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# FIXED POINTS AND STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN $G$-NORMED SPACES

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Abstract. In this paper, we introduce functional equations in $G$-normed spaces and we prove the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in complete $G$-normed spaces by using the fixed point method.

## 1. Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam [44] concerning the stability of group homomorphisms. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [37] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [37] has provided a lot of influence in the development of what we call *Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [43] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space.

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Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation.

In [21], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [25], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

The following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y)-6f(x) \\ &\quad +f(2y)+f(-2y)-4f(y)-4f(-y) \end{aligned} \tag{1.3}$$

has been investigated in [24, 33, 34]. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 7, 9, 11, 15, 16], [19, 22, 23, 28, 30], [38]–[42]).

**Definition 1.1** ([29])**.** Let $X$ be a vector space. A function $G : X^3 \to [0,\infty)$ is called a *$G$-metric* if the following conditions are satisfied:

(1) $G(x,y,z)=0$ if $x=y=z$,
(2) $G(x,x,z)>0$ for all $x,z\in X$ with $x\neq z$,
(3) $G(x,x,z)\le G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$,
(4) $G(x,y,z)=G(p(x),p(y),p(z))$, where $p$ is a permutation of $x,y,z$,
(5) $G(x,y,z)\le G(x,w,w)+G(w,y,z)$ for all $x,y,z,w\in X$.

The pair $(X,G)$ is called a *$G$-metric space*.

**Definition 1.2** ([29])**.** Let $(X,G)$ be a $G$-metric space.

(1) A sequence $\{x_n\}$ in $X$ is said to be a *$G$-Cauchy sequence* if, for each $\varepsilon>0$, there exists an integer $N$ such that, for all $m,n,l\ge N$,

$$G(x_m, x_n, x_l) < \varepsilon.$$

(2) A sequence $\{x_n\}$ in $X$ is said to be *$G$-convergent to a point $x$* if, for each $\varepsilon > 0$, there exists an integer $N$ such that, for all $m, n \geq N$,

$$G(x_m, x_n, x) < \varepsilon.$$

A $G$-metric space $(X, G)$ is called *complete* if every $G$-Cauchy sequence is $G$-convergent.

**Example 1.3** ([29])**.** Let $(X, d)$ be a metric space. Then $G : X^3 \to [0, \infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \qquad x, y, z \in X,$$

is a $G$-metric.

One can define the following.

**Definition 1.4** ([35])**.** Let $X$ be a vector space over a field $F = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot, \cdot\| : X^2 \to [0, \infty)$ is called a *$G$-norm* if the following conditions are satisfied:

(1) $\|x, y\| = 0$ if and only if $x = y = 0$,

(2) $\|x - y, 0\| \leq \|x - w, 0\| + \|w - y, 0\|$ for all $x, y, w \in X$,

(3) $\|x - y, x - z\| \leq \|x - w, x - w\| + \|w - y, w - z\|$ for all $x, y, z, w \in X$,

(4) $\|\lambda x, \lambda y\| = |\lambda| \; \|x, y\|$ for all $x, y \in X$ and all $\lambda \in F$.

The pair $(X, \|\cdot, \cdot\|)$ is called a *$G$-normed space.*

**Definition 1.5** ([35])**.** Let $(X, \|\cdot, \cdot\|)$ be a $G$-normed space.

(1) A sequence $\{x_n\}$ in $X$ is said to be a *$G$-Cauchy sequence* if, for each $\varepsilon > 0$, there exists an integer $N$ such that, for all $m, l \geq N$,

$$\|x_l - x_m, x_l - x_m\| < \varepsilon \qquad \& \qquad \|x_l - x_m, 0\| < \varepsilon.$$

(2) A sequence $\{x_n\}$ in $X$ is said to be *$G$-convergent to a point $x \in X$* if, for each $\varepsilon > 0$, there exists an integer $N$ such that, for all $m \geq N$,

$$\|x - x_m, x - x_m\| < \varepsilon \qquad \& \qquad \|x - x_m, 0\| < \varepsilon.$$

We will denote $x$ by $G\text{-}\lim_{n\to\infty} x_n$.

A $G$-normed space $(X, \|\cdot, \cdot\|)$ is called *complete* if every $G$-Cauchy sequence is $G$-convergent.

