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ON EVALUATIONS OF THE CUBIC CONTINUED FRACTION BY A MODULAR EQUATION OF DEGREE 9

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Abstract. We show how to evaluate the cubic continued fraction $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 4^m$, 4^{-m}, $2 \cdot 4^m$, and $2^{-1} \cdot 4^{-m}$ for some nonnegative integer m by using modular equations of degree 9. We then find some explicit values of them.

1. Introduction

The cubic continued fraction $G(q)$ is defined by

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1.$$

Ramanujan investigated $G(q)$ and claimed that there are many results which are similar to those for the Rogers-Ramanujan continued fraction $F(q)$, where

$$F(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots, \quad |q| < 1.$$

Motivated by Ramanujan's claim, there has been interest by number theorists in evaluating the numerical values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for some positive real number n. Using Ramanujan's class invariants, Berndt, Chan, and Zhang [3] evaluated the values of $G(e^{-\pi\sqrt{n}})$ for $n = 2, 10, 22, 58$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5, 13, 37$. In addition, Chan [4] established the values of $G(e^{-\pi\sqrt{n}})$ for $n = 1, 2, 4, \frac{2}{9}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5$ by using some reciprocity theorems for $G(q)$. Yi [6] found the values of $G(e^{-\pi\sqrt{n}})$ for $n = 2, 3, 4, 6, 7, 8, 10, 12, 16, 28, \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 2, 3, 4, 5, 7, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}$ by employing relations among $G(q)$, Ramanujan-Weber class invariants, and some

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parameters for eta function. In [8], the values of $G(e^{-\pi\sqrt{n}})$ for $n = 1, 4, 9, \frac{1}{3}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 4, 9$ were evaluated by using some modular equations of degrees 3 and 9. Moreover, Paek and Yi [5] evaluated the values of $G(e^{-\pi\sqrt{n}})$ for $n = 36, 81, 144, 324, \frac{4}{3}, \frac{16}{3}, \frac{64}{3}$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 36, \frac{4}{3}, \frac{16}{3}$ by employing some modular equations of degrees 3 and 9.

In this paper, we further show how to evaluate explicit values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 4^m, 4^{-m}, 2\cdot 4^m$, and $2^{-1}\cdot 4^{-m}$ for some nonnegative integer m by using modular equations of degree 9.

Since a modular equation is crucial for evaluating such cubic continued fraction, we now give a definition of a modular equation. For $|ab| < 1$, define Ramanujan's general theta function f by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Moreover, define the theta functions φ and ψ by, for $|q| < 1$,

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty$$

and

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

Let a, b, and c be arbitrary complex numbers except that c cannot be a nonpositive integer. Then, for $|z| < 1$, the Gaussian or ordinary hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\dots(a+n-1)$ for each positive integer n.

Now the complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;k^2\right) = \frac{\pi}{2}\varphi^2\left(e^{-\pi\frac{K'}{K}}\right), \tag{1.1}$$

where $0 < k < 1$, $K' = K(k')$, and $k' = \sqrt{1-k^2}$. The number k is called the modulus of K and k' is called the complementary modulus.

Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0 < k < 1$ and $0 < l < 1$. Suppose that

$$\frac{L'}{L} = n\frac{K'}{K} \tag{1.2}$$

holds for some positive integer n. A relation between k and l induced by (1.2) is called a modular equation of degree n.

If we set

$$q = \exp\left(-\pi\frac{K'}{K}\right) \quad \text{and} \quad q' = \exp\left(-\pi\frac{L'}{L}\right),$$

we see that (1.2) is equivalent to the relation $q^n = q'$. Hence a modular equation can be viewed as an identity involving theta functions at the arguments q and q^n. Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$, then we say that β has degree n over α. By the relationship between complete elliptic integrals of the first kind and hypergeometric function, we have

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}.$$

Let $z_n = \varphi^2(q^n)$. Then the multiplier m for degree n is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{z_1}{z_n}.$$

We recall the definition of the parameterizations $h'_{k,n}$ and $l'_{k,n}$ for the theta functions φ and ψ from [7, 9]. For any positive real numbers k and n, define $h'_{k,n}$ by

$$h'_{k,n} = \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)}, \tag{1.3}$$

where $q = e^{-2\pi\sqrt{n/k}}$, and define $l'_{k,n}$ by

$$l'_{k,n} = \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \tag{1.4}$$

where $q = e^{-\pi\sqrt{n/k}}$. For convenience, we write h'_n and l'_n instead of $h'_{9,n}$ and $l'_{9,n}$, respectively, throughout this paper. We end this section by noting that

$$h'_1 = \frac{1+\sqrt{3}-\sqrt{2}\,\sqrt[4]{3}}{2} \tag{1.5}$$

and

$$l'_2 = \sqrt{2}+\sqrt{3} \tag{1.6}$$

from [8, Theorem 4.1] and [9, Theorem 3.4], which will play crucial roles in evaluating the values of cubic continued fraction.

