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# ADDITIVE $\rho$-FUNCTIONAL INEQUALITIES

Sung Jin Lee [a], Jung Rye Lee [b,*] and Jeong Pil Seo [c]

Abstract. In this paper, we solve the additive $\rho$-functional inequalities

$$(0.1)\quad \|f(x+y)+f(x-y)-2f(x)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right)\right\|,$$

where $\rho$ is a fixed complex number with $|\rho|<1$, and

$$(0.2)\quad \left\|2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right\| \le \|\rho(f(x+y)+f(x-y)-2f(x))\|,$$

where $\rho$ is a fixed complex number with $|\rho|<1$.

Furthermore, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

## 1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [11] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The stability of quadratic functional equation was proved by Skof [10] for mappings $f: E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. See [2, 4, 7, 9, 12] for more information on the stability problems of functional equations.

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*Corresponding author.

In Section 2, we solve the additive $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the additive $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let $G$ be a 2-divisible abelian group. Assume that $X$ is a real or complex normed space with norm $\|\cdot\|$ and that $Y$ is a complex Banach space with norm $\|\cdot\|$.

## 2. Additive $\rho$-functional Inequality (0.1)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive $\rho$-functional inequality (0.1) in complex Banach spaces.

**Lemma 2.1.** *If a mapping $f : G \to Y$ satisfies $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \tag{2.1}$$

*for all $x, y \in G$, then $f : G \to Y$ is additive.*

*Proof.* Assume that $f : G \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so $f(2x) = 2f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \\ &= |\rho|\|f(x+y) + f(x-y) - 2f(x)\| \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in G$. It is easy to show that $f$ is additive. □

We prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (2.1) in complex Banach spaces.

**Theorem 2.2.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} (2.3) \qquad & \|f(x+y) + f(x-y) - 2f(x)\| \\ & \leq \left\| \rho \left( 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \right) \right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned}$$

*for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that*

$$(2.4) \qquad \|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ in (2.3), we get

$$(2.5) \qquad \|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| & \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ (2.6) \qquad & \leq \frac{2}{2^r} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \theta\|x\|^r \end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} & \|h(x+y) + h(x-y) - 2h(x)\| \\ & = \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) \right\| \\ & \leq \lim_{n\to\infty} 2^n|\rho| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) \right\| + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ & = |\rho| \left\| 2h\left(\frac{x+y}{2}\right) + h(x-y) - 2h(x)) \right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y)+h(x-y)-2h(x)\| \le \left\|\rho\left(2h\left(\frac{x+y}{2}\right)+h\left(x-y\right)-2h(x)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned}\|h(x)-T(x)\| &= 2^n\left\|h\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\| \\ &\le 2^n\left(\left\|h\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|+\left\|T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\le \frac{4\cdot 2^n}{(2^r-2)2^{nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.4). □

**Theorem 2.3.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (2.3). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x)-h(x)\| \le \frac{2\theta}{2-2^r}\|x\|^r \tag{2.7}$$

*for all $x \in X$.*

*Proof.* It follows from (2.5) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x)-\frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

**Remark 2.4.** If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## 3. Additive $\rho$-functional Inequality (0.2)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive $\rho$-functional inequality (0.2) in complex Banach spaces.

**Lemma 3.1.** *If a mapping $f : G \to Y$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \right\| \le \|\rho(f(x+y) + f(x-y) - 2f(x))\| \tag{3.1}$$

*for all $x, y \in G$, then $f : G \to Y$ is additive.*

*Proof.* Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le 0$ and so

$$2f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &= \left\| 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \right\| \\ &\le |\rho| \|f(x+y) + f(x-y) - 2f(x)\| \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in G$. . It is easy to show that $f$ is additive. $\square$

We prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (3.1) in complex Banach spaces.

**Theorem 3.2.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\| \\ &\qquad \le \|\rho(f(x+y) + f(x-y) - 2f(x))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.3}$$

*for all* $x, y \in X$*. Then there exists a unique additive mapping* $h : X \to Y$ *such that*

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{2^r - 2}\|x\|^r \tag{3.4}$$

*for all* $x \in X$*.*

*Proof.* Letting $x = y = 0$ in (3.3), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{3.6}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

It follows from (3.3) that

$$\begin{aligned}&\left\|2h\left(\frac{x+y}{2}\right) + h\left(x-y\right) - 2h(x)\right\| \\ &= \lim_{n\to\infty} 2^n\left\|2f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^n\left\|\rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right)\right\| + \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ &= \|\rho(h(x+y) + h(x-y) - 2h(x))\|\end{aligned}$$

for all $x, y \in X$. So

$$\left\|2h\left(\frac{x+y}{2}\right) + h\left(x-y\right) - 2h(x)\right\| \le \|\rho(h(x+y) + h(x-y) - 2h(x))\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.4). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\ &\le 2^n \left( \left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\ &\le \frac{2 \cdot 2^n \cdot 2^r}{(2^r - 2)2^{nr}} \theta \|x\|^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.4). □

**Theorem 3.3.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3)*. Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2^r \theta}{2 - 2^r} \|x\|^r \tag{3.7}$$

*for all $x \in X$.*

*Proof.* It follows from (3.5) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{2^r \theta}{2} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \\ &\le \frac{2^r \theta}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|^r \end{aligned} \tag{3.8}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 3.2. □

**Remark 3.4.** If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

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[a]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
*Email address*: hyper@daejin.ac.kr

[b]Department of Mathematics, Daejin University, Kyeonggi 11159, Korea
*Email address*: jrlee@daejin.ac.kr

[c]Ohsang High School, Gumi, Kyeongsangbuk-do 730-842, Korea
*Email address*: sjp4829@hanmail.net