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BOUNDEDNESS IN THE NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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Abstract. This paper shows that the solutions to the nonlinear perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s),T_1y(s))ds + h(t,y(t),T_2y(t)),$$

have the bounded property by imposing conditions on the perturbed part

$$\int_{t_0}^{t} g(s,y(s),T_1y(s))ds, h(t,y(t),T_2y(t)),$$

and on the fundamental matrix of the unperturbed system $y' = f(t,y)$ using the notion of h-stability.

1. Introduction and Preliminaries

We are interested in the relations between the solutions of the unperturbed nonlinear nonautonomous differential system

$$x'(t) = f(t,x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

and the solutions of the perturbed differential system of (1.1) including two operators T_1, T_2 such that

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s),T_1y(s))ds + h(t,y(t),T_2y(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0,\infty)$, $f(t,0) = 0$, $g(t,0,0) = h(t,0,0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are a continuous operator and $\mathbb{R}^n$ is an n-dimensional Euclidean space. We always assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$.

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Let $x(t,t_0,x_0)$ denote the unique solution of (1.1) with $x(t_0,t_0,x_0)=x_0$, existing on $[t_0,\infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t)=f_x(t,0)v(t),\ v(t_0)=v_0 \tag{1.3}$$

and

$$z'(t)=f_x(t,x(t,t_0,x_0))z(t),\ z(t_0)=z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t,t_0,x_0)$ of (1.4) is given by

$$\Phi(t,t_0,x_0)=\frac{\partial}{\partial x_0}x(t,t_0,x_0),$$

and $\Phi(t,t_0,0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [16].

Definition 1.1. The system (1.1) (the zero solution $x=0$ of (1.1)) is called an *h-system* if there exist a constant $c\geq 1$ and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)|\leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t\geq t_0\geq 0$ and $|x_0|$ small enough (here $h(t)^{-1}=\frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x=0$ of (1.1)) is called (hS)*h-stable* if there exists $\delta>0$ such that (1.1) is an h-system for $|x_0|\leq\delta$ and h is bounded.

Pachpatte[14, 15] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T. The purpose of this paper is to investigate bounds for solutions of the nonlinear differential systems

The notion of h-stability (hS) was introduced by Pinto [16,17] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [5] and Choi, Koo, and Ryu [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10] and Goo et al. [3,4] studied the boundedness of solutions for the perturbed differential systems.

Let $\mathcal{M}$ denote the set of all $n\times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1

with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [7].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [7, 12].

We give some related properties that we need in the sequal.

Lemma 1.4 ([17])**.** *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5 ([2])**.** *Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.6 ([5]). *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

Theorem 1.7 ([6]). *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

Lemma 1.8. (*Bihari − type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s)w(u(s))ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s)ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 1.9 ([11]). *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau)u(\tau)$$
$$+\lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t} \lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau$$
$$+\lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)d\tau\Big)ds\Big],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau)$$
$$+\lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau + \lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

For the proof we prepare the following lemma.

Corollary 1.10. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)w(u(r))dr\Big)d\tau ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.11 ([3]). *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \;\; 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds \in \mathrm{domW^{-1}}\Big\}.$$

2. Main Results

In this section, we investigate boundedness for solutions of perturbed functional differential systems using the notion of t_∞-similarity.

We need the lemma to prove the following theorem.

Lemma 2.1. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) \\ &+ \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)u(r)dr)d\tau ds + \int_{t_0}^t \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)w(u(\tau))d\tau ds. \end{aligned} \tag{2.1}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) &+ \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr)d\tau \\ &+ \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)d\tau\Big)ds\Big], \end{aligned} \tag{2.2}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \\ &+ \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Define a function $v(t)$ by the right member of (2.1) and let us differentiate $v(t)$ to obtain

$$\begin{aligned} v'(t) = \lambda_1(t)u(t) &+ \lambda_2(t)\int_{t_0}^t \Big(\lambda_3(s)u(s) + \lambda_4(s)w(u(s)) \\ &+ \lambda_5(s)\int_{t_0}^s \lambda_6(\tau)u(\tau)d\tau\Big)ds + \lambda_7(t)\int_{t_0}^t \lambda_8(s)w(u(s))ds. \end{aligned}$$