2. Preliminary Results

In this section, we introduce fundamental theta function identities that will play key roles in deriving a modular equation of degree 9. Let k be the modulus as in (1.1). Set $x = k^2$ and also set

$$k^2 = x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{2.1}$$

Then

$$\varphi^2(q) = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right) =: z, \tag{2.2}$$

where

$$q = e^{-y} := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right) = \exp\left(-\pi \frac{K(k')}{K(k)}\right). \tag{2.3}$$

Lemma 2.1 ([1, Theorems 5.4.1 and 5.4.2]). *If x, q, and z are related by (2.1), (2.2), and (2.3), then*

(i) $\varphi(-q) = \sqrt{z}(1-x)^{1/4}$,
(ii) $\varphi(-q^2) = \sqrt{z}(1-x)^{1/8}$,
(iii) $\psi(q) = \sqrt{\frac{1}{2}z}\left(\frac{x}{q}\right)^{1/8}$.

Lemma 2.2 ([2, Entry 3, Chapter 20]). *Let γ be the ninth degree and $m = \dfrac{z_1}{z_9}$, then*

(i) $\left(\dfrac{\gamma}{\alpha}\right)^{1/8} + \left(\dfrac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\dfrac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}\,,$
(ii) $\left(\dfrac{\alpha}{\gamma}\right)^{1/8} + \left(\dfrac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\dfrac{\alpha\,(1-\alpha)}{\gamma\,(1-\gamma)}\right)^{1/8} = \dfrac{3}{\sqrt{m}}.$

The following results exhibit formulas for evaluating $G(e^{-2\pi\sqrt{n}})$ in terms of h'_n and $G(e^{-\pi\sqrt{n}})$ in terms of l'_n.

Lemma 2.3 ([9, Theorem 6.2]). *For any positive real number n, we have*

(i) $G(e^{-2\pi\sqrt{n}}) = \dfrac{1-\sqrt{3}\,h'_n}{2}$,
(ii) $G(e^{-\pi\sqrt{n}}) = \dfrac{1}{\sqrt{3}\,l'_n - 1}$.

We end this section by stating the following identity:

Lemma 2.4 ([6, Lemma 6.3.6]). *We have*

$$G(e^{-2\pi\sqrt{n}}) = -G(e^{-\pi\sqrt{n}})G(-e^{-\pi\sqrt{n}})$$

for any positive real number n.

3. Modular Equations

In this section, we first derive a modular equation of degree 9 to establish some explicit relation for h'_n and $h'_{n/4}$ for any positive real number n.

Theorem 3.1. *If $P = \dfrac{\varphi(-q)}{\varphi(-q^9)}$ and $Q = \dfrac{\varphi(-q^2)}{\varphi(-q^{18})}$, then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = Q + \frac{3}{Q}. \tag{3.1}$$

Proof. By Lemma 2.1,

$$P = \sqrt{\frac{z_1}{z_9}} \left(\frac{1-\alpha}{1-\gamma}\right)^{1/4} \quad \text{and} \quad Q = \sqrt{\frac{z_1}{z_9}} \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8},$$

where γ has degree 9 over α. Thus

$$\frac{P}{Q} = \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}.$$

By Lemma 2.2,

$$\left(\frac{\gamma}{\alpha}\right)^{1/8} + \frac{Q}{P} - \left(\frac{\gamma}{\alpha}\right)^{1/8} \frac{Q}{P} = \frac{Q^2}{P}$$

and

$$\left(\frac{\alpha}{\gamma}\right)^{1/8} + \frac{P}{Q} - \left(\frac{\alpha}{\gamma}\right)^{1/8} \frac{P}{Q} = \frac{3P}{Q^2}.$$