It is easy to show that if there exists a $G$-limit $x \in X$ of a sequence $\{x_n\}$ in $X$ then the $G$-limit is *unique.*

**Example 1.6** ([35]). Let $(X, \|\cdot\|)$ be a normed space. It is easy to show that $\|\cdot,\cdot\| : X^2 \to [0,\infty)$, defined by

$$\|x, y\| = \max\{\|x\|, \|y\|\}, \qquad x, y \in X,$$

is a $G$-norm.

Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

**Theorem 1.7** ([4, 12]). *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) *$d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 27, 31, 32, 36]).

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1.3) in complete $G$-normed spaces.

Throughout this paper, let $X$ be a $G$-normed space and let $Y$ be a complete $G$-normed space.

## 2. STABILITY OF THE AQCQ-FUNCTIONAL EQUATION (1.3) IN $G$-NORMED SPACES: ODD CASE

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x).$$

It was shown in [14, Lemma 2.2] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in [13, Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y) : \ = \ & f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) \\ & -f(2y) - f(-2y) + 4f(y) + 4f(-y) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in $G$-normed spaces: odd case.

**Theorem 2.1.** *Let $\varphi : X^4 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z, w) \le \frac{L}{8}\varphi(2x, 2y, 2z, 2w) \tag{2.1}$$

*for all $x, y, z, w \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y), Df(z, w)\| \le \varphi(x, y, z, w) \tag{2.2}$$

*for all $x, y, z, w \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned} &\|f(2x) - 2f(x) - C(x), f(2x) - 2f(x) - C(x)\| \\ &\quad \le \frac{L}{8 - 8L}\left(4\varphi(x, x, x, x) + \varphi(2x, x, 2x, x)\right), \end{aligned} \tag{2.3}$$

$$\|f(2x) - 2f(x) - C(x), 0\| \le \frac{L}{8 - 8L}\left(4\varphi(x, x, 0, 0) + \varphi(2x, x, 0, 0)\right) \tag{2.4}$$

*for all $x \in X$.*

*Proof.* Since $f$ is odd, $f(0) = 0$.

Letting $x = z = w = y$ in (2.2), we get

$$\|f(3y) - 4f(2y) + 5f(y), f(3y) - 4f(2y) + 5f(y)\| \le \varphi(y, y, y, y) \tag{2.5}$$

for all $y \in X$.

Replacing $x$, $z$, $w$ by $2y$, $2y$, $y$ in (2.2), respectively, we get

$$\begin{aligned}(2.6)\qquad &\|f(4y)-4f(3y)+6f(2y)-4f(y), f(4y)-4f(3y)+6f(2y)-4f(y)\| \\ &\leq \varphi(2y,y,2y,y)\end{aligned}$$

for all $y \in X$.

Since $\|x-y, x-z\| \leq \|x-w, x-w\| + \|w-y, w-z\|$ for all $x, y, z, w \in X$, it follows from (2.5) and (2.6) that

$$\begin{aligned}(2.7)\qquad &\|f(4y)-10f(2y)+16f(y), f(4y)-10f(2y)+16f(y)\| \\ &\leq \|4(f(3y)-4f(2y)+5f(y)), 4(f(3y)-4f(2y)+5f(y))\| \\ &\quad + \|f(4y)-4f(3y)+6f(2y)-4f(y), f(4y)-4f(3y)+6f(2y)-4f(y)\| \\ &\leq 4\varphi(y,y,y,y)+\varphi(2y,y,2y,y)\end{aligned}$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x)-2f(x)$ for all $x \in X$, we get

$$\begin{aligned}(2.8)\quad \left\|g(x)-8g\left(\frac{x}{2}\right), g(x)-8g\left(\frac{x}{2}\right)\right\| &\leq 4\varphi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2},x,\frac{x}{2}\right) \\ &\leq \frac{L}{8}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right)\end{aligned}$$

for all $x \in X$.

Letting $x = y$ and $z = w = 0$ in (2.2), we get

$$(2.9)\qquad \|f(3y)-4f(2y)+5f(y), 0\| \leq \varphi(y,y,0,0)$$

for all $y \in X$.