This reduces to

$$\begin{aligned} v'(t) \le \Big(\lambda_1(t) &+ \lambda_2(t)\int_{t_0}^t (\lambda_3(s) + \lambda_4(s) + \lambda_5(s)\int_{t_0}^s \lambda_6(\tau)d\tau)ds \\ &+ \lambda_7(t)\int_{t_0}^t \lambda_8(s)ds\Big)w(v(t)), \end{aligned}$$

$t \ge t_0$, since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$\begin{aligned} v(t) \le c &+ \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr)d\tau \\ &+ \lambda_7(s)\int_{t_0}^s \lambda_8(\tau)d\tau\Big)w(v(s))ds. \end{aligned} \tag{2.3}$$

By view of Lemma 1.8, (2.3) yields the estimate (2.2). □

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(H3) $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

Theorem 2.2. *Let $a, b, c, k, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

(2.4)
$$|g(t,y,T_1y)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, |T_1y(t)| \le b(t)\int_{t_0}^t k(s)|y(s)|ds$$

and

$$|h(t,y(t),T_2y(t))| \le c(t)\Big(|y(t)| + |T_2y(t)|\Big), |T_2y(t)| \le \int_{t_0}^t q(s)w(|y(s)|)ds, \tag{2.5}$$

where $a, b, c, k, q, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are a continuous operator. Then, any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on $[t_0,\infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(c(s) + \int_{t_0}^s (a(\tau) + b(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s)\int_{t_0}^\tau q(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t \Big(c(s) + \int_{t_0}^s (a(\tau) + b(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s)\int_{t_0}^\tau q(\tau)d\tau\Big)ds \in \mathrm{dom}W^{-1}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, from (H1), by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula Lemmma 1.5, together with (2.4) and (2.5), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))|\Big(\int_{t_0}^s |g(\tau,y(\tau),T_1y(s))|d\tau + |h(s,y(s),T_2y(s))|\Big)ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1}\Big(\int_{t_0}^s (a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|) \\ &\quad +b(\tau)\int_{t_0}^\tau k(r)|y(r)|dr)d\tau + c(s)(|y(s)| + \int_{t_0}^s q(\tau)w(|y(\tau)|)d\tau)\Big)ds.\end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)\Big(c(s)\frac{|y(s)|}{h(s)} \\ &\quad + \int_{t_0}^s \Big(a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^\tau k(r)\frac{|y(r)|}{h(r)}dr)d\tau \\ &\quad +c(s)\int_{t_0}^s q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.\end{aligned}$$

Define $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.1, we have

$$\begin{aligned}|y(t)| &\le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(c(s) + \int_{t_0}^s (a(\tau) + b(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau \\ &\quad +c(s)\int_{t_0}^\tau q(\tau)d\tau\Big)ds\Big],\end{aligned}$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 2.3. Letting $c(t) = 0$ in Theorem 2.2, we obtain the same result as that of Theorem 3.1 in [10].

Theorem 2.4. *Let $a, b, c, d, k, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^t |g(s,y(s),T_1y(s))|ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, |T_1y(t)| \le b(t)\int_{t_0}^t k(s)w(|y(s)|)ds \tag{2.6}$$

and

$$|h(t,y(t),T_2y(t))| \le \Big(c(t)w(|y(t)|) + |T_2y(t)|\Big), |T_2y(t)| \le d(t)\int_{t_0}^t q(s)|y(s)|ds \tag{2.7}$$