Combining the last two identities in terms of P and Q, we deduce that

$$\left(1 - \frac{Q}{P}\right)\left(1 - \frac{P}{Q}\right) = (Q-1)\left(\frac{3}{Q} - 1\right).$$

This then completes the proof. □

Corollary 3.2. *For any positive real number n, we have*

$$\sqrt{3}\left(h'_n + \frac{1}{h'_n}\right) = \frac{h'_{n/4}}{h'_n} + \frac{h'_n}{h'_{n/4}} + 2. \tag{3.2}$$

Proof. Let $q = e^{-\pi\sqrt{n/9}}$ in (1.3). Then P and Q in Theorem 3.1 can be written as $P = \sqrt{3}\, h'_{n/4}$ and $Q = \sqrt{3}\, h'_n$. Rewrite (3.1) in terms of $h'_{n/4}$ and h'_n to complete the proof. □

We next recall a modular equation of degree 9 given in [8] to employ an explicit relation for l'_n and l'_{4n} for any positive real number n.

Theorem 3.3 ([8, Theorem 3.15]). *If* $P = \dfrac{\psi(q)}{q\psi(q^9)}$ *and* $Q = \dfrac{\psi(q^2)}{q^2\psi(q^{18})}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = P + \frac{3}{P}. \tag{3.3}$$

Corollary 3.4 ([8, Corollary 3.16]). *For any positive real number n, we have*

$$\sqrt{3}\left(l'_n + \frac{1}{l'_n}\right) = \frac{l'_n}{l'_{4n}} + \frac{l'_{4n}}{l'_n} + 2. \tag{3.4}$$

4. EVALUATIONS OF h'_{4^m}, $h'_{1/4^m}$, $l'_{2\cdot 4^m}$, AND $l'_{2/4^m}$

In this section, we show how to evaluate the values of h'_{4^m}, $h'_{1/4^m}$, $l'_{4^m/2}$, and $l'_{2/4^m}$ for every positive integer m by employing the relations (3.2) and (3.4). We first need the following:

Lemma 4.1. *For any nonnegative integer m,*

$$0 < \sqrt{3}\, h'_{4^m} < 1. \tag{4.1}$$

Proof. Let $a_m = h'_{4^m}$ for brevity. Since $a_m > 0$ for any nonnegative integer m from the definition of h'_k, we have $\sqrt{3}\, a_m > 0$. Hence it is enough to show that $\sqrt{3}\, a_m < 1$ for any nonnegative integer m. We prove by induction on m. For $m = 0$, since $a_0 = \dfrac{1+\sqrt{3}-\sqrt{2}\,\sqrt[4]{3}}{2}$ from [8, Theorem 4.1], which is approximately equal to 0.44, it follows that $\sqrt{3}\, a_0 < 1$. Now assume that $\sqrt{3}\, a_k < 1$ for some nonnegative integer k. Then, by (3.2),

$$\sqrt{3}\left(a_{k+1} + \frac{1}{a_{k+1}}\right) = \frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_k} + 2.$$

Solving the last equality for a_{k+1} and using the fact that $0 < \sqrt{3}\, a_k < 1$ and $a_{k+1} > 0$, we have

$$a_{k+1} = \frac{-a_k + \sqrt[4]{3}\sqrt{a_k(a_k^2 - \sqrt{3}\, a_k + 1)}}{1 - \sqrt{3}\, a_k}.$$

Since

$$\left(\sqrt{3}\,a_k - 1\right)^3 < 0$$

or equivalently,

$$3\sqrt{3}\,a_k(a_k^2 - \sqrt{3}\,a_k + 1) < 1\,,$$

it follows that

$$-a_k + \sqrt[4]{3}\sqrt{a_k(a_k^2 - \sqrt{3}\,a_k + 1)} < -a_k + \frac{1}{\sqrt{3}} = \frac{1-\sqrt{3}\,a_k}{\sqrt{3}}.$$

Dividing both sides of the last inequality by $\dfrac{1-\sqrt{3}\,a_k}{\sqrt{3}}$, we conclude that

$$\sqrt{3}\,a_{k+1} < 1,$$

which completes the proof. □

The following result exhibits an algorithm for evaluating the values of h'_{4m} for all positive integers m.

Theorem 4.2. *We have*

$$h'_{4m+1} = \frac{-h'_{4m} + \sqrt[4]{3}\sqrt{h'_{4m}(h'^2_{4m} - \sqrt{3}\,h'_{4m} + 1)}}{1-\sqrt{3}\,h'_{4m}}. \tag{4.2}$$

for any nonnegative integer m.