Replacing $x$ by $2y$ and $z = w = 0$ in (2.2), we get

$$(2.10)\qquad \|f(4y)-4f(3y)+6f(2y)-4f(y), 0\| \leq \varphi(2y,y,0,0)$$

for all $y \in X$.

Since $\|x-y, 0\| \leq \|x-w, 0\| + \|w-y, 0\|$ for all $x, y, w \in X$, it follows from (2.9) and (2.10) that

$$\begin{aligned}(2.11)\quad &\|f(4y)-10f(2y)+16f(y), 0\| \\ &\leq \|4(f(3y)-4f(2y)+5f(y)), 0\| + \|f(4y)-4f(3y)+6f(2y)-4f(y), 0\| \\ &\leq 4\varphi(y,y,0,0)+\varphi(2y,y,0,0)\end{aligned}$$

for all $y \in X$. So

$$\left\|g(x) - 8g\left(\frac{x}{2}\right), 0\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}, 0, 0\right) + \varphi\left(x, \frac{x}{2}, 0, 0\right) \leq \frac{L}{8}\left(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\right) \tag{2.12}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g, h) = \inf\big\{\mu \in \mathbb{R}_+ : \|g(x) - h(x), g(x) - h(x)\| \leq \mu\big(4\varphi\left(x, x, x, x\right) + \varphi\left(2x, x, 2x, x\right)\big),\ \big\|g(x) - h(x), 0\big\| \leq \mu\big(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\big),\ \forall x \in X\big\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see [26]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x), 0\| \leq \left(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\right)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x), 0\| = \|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), 0\| \leq L\left(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\right)$$

for all $x \in X$.

Similarly, one can show that

$$\|Jg(x) - Jh(x), Jg(x) - Jh(x)\| = \|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), 8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\| \leq L\left(4\varphi\left(x, x, x, x\right) + \varphi\left(2x, x, 2x, x\right)\right)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.8) and (2.12) that $d(g, Jg) \leq \frac{L}{8}$.

By Theorem 1.7, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.13}$$

for all $x \in X$. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\begin{aligned} \|g(x) - C(x), g(x) - C(x)\| &\leq \mu\left(4\varphi(x, x, x, x) + \varphi(2x, x, 2x, x)\right), \\ \|g(x) - C(x), 0\| &\leq \mu\left(4\varphi(x, x, 0, 0) + \varphi(2x, x, 0, 0)\right) \end{aligned}$$

for all $x \in X$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{L}{8 - 8L}.$$

This implies that the inequalities (2.3) and (2.4) hold.

By (2.1) and (2.2),

$$\begin{aligned} 8^n \left\| Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 0 \right\| &\leq 8^n \varphi\left(\frac{2x}{2^n}, \frac{2y}{2^n}, 0, 0\right) + 2 \cdot 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0\right) \\ &\leq L^n(\varphi(2x, 2y, 0, 0) + 2\varphi(2x, 2y, 0, 0)), \end{aligned}$$

which tends to zero as $n \to \infty$. So

$$\|DC(x, y), 0\| = 0$$

for all $x, y \in X$. Thus the mapping $C : X \to Y$ is a cubic mapping, as desired. $\square$

**Corollary 2.2.** *Let $\theta$ be a positive real number and $p$ a real number with $0 < p < 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying*

$$\|Df(x, y), Df(z, w)\| \leq \theta(\|x, x\|^p + \|y, y\|^p + \|z, z\|^p + \|w, w\|^p) \tag{2.14}$$

*for all $x, y, z, w \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned} \|f(2x) - 2f(x) - C(x), f(2x) - 2f(x) - C(x)\| &\leq \frac{2^p(9 + 2^p)}{4(8 - 2^p)} \theta \|x, x\|^p, \\ \|f(2x) - 2f(x) - C(x), 0\| &\leq \frac{2^p(9 + 2^p)}{8(8 - 2^p)} \theta \|x, x\|^p \end{aligned}$$

*for all* $x \in X$.