where $a,b,c,d,k,q,w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 *are a continuous operator. Then, any solution* $y(t) = y(t,t_0,y_0)$ *of (1.2) is bounded on* $[t_0,\infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c)+c_2\int_{t_0}^{t}\Big(a(s)+b(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau+d(s)\int_{t_0}^{s}q(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}\Big(a(s)+b(s)+c(s)$$
$$+b(s)\int_{t_0}^{s}k(\tau)d\tau + d(s)\int_{t_0}^{s}q(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula Lemma 1.5, together with (2.6) and (2.7), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}\Big(a(s)|y(s)| + (b(s)+c(s))w(|y(s)|)$$
$$+b(s)\int_{t_0}^{s}k(\tau)w(|y(\tau)|)d\tau + d(s)\int_{t_0}^{s}q(\tau)|y(\tau)|d\tau\Big)ds.$$

It follows from (H2) and (H3) that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)\Big(a(s)\frac{|y(s)|}{h(s)} + (b(s)+c(s))w(\frac{|y(s)|}{h(s)})$$
$$+b(s)\int_{t_0}^{s}k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + d(s)\int_{t_0}^{s}q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.11, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\big(a(s)+b(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau$$
$$+d(s)\int_{t_0}^{s}q(\tau)d\tau\Big)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on $[t_0,\infty)$, and so the proof is complete. □

Remark 2.5. Letting $c(t) = d(t) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.7 in [10].

Theorem 2.6. *Let $a, b, c, d, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

(2.8)

$$|g(t,y,T_1y)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, |T_1y(t)| \le b(t)\int_{t_0}^t k(s)w(|y(s)|)ds$$

and

$$|h(t,y(t),T_2y(t))| \le \Big(\int_{t_0}^t c(s)w(|y(s)|)ds + |T_2y(t)|\Big), |T_2y(t)| \le d(t)|y(t)| \tag{2.9}$$

where $a, b, c, d, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(d(s) + \int_{t_0}^s (a(\tau) + b(\tau) + c(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau\Big)ds\Big]$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t \Big(d(s) + \int_{t_0}^s (a(\tau) + b(\tau) + c(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5, together with (2.8) and (2.9), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1}\Big(\int_{t_0}^s (a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|) + b(\tau)\int_{t_0}^\tau k(r)w(|y(r)|)dr)d\tau + \int_{t_0}^s c(\tau)w(|y(\tau)|)d\tau + d(s)|y(s)|\Big)ds.$$

Using the assumptions (H2) and (H3), we obtain

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)\Big(d(s)\frac{|y(s)|}{h(s)} + \int_{t_0}^s (a(\tau)\frac{|y(\tau)|}{h(\tau)} + (b(\tau)+c(\tau))w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^\tau k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau\Big)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Corollary 1.10 that we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(d(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau\Big)ds\Big],$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 2.7. Letting $c(t) = d(t) = 0$ in Theorem 2.6, we obtain the same result as that of Theorem 3.5 in [10].

Theorem 2.8. *Let $a, b, c, k, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1y(s))|ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, |T_1y(t)| \le b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{2.10}$$

and

$$|h(t, y(t), T_2y(t))| \le c(t)\Big(|y(t)|) + |T_2y(t)|\Big), |T_2y(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds \tag{2.11}$$

where $a, b, c, k, q, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are a continuous operator. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0,\infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(a(s) + b(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} \Big(a(s) + b(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula Lemma 1.5, together with (2.10) and (2.11), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big((a(s)+c(s))|y(s)| + b(s)w(|y(s)|)$$
$$+b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + c(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds.$$

Using (H2) and (H3), we obtain

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big((a(s)+c(s))\frac{|y(s)|}{h(s)} + b(s)w(\frac{|y(s)|}{h(s)})$$
$$+b(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Put $u(t) = |y(t)||h(t)|^{-1}$.Then, an application of Lemma 1.11 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s)+b(s)+c(s)+b(s)\int_{t_0}^{s} k(\tau)d\tau$$
$$+c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.9. Letting $c(t) = 0$ in Theorem 2.8, we obtain the same result as that of Theorem 3.7 in [10].

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