Proof. It is an immediate consequence of Corollary 3.2 and Lemma 4.1. □

We are now ready to show how to evaluate the values of h'_{4m} for every positive integer m. We only exhibit the cases when $m = 1$, 2, and 3.

Corollary 4.3. *We have*

(i) $h'_4 = \dfrac{1}{2}\left(-1 - \sqrt{3+2\sqrt{3}} + \sqrt{2\,(3+2\sqrt{3}+\sqrt{9+6\sqrt{3}}\,)}\right)$,

(ii) $h'_{16} = \dfrac{1-\sqrt{2}\sqrt[4]{3}-\sqrt{3}+\sqrt{(6-2\sqrt{6}\,)(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6}\,)}}{-1+2\sqrt[4]{3}+\sqrt{3}-\sqrt{6}\sqrt[4]{3}}$,

(iii) $h'_{64} = \dfrac{-a+\sqrt[4]{3}\sqrt{a(a^2-\sqrt{3}\,a+1)}}{1-\sqrt{3}\,a}$,

where

$$a = \frac{1-\sqrt{2}\sqrt[4]{3}-\sqrt{3}+\sqrt{(6-2\sqrt{6}\,)(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6}\,)}}{-1+2\sqrt[4]{3}+\sqrt{3}-\sqrt{6}\sqrt[4]{3}}.$$

Proof. For (i), letting $m = 0$ in (4.2) and putting the value of

$$h'_1 = \frac{1+\sqrt{3}-\sqrt{2}\sqrt[4]{3}}{2}$$

from [8, Theorem 4.1], we complete the proof.

For (ii), letting $m = 1$ in (4.2) and putting the value of h'_4 from the previous result of (i), we complete the proof.

Part (iii) is clear from Theorem 4.2. □

The following results show a method for evaluating the values of $h'_{1/4^m}$ for all positive integers m. We only exhibit the cases when $m = 1$, 2, and 3.

Theorem 4.4. *We have*

(i) $h'_{1/4} = 1+\sqrt{3}-\sqrt{3+2\sqrt{3}}$,

(ii) $h'_{1/16} = 5+3\sqrt{3}-\sqrt{45+26\sqrt{3}}-\sqrt{90+52\sqrt{3}-4\sqrt{6(168+97\sqrt{3})}}$,

(iii) $h'_{1/64} = \frac{1}{2}\left(\sqrt{3}\,b^2-2b+\sqrt{3}-\sqrt{(b^2+1)(b-\sqrt{3})(3b-\sqrt{3}\,)}\right)$,

where

$$b = 5+3\sqrt{3}-\sqrt{45+26\sqrt{3}}-\sqrt{90+52\sqrt{3}-4\sqrt{6(168+97\sqrt{3})}}.$$

Proof. For (i), letting $n = 1$ in (3.2), putting the value of

$$h'_1 = \frac{1+\sqrt{3}-\sqrt{2}\sqrt[4]{3}}{2}$$

from [8, Theorem 4.1], solving for $h'_{1/4}$, and using the fact that $h'_{1/4}$ has a positive value less than 1, we complete the proof.

For (ii), letting $n = \frac{1}{4}$ in (3.2), putting the value of $h'_{1/4}$ from the previous result of (i), solving for $h'_{1/16}$, and using the fact that $h'_{1/16}$ has a positive value less than 1, we complete the proof.

For (iii), repeat the same argument as in the proof of (ii). □

We next show how to evaluate the values of $l'_{2\cdot 4^m}$ for every positive integer m. We only exhibit the cases when $m = 1$, 2, and 3.

Theorem 4.5. *We have*

(i) $l'_8 = (2+\sqrt{3}\,)(\sqrt{2}+\sqrt{3}\,)$,

(ii) $l'_{32} = 27+19\sqrt{2}+16\sqrt{3}+11\sqrt{6}+\sqrt{6(485+343\sqrt{2}+280\sqrt{3}+198\sqrt{6}\,)}$,

(iii) $l'_{128} = \frac{1}{2}\left(\sqrt{3}\,c^2-2c+\sqrt{3}+\sqrt{(c^2+1)(c-\sqrt{3}\,)(3c-\sqrt{3}\,)}\right)$, *where*

$$c = 27 + 19\sqrt{2} + 16\sqrt{3} + 11\sqrt{6} + \sqrt{6(485 + 343\sqrt{2} + 280\sqrt{3} + 198\sqrt{6}\,)}\,.$$

Proof. For (i), letting $n = 2$ in (3.4) and putting the value of

$$l'_2 = \sqrt{2} + \sqrt{3}$$

from [9, Theorem 3.4], solving for l'_8, and using the fact that l'_8 has a positive value greater than 1, we complete the proof.