*Proof.* The proof follows from Theorem 2.1 by taking $\varphi(x,y,z,w) = \theta(\|x,x\|^p + \|y,y\|^p + \|z,z\|^p + \|w,w\|^p)$ for all $x,y,z,w \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

**Theorem 2.3.** *Let* $\varphi : X^4 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y,z,w) \le 8L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right)$$

*for all* $x,y,z,w \in X$. *Let* $f : X \to Y$ *be an odd mapping satisfying* (2.2). *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\begin{aligned}&\|f(2x) - 2f(x) - C(x), f(2x) - 2f(x) - C(x)\|\\ &\quad \le \frac{1}{8-8L}\left(4\varphi(x,x,x,x) + \varphi(2x,x,2x,x)\right),\\ &\|f(2x) - 2f(x) - C(x), 0\| \le \frac{1}{8-8L}\left(4\varphi(x,x,0,0) + \varphi(2x,x,0,0)\right)\end{aligned}$$

*for all* $x \in X$.

*Proof.* Let $(S,d)$ be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$.

It follows from (2.8) and (2.12) that

$$\begin{aligned}\left\|g(x) - \frac{1}{8}g(2x), g(x) - \frac{1}{8}g(2x)\right\| &\le \frac{1}{8}\left(4\varphi(x,x,x,x) + \varphi(2x,x,2x,x)\right),\\ \left\|g(x) - \frac{1}{8}g(2x), 0\right\| &\le \frac{1}{8}\left(4\varphi(x,x,0,0) + \varphi(2x,x,0,0)\right)\end{aligned}$$

for all $x \in X$. So $d(g,Jg) \le \frac{1}{8}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.4.** *Let* $\theta$ *be a positive real number and* $p$ *a real number with* $p > 3$. *Suppose that* $f : X \to Y$ *is an odd mapping satisfying* (2.14). *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\begin{aligned}\|f(2x)-2f(x)-C(x), f(2x)-2f(x)-C(x)\| &\le \frac{2^p(9+2^p)}{4(2^p-8)}\theta\|x,x\|^p,\\ \|f(2x)-2f(x)-C(x), 0\| &\le \frac{2^p(9+2^p)}{8(2^p-8)}\theta\|x,x\|^p\end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.3 by taking $\varphi(x,y,z,w) = \theta(\|x,x\|^p + \|y,y\|^p + \|z,z\|^p + \|w,w\|^p)$ for all $x,y,z,w \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

**Theorem 2.5.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z,w) \le \frac{L}{2}\varphi(2x,2y,2z,2w)$$

*for all $x,y,z,w \in X$. Let $f : X \to Y$ be an odd mapping satisfying* (2.2)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned}&\|f(2x)-8f(x)-A(x), f(2x)-8f(x)-A(x)\|\\ &\quad\le \frac{L}{2-2L}\left(4\varphi(x,x,x,x)+\varphi(2x,x,2x,x)\right),\\ &\|f(2x)-8f(x)-A(x),0\| \le \frac{L}{2-2L}\left(4\varphi(x,x,0,0)+\varphi(2x,x,0,0)\right)\end{aligned}$$

*for all $x \in X$.*

*Proof.* Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.7), we get

$$\begin{aligned}\left\|h(x)-2h\left(\frac{x}{2}\right), h(x)-2h\left(\frac{x}{2}\right)\right\| &\le 4\varphi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2},x,\frac{x}{2}\right)\\ &\le \frac{L}{2}\left(4\varphi(x,x,x,x)+\varphi(2x,x,2x,x)\right)\end{aligned} \tag{2.15}$$

for all $x \in X$.

It follows from (2.11) that

$$\begin{aligned}\left\|h(x)-2h\left(\frac{x}{2}\right),0\right\| &\le 4\varphi\left(\frac{x}{2},\frac{x}{2},0,0\right)+\varphi\left(x,\frac{x}{2},0,0\right)\\ &\le \frac{L}{2}\left(4\varphi(x,x,0,0)+\varphi(2x,x,0,0)\right)\end{aligned} \tag{2.16}$$

for all $x \in X$.

