{-1}(v)w(|x(v)|)\frac{1}{\alpha'\circ\alpha^{-1}(v)}\,\mathrm{d}v \le \int_{\alpha(t_{i-1})}^{\alpha(t_i)} b(v)w(|x(v)|)\,\mathrm{d}v,$$

and similarly we have

$$\int_{t_k}^{t} j(s)w(|x(\alpha(s))|)\,\mathrm{d}s \le \int_{\alpha(t_k)}^{\alpha(t)} b(v)w(|x(v)|)\,\mathrm{d}v.$$

So we get

$$\int_0^t j(s)w(|x(\alpha(s))|)\,\mathrm{d}s \le \int_0^{\alpha(t_1)} b(s)w(|x(s)|)\,\mathrm{d}s$$

$$+\sum_{i=2}^{k} \int_{\alpha(t_{i-1})}^{\alpha(t_i)} b(s)w(|x(s)|)\,\mathrm{d}s + \int_{\alpha(t_k)}^{\alpha(t)} b(s)w(|x(s)|)\,\mathrm{d}s$$

$$= \int_0^{\alpha(t)} b(s)w(|x(s)|)\,\mathrm{d}s.$$

The proof is complete. □

Theorem 4.6. *If x satisfies the equation* (4.3), *then we have for some $T \ge 0$*

$$|x(t)| \le E^{-1}[\gamma(t)], \tag{4.4}$$

where

$$\gamma(t) = \int_0^t \left(A[a+(b\circ\alpha)\,\alpha_m'] + B[a+M_b\,\alpha_m']\right)\mathrm{d}m + E\left(|x_0| + \int_0^t c(s)\,\mathrm{d}s\right),$$

where the number T is chosen so that $\gamma(t)\in \mathrm{Dom}(E^{-1})$ for all $t\in[0,T]$.

Proof. Since $c(t_k)=0, k\in\mathbf{N}$, by Lemma 2.6, we have

$$\int_0^t c(s)\,\mathrm{d}m(s) = \int_0^t c(s)\,\mathrm{d}s + \sum_{0<t_k<t} c(t_k) = \int_0^t c(s)\,\mathrm{d}s.$$

So by (4.3) and Lemma 4.5 we have

$$|x(t)| \le e^{-At}|x_0| + \int_0^t e^{-A(t-s)}|F(s,x(s))|\,\mathrm{d}m(s) + \int_0^t e^{-A(t-s)}|G(s,x(\alpha(s)))|\,\mathrm{d}s$$

$$\le e^{-At}|x_0| + \int_0^t e^{-A(t-s)}[a(s)w(|x(s)|)+c(s)]\,\mathrm{d}m(s) \tag{4.5}$$

$$+ \int_0^t e^{-A(t-s)}j(s)w(|x(\alpha(s))|)\,\mathrm{d}s$$

$$\le |x_0| + \int_0^t [a(s)w(|x(s)|)+c(s)]\,\mathrm{d}m(s) + \int_0^t j(s)w(|x(\alpha(s))|)\,\mathrm{d}s$$

$$\le |x_0| + \int_0^t c(s)\,\mathrm{d}s + \int_0^t a(s)w(|x(s)|)\,\mathrm{d}m(s) + \int_0^{\alpha(t)} b(s)w(|x(s)|)\,\mathrm{d}s.$$

Thus by Theorem 3.4 we obtain (4.4). The proof is complete. □

Theorem 4.7. *If, in Theorem 4.6, we have $\gamma(t) \in \mathrm{Dom}(E^{-1})$ for all $t \in \mathbf{R}^+$ and for some $M \ge 0, \gamma(t) \le M$, and*

$$\int_0^\infty e^{As}c(s)\,\mathrm{d}s + \int_0^\infty e^{As}a(s)\,\mathrm{d}m(s) + \int_0^\infty e^{As}j(s)\,\mathrm{d}s < \infty,$$

then we have

$$\lim_{t\to\infty} x(t) = 0. \tag{4.6}$$

Proof. By Theorem 4.6, $|x(t)| \le E^{-1}(M) \equiv K$ for all $t \in \mathbf{R}^+$. So by (4.5) we have

$$|x(t)| \le e^{-At}\left[|x_0| + \int_0^\infty e^{As}c(s)\,\mathrm{d}s + w(K)\int_0^\infty e^{As}a(s)\,\mathrm{d}m(s) + w(K)\int_0^\infty e^{As}j(s)\,\mathrm{d}s\right].$$

This gives (4.6). □

Theorem 4.8. *If $\gamma(t) \in \mathrm{Dom}(E^{-1})$ for all $t \in \mathbf{R}^+$, then the equation (4.1) has a solution in $G(\mathbf{R}^+)$.*

Proof. We define semi-norms for every positive integer n as follows:

$$|x|_n = \sup_{s\in[0,n]} |x(s)|.$$

We define an operator $T : G(\mathbf{R}^+) \longrightarrow G(\mathbf{R}^+)$ as, for every $t \in \mathbf{R}^+$,

$$Tx(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)}F(s, x(s))\,\mathrm{d}m(s) + \int_0^t e^{-A(t-s)}G(s, x(\alpha(s)))\,\mathrm{d}s.$$

Then, by Lemma 4.4, if x satisfies $x = Tx$, then x is a solution of the equation (4.1).