For (ii), letting $n = 8$ in (3.4), putting the value of l'_8 from the previous result of (i), solving for l'_{32}, and using the fact that l'_{32} has a value greater than 1, we complete the proof.

For (iii), repeat the same argument as in the proof of (ii). $\square$

The following results show a method for evaluating the values of $l'_{2/4^m}$ for every positive integer m. We only exhibit the cases when $m = 1$, 2, 3, and 4.

Theorem 4.6. *We have*

(i) $l'_{1/2} = \dfrac{1+\sqrt{3}}{\sqrt{2}}$,

(ii) $l'_{1/8} = \dfrac{\sqrt{2} + \sqrt{6(-1+\sqrt{2}\,)}}{1-\sqrt{3}+\sqrt{6}}$,

(iii) $l'_{1/32} = \dfrac{2-\sqrt{2}+\sqrt{6(-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}\,)}}{(\sqrt{3}-\sqrt{1+\sqrt{2}}\,)\left(3+\sqrt{(1+\sqrt{2}\,)(2-\sqrt{3}\,)}\right)}$,

(iv) $l'_{1/128} = \dfrac{d+\sqrt{\sqrt{3}\,d\,(d^2-\sqrt{3}\,d+1)}}{\sqrt{3}\,d-1}$,

where

$$d = \frac{2-\sqrt{2}+\sqrt{6(-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}\,)}}{(\sqrt{3}-\sqrt{1+\sqrt{2}}\,)\left(3+\sqrt{(1+\sqrt{2}\,)(2-\sqrt{3}\,)}\right)}.$$

Proof. For (i), letting $n = \frac{1}{2}$ in (3.4), putting the value of

$$l'_2 = \sqrt{2} + \sqrt{3}$$

from [7, Theorem 4.16], solving for $l'_{1/2}$, and using the fact that $l'_{1/2}$ has a positive value, we complete the proof.

For (ii), letting $n = \frac{1}{8}$ in (3.4), putting the value of $l'_{1/2}$ from the previous result of (i), solving for $l'_{1/8}$, and using the fact that $l'_{1/8}$ has a positive value, we complete the proof.

For (iii) and (iv), repeat the same argument as in the proof of (ii). □

5. Evaluations of $G(q)$

We turn to evaluations of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 4^m$, 4^{-m}, $2 \cdot 4^m$, and $2^{-1} \cdot 4^{-m}$ for some positive integer m. We first find $G(e^{-2^m\pi})$ for $m = 2$, 3, and 4 and $G(-e^{-2^m\pi})$ for $m = 2$ and 3.

Theorem 5.1. *We have*

$$\text{(i)}\ G(e^{-4\pi}) = \frac{1}{4}\left(2+\sqrt{3}+\sqrt{9+6\sqrt{3}}-\sqrt{6(3+2\sqrt{3}+\sqrt{9+6\sqrt{3}})}\right),$$

$$\text{(ii)}\ G(e^{-8\pi}) = \frac{-2\left(1+\sqrt[4]{3}\right)+\sqrt{3(6-2\sqrt{6})(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6})}}{2\left(1-2\sqrt[4]{3}-\sqrt{3}+\sqrt{6}\sqrt[4]{3}\right)},$$

$$\text{(iii)}\ G(e^{-16\pi}) = \frac{-1+\sqrt[4]{3}\sqrt{3a(a^2-\sqrt{3}\,a+1)}}{2(\sqrt{3}\,a-1)},$$

where

$$a = \frac{1-\sqrt{2}\sqrt[4]{3}-\sqrt{3}+\sqrt{(6-2\sqrt{6})(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6})}}{-1+2\sqrt[4]{3}+\sqrt{3}-\sqrt{6}\sqrt[4]{3}}.$$

Proof. For (i), letting $n = 4$ in Lemma 2.3(i) and putting the value of h'_4 from Corollary 4.3(i), we complete the proof.