Let $(S,d)$ be the generalized metric space defined in the proof of Theorem 2.1. The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.6.** *Let $\theta$ be a positive real number and $p$ a real number with $0 < p < 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying* (2.14)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} \|f(2x) - 8f(x) - A(x), f(2x) - 8f(x) - A(x)\| &\leq \frac{2^p(9+2^p)}{4(2-2^p)}\theta\|x,x\|^p, \\ \|f(2x) - 8f(x) - A(x), 0\| &\leq \frac{2^p(9+2^p)}{8(2-2^p)}\theta\|x,x\|^p \end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.5 by taking $\varphi(x,y,z,w) = \theta(\|x,x\|^p + \|y,y\|^p + \|z,z\|^p + \|w,w\|^p)$ for all $x,y,z,w \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

**Theorem 2.7.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z,w) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right)$$

*for all $x,y,z,w \in X$. Let $f : X \to Y$ be an odd mapping satisfying* (2.2)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} &\|f(2x) - 8f(x) - A(x), f(2x) - 8f(x) - A(x)\| \\ &\quad \leq \frac{1}{2-2L}\left(4\varphi(x,x,x,x) + \varphi(2x,x,2x,x)\right), \\ &\|f(2x) - 8f(x) - A(x), 0\| \leq \frac{1}{2-2L}\left(4\varphi(x,x,0,0) + \varphi(2x,x,0,0)\right) \end{aligned}$$

*for all $x \in X$.*

*Proof.* Let $(S,d)$ be the generalized metric space defined in the proof of Theorem 2.1.

It follows from (2.15) and (2.16) that

$$\begin{aligned} \left\|h(x) - \frac{1}{2}h(2x), h(x) - \frac{1}{2}h(2x)\right\| &\leq \frac{1}{2}\left(4\varphi(x,x,x,x) + \varphi(2x,x,2x,x)\right), \\ \left\|h(x) - \frac{1}{2}h(2x), 0\right\| &\leq \frac{1}{2}\left(4\varphi(x,x,0,0) + \varphi(2x,x,0,0)\right) \end{aligned}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.8.** *Let $\theta$ be a positive real number and $p$ a real number with $p > 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying* (2.14). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} \|f(2x) - 8f(x) - A(x), f(2x) - 8f(x) - A(x)\| &\leq \frac{2^p(9+2^p)}{4(2^p-2)}\theta\|x,x\|^p, \\ \|f(2x) - 8f(x) - A(x), 0\| &\leq \frac{2^p(9+2^p)}{8(2^p-2)}\theta\|x,x\|^p \end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.7 by taking $\varphi(x,y,z,w) = \theta(\|x,x\|^p + \|y,y\|^p + \|z,z\|^p + \|w,w\|^p)$ for all $x,y,z,w \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

## 3. STABILITY OF THE AQCQ-FUNCTIONAL EQUATION (1.3) IN $G$-NORMED SPACES: EVEN CASE

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in $G$-normed spaces: even case.

**Theorem 3.1.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z,w) \leq \frac{L}{16}\varphi(2x,2y,2z,2w)$$

*for all $x,y,z,w \in X$. Let $f : X \to Y$ be an even mapping satisfying* (2.2) *and $f(0) = 0$. Then there exists a unique quartic mapping $R : X \to Y$ such that*

$$\begin{aligned} &\|f(2x) - 4f(x) - R(x), f(2x) - 4f(x) - R(x)\| \\ &\quad \leq \frac{L}{16-16L}\left(4\varphi(x,x,x,x) + \varphi(2x,x,2x,x)\right), \\ &\|f(2x) - 4f(x) - R(x), 0\| \leq \frac{L}{16-16L}\left(4\varphi(x,x,0,0) + \varphi(2x,x,0,0)\right) \end{aligned}$$

*for all $x \in X$.*

*Proof.* Letting $x = z = w = y$ in (2.2), we get

$$\|f(3y) - 6f(2y) + 15f(y), f(3y) - 6f(2y) + 15f(y)\| \leq \varphi(y,y,y,y) \tag{3.1}$$

for all $y \in X$.

Replacing $x$, $z$, $w$ by $2y$, $2y$, $y$ in (2.2), respectively, we get

$$\begin{aligned}(3.2)\qquad &\|f(4y)-4f(3y)+4f(2y)+4f(y), f(4y)-4f(3y)+4f(2y)+4f(y)\|\\ &\le \varphi(2y,y,2y,y)\end{aligned}$$

for all $y\in X$.

Since $\|x-y, x-z\| \le \|x-w, x-w\| + \|w-y, w-z\|$ for all $