First, we will show that T is completely continuous on the semi-normed space $(G(\mathbf{R}^+), |\cdot|_n)$.

Assume that $x_k \to x$ in $G(\mathbf{R}^+)$. Then for every positive integer n we have

$$\lim_{k\to\infty} |x_k - x|_n = 0,$$

and this implies that there is a nonnegative number M_n such that $|x_k|_n, |x|_n \le M_n$.

By our assumption, for every $s \in [0, n]$, we get

$$\begin{aligned} |F(s, x_k(s)) - F(s, x(s))| &\le |F(s, x_k(s))| + |F(s, x(s))| \\ &\le a(s)w(M_n) + c(s) + a(s)w(M_n) + c(s) = 2[a(s)w(M_n) + c(s)], \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &|G(s, x_k(\alpha(s))) - G(s, x(\alpha(s)))| \\ &\le |G(s, x_k(\alpha(s)))| + |G(s, x(\alpha(s)))| \\ &\le j(s)w(|x_k(\alpha(s))|) + j(s)w(|x_k(\alpha(s))|) \\ &\le j(s)w(M_n) + j(s)w(M_n) = 2j(s)w(M_n). \end{aligned} \tag{4.8}$$

So by [10, 1.32 Corollary] we have

$$\begin{aligned} |Tx_k - Tx|_n \le& \int_0^n |F(s, x_k(s)) - F(s, x(s))|\,\mathrm{d}m(s) \\ &+ \int_0^n |G(s, x_k(\alpha(s))) - G(s, x(\alpha(s)))|\,\mathrm{d}s \to 0, \text{as} \quad k \to \infty. \end{aligned}$$

This implies that T is continuous.

Let $\mathcal{M}$ be a bounded subset in $G(\mathbf{R}^+)$. Then for every $n \in \mathbf{N}$ there is a nonnegative number M_n such that $|x|_n \le M_n$ for all $x \in \mathcal{M}$. Then for every $x \in \mathcal{M}$. and for every $t \in [0,n]$, we have

$$|Tx(t)| \le |x_0| + \int_0^n [a(s)w(M_n) + c(s)]\, \mathrm{d}m(s) + \int_0^n j(s)w(M_n)\, \mathrm{d}s.$$

This implies that $\{Tx : x \in \mathcal{M}\}$ is equi-bounded on $[0,n]$.

Let $t_0 \in [0,n)$ and assume that $t_j, t_k \to t_0+$ $(t_j < t_k)$ as $j,k \to \infty$. Then using Theorem 2.4 and the method in the proof of [3, Theorem 5.3.] we see that, uniformly for all $x \in \mathcal{M}$,

$$|Tx(t_j) - Tx(t_k)| \longrightarrow 0+ \quad \text{as} \quad j,k \to \infty.$$

Now let $t_j, t_k \to t_0-$ as $j,k \to \infty$. Then similarly we can show that

$$\lim_{j,k\to\infty} |Tx(t_j) - Tx(t_k)| = 0,$$

uniformly for all $x \in \mathcal{M}$. So the set $\{Tx : x \in \mathcal{M}\}$ is equi-regulated on $[0,n]$.

Thus, by Theorem 4.2, we have shown that T is completely continuous for the semi-norm $|\cdot|_n$. This implies that T is completely continuous on the semi-normed space $G(\mathbf{R}^+)$.

Finally we show that the set

$$\Lambda = \{x \in G(\mathbf{R}^+) : x = \lambda\, Tx \quad \text{for some} \quad \lambda \in (0,1)\}$$

is bounded. Here every $x \in \Lambda$ has to satisfy $|x(t)| = |Tx(t)|$, $\forall t \in [0,n]$. Then by Theorem 4.6 we have $|x|_n \le E^{-1}[\gamma(n)]$.

Thus we conclude that Λ is a bounded set in the semi-normed space $(G(\mathbf{R}^+), |\cdot|_n)$. Thus the operator T satisfies all conditions in Theorem 4.1. So there is an $x \in G(\mathbf{R}^+)$ such that $x = Tx$. This completes the proof. □

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