For (ii) and (iii), repeat the same argument as in the proof of (i). □

See [6, Theorem 6.3.7(iii)] for an alternative proof for Theorem 5.1(i), where $G(e^{-4\pi})$ was given by

$$G(e^{-4\pi}) = \frac{(\sqrt{2}\,3^{3/4}-1-\sqrt{3})^2}{4(2+3\cdot 3^{1/4}-3\sqrt{2}+3^{3/4})}.$$

Corollary 5.2. *We have*

$$\text{(i)}\ G(-e^{-4\pi}) = \frac{2(1+\sqrt[4]{3})-\sqrt{6(3-\sqrt{6})(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6})}}{7+\sqrt[4]{3}\left(-5+\sqrt[4]{3}-3\sqrt{3}+\sqrt{2}\left(3+2\sqrt{3}-3\sqrt[4]{3}\right)\right)},$$

$$\text{(ii)}\ G(-e^{-8\pi}) = \frac{(1-2\sqrt[4]{3}-\sqrt{3}+\sqrt{6}\sqrt[4]{3})^2\left(\sqrt{3\sqrt{3}a(a^2-\sqrt{3}\,a+1)}-1\right)}{\left(2+2\sqrt[4]{3}-\sqrt{6(3\sqrt{2}-2\sqrt{3})\left(1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}\right)}\right)^2},$$

where

$$a = \frac{1-\sqrt{2}\sqrt[4]{3}-\sqrt{3}+\sqrt{(6-2\sqrt{6}\,)(\sqrt{2}+2\sqrt[4]{3}+\sqrt{6}\,)}}{-1+2\sqrt[4]{3}+\sqrt{3}-\sqrt{6}\sqrt[4]{3}}.$$

Proof. Parts (i) and (ii) follow directly from Lemma 2.4 and Theorem 5.1. □

We next find $G(e^{-\pi/2^m})$ for $m = 0$, 1, and 2 and $G(-e^{-\pi/2^m})$ for $m = 1$ and 2.

Theorem 5.3. *We have*

(i) $G(e^{-\pi}) = \frac{1}{2}\left(-2-\sqrt{3}+\sqrt{9+6\sqrt{3}}\right)$,

(ii) $G(e^{-\pi/2}) = -4+\frac{\sqrt{3}}{2}\left(-5+\sqrt{45+26\sqrt{3}}+\frac{\sqrt{6+4\sqrt{3}-4\sqrt{6}\sqrt[4]{3}}}{2-\sqrt{3}}\right)$,

(iii) $G(e^{-\pi/4}) = -\frac{1}{4}\left((\sqrt{3}\,b-1)^2-\sqrt{3(b^2+1)(b-\sqrt{3}\,)(3b-\sqrt{3}\,)}\right)$,

where

$$b = 5+3\sqrt{3}-\sqrt{45+26\sqrt{3}}-\sqrt{90+52\sqrt{3}-4\sqrt{6(168+97\sqrt{3})}}.$$

Proof. Part (i) follows directly from Lemma 2.3(i) and Theorem 4.4(i). The proofs of Parts (ii) and (iii) are similar to that of Part (i). □

See [4] for a different proof for Theorem 5.3(i), where $G(e^{-\pi})$ was given by

$$G(e^{-\pi}) = \frac{(1+\sqrt{3}\,)\,(-1-\sqrt{3}+\sqrt{6\sqrt{3}}\,)}{4}.$$

See also [6, Theorem 6.3.3(vii)] for an alternative proof for Theorem 5.3(ii), where $G(e^{-\pi/2})$ was given by

$$G(e^{-\pi/2}) = \frac{-1-\sqrt{2}+\sqrt{3}+3^{3/4}\sqrt{2}-\sqrt{6}}{2(\sqrt{2}-1)(\sqrt{3}-1)(\sqrt{3}-\sqrt{2}\,)}.$$

Corollary 5.4. *We have*

(i) $G(-e^{-\pi/2}) = \dfrac{-1+\sqrt{-9+6\sqrt{3}}}{1+2\sqrt{3}-\sqrt{9+6\sqrt{3}}-\sqrt{6(3+2\sqrt{3}-2\sqrt{6}\sqrt[4]{3}\,)}}$,

(ii) $G(-e^{-\pi/4}) = -\frac{1}{4}\left((\sqrt{3}\,b-1)^2+\sqrt{3(b^2+1)(b-\sqrt{3}\,)(3b-\sqrt{3}\,)}\right)$,

where

$$b = 5+3\sqrt{3}-\sqrt{45+26\sqrt{3}}-\sqrt{90+52\sqrt{3}-4\sqrt{6(168+97\sqrt{3})}}.$$

Proof. The results follow directly from Lemma 2.4 and Theorem 5.3. □

See also [6, Theorem 6.3.5(vii)] for an alternative proof for Theorem 5.4(i), where $G(-e^{-\pi/2})$ was given by

$$G(-e^{-\pi/2}) = \frac{2(1+3^{1/4})}{1-2\cdot 3^{1/4}-\sqrt{3}-\sqrt{2}\,3^{3/4}}\,.$$

We now find $G(e^{-2^m\sqrt{2}\pi})$ for $m = 1$, 2, and 3 and $G(-e^{-2^m\sqrt{2}\pi})$ for $m = 1$ and 2.

Theorem 5.5. *We have*

(i) $G(e^{-2\sqrt{2}\pi}) = \frac{1}{4}\left(2-3\sqrt{2}+\sqrt{6}\,\right)$,

(ii) $G(e^{-4\sqrt{2}\pi})$

$$= \frac{47+33\sqrt{2}+27\sqrt{3}+19\sqrt{6}-3\sqrt{(970+686\sqrt{2}+560\sqrt{3}+396\sqrt{6}\,)}}{10+6\sqrt{2}+6\sqrt{3}+4\sqrt{6}},$$

(iii) $G(e^{-8\sqrt{2}\pi}) = \frac{1}{4}\left(\sqrt{3}\,c-1-3^{3/4}\sqrt{\frac{(c^2+1)(c-\sqrt{3}\,)}{\sqrt{3}\,c-1}}\right)$,

where

$$c = 27+19\sqrt{2}+16\sqrt{3}+11\sqrt{6}+\sqrt{6(485+343\sqrt{2}+280\sqrt{3}+198\sqrt{6}\,)}\,.$$

Proof. For (i), letting $n = 8$ in Lemma 2.3(ii) and putting the value of l_8' from Corollary 4.5(i), we complete the proof.

For (ii) and (iii), repeat the same argument as in the proof of (i). □

See also [6, Theorem 6.3.7(i)] for an alternative proof for Theorem 5.5(i), where $G(e^{-2\sqrt{2}\pi})$ was given by

$$G(e^{-2\sqrt{2}\pi}) = \frac{(1+\sqrt{2}\,)^{1/8}(\sqrt{3}-\sqrt{2}\,)^2}{2^{3/4}(1+35\sqrt{2}-28\sqrt{3}\,)^{3/8}}\,.$$

Corollary 5.6. *We have*

(i) $G(-e^{-2\sqrt{2}\pi})$

$$= -\frac{1}{2}\left(47+33\sqrt{2}+27\sqrt{3}+19\sqrt{6}-3\sqrt{970+686\sqrt{2}+560\sqrt{3}+396\sqrt{6}}\right),$$

(ii) $G(-e^{-4\sqrt{2}\pi}) = -\frac{1}{4}\left((\sqrt{3}\,c-1)^2-3^{3/4}\sqrt{(c^2+1)(c-\sqrt{3}\,)(\sqrt{3}\,c-1)}\right)$,

where

$$c = 27+19\sqrt{2}+16\sqrt{3}+11\sqrt{6}+\sqrt{6(485+343\sqrt{2}+280\sqrt{3}+198\sqrt{6}\,)}\,.$$

Proof. Parts (i) and (ii) follow directly from Lemma 2.4 and Theorem 5.5. □

We end this section by evaluating $G(e^{-\pi/2^m\sqrt{2}})$ for $m = 0, 1, 2$ and 3 and $G(-e^{-\pi/2^m\sqrt{2}})$ for $m = 1, 2$, and 3.

Theorem 5.7. *We have*

(i) $G(e^{-\pi/\sqrt{2}}) = \dfrac{\sqrt{2}}{3+\sqrt{3}-\sqrt{2}}$,

(ii) $G(e^{-\pi/2\sqrt{2}}) = \dfrac{1-\sqrt{3}+\sqrt{6}}{-1+\sqrt{3}+3\sqrt{2(-1+\sqrt{2}\,)}}$,

(iii) $G(e^{-\pi/4\sqrt{2}})$

$$= \frac{-1-\sqrt{2}+\sqrt{3}-2\sqrt{6}+\sqrt{6(-1+2\sqrt{2}+\sqrt{3}\,)}}{1+\sqrt{2}+\sqrt{3}-\sqrt{6(-1+2\sqrt{2}+\sqrt{3}\,)}-6\sqrt{-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}}},$$

(iv) $G(e^{-\pi/8\sqrt{2}}) = \dfrac{\sqrt{3}\,d-1}{1+3^{3/4}\sqrt{d\,(d^2-\sqrt{3}\,d+1)}}$,

where

$$d = \frac{2-\sqrt{2}+\sqrt{6(-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}\,)}}{(\sqrt{3}-\sqrt{1+\sqrt{2}}\,)\left(3+\sqrt{(1+\sqrt{2}\,)(2-\sqrt{3}\,)}\right)}.$$

Proof. Part (i) follows directly from Lemma 2.3 and Theorem 4.6(i). The proofs of Parts (ii), (iii), and (iv) are similar to that of Part (i). □

Corollary 5.8. *We have*

(i) $G(-e^{-\pi/2\sqrt{2}}) = \dfrac{1-\sqrt{3}-3\sqrt{2}\sqrt{-1+\sqrt{2}}}{2+4\sqrt{3}-2\sqrt{6}}$,

(ii) $G(-e^{-\pi/4\sqrt{2}}) = \dfrac{1-\sqrt{3}+\sqrt{6}}{-1+\sqrt{3}+3\sqrt{2(-1+\sqrt{2})}}$

$$\times\frac{1+\sqrt{2+\sqrt{3}}-\sqrt{3(-1+2\sqrt{2}+\sqrt{3}\,)}-3\sqrt{2(-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}\,)}}{1+2\sqrt{3}-\sqrt{2-\sqrt{3}}-\sqrt{3(-1+2\sqrt{2}+\sqrt{3})}},$$

(iii) $G(-e^{-\pi/8\sqrt{2}}) = \dfrac{1+3^{3/4}\sqrt{d(d^2-\sqrt{3}\,d+1)}}{\sqrt{3}\,d-1}$

$$\times \frac{1+2\sqrt{3}-\sqrt{2-\sqrt{3}}-\sqrt{3(-1+2\sqrt{2}+\sqrt{3})}}{1+\sqrt{2+\sqrt{3}}-\sqrt{3(-1+2\sqrt{2}+\sqrt{3}\,)}-3\sqrt{2(-2+\sqrt{2}+\sqrt{-1+\sqrt{2}}\,)}},$$

where

$$d=\frac{2-\sqrt{2}+\sqrt{6(-2+\sqrt{2}+\sqrt{\sqrt{2}-1}\,)}}{(\sqrt{3}-\sqrt{1+\sqrt{2}}\,)\left(3+\sqrt{(1+\sqrt{2}\,)(2-\sqrt{3}\,)}\right)}.$$

Proof. The results follow directly from Lemma 2.4 and Theorem 5.7. □

REFERENCES

1. B.C. Berndt: *Number Theory in the Spirit of Ramanujan.* American Mathematical Society, 2006.
2. ______: *Ramanujan's Notebooks, Part III.* Springer–Verlag, New York, 1991.
3. B.C. Berndt, H.H. Chan & L.-C. Zhang: Ramanujan's class invariants and cubic continued fraction. *Acta Arith.* **73** (1995), 67-85.
4. H.H. Chan: On Ramanujan's cubic continued fraction. *Acta Arith.* **73** (1995), 343-355.
5. D.H. Paek & J. Yi: On some modular equations and their applications II. *Bull. Korean Math. Soc.* **50** (2013), no. 4, 1221-1233.
6. J. Yi: *The Construction and Applications of Modular Equations.* Ph. D. Thesis, University of Illinois at Urbana-Champaign, 2001.
7. ______: Theta-function identities and the explicit formulas for theta-function and their applications. *J. Math. Anal. Appl.* **292** (2004), 381-400.
8. J. Yi, M.G. Cho, J.H. Kim, S.H. Lee, J.M. Yu & D.H. Paek: On some modular equations and their applications I. *Bull. Korean Math. Soc.* **50** (2013), no. 3, 761-776.
9. J. Yi, Y. Lee & D.H. Paek: The explicit formulas and evaluations of Ramanujan's theta-function ψ. *J. Math. Anal. Appl.* **321** (2006), 157-181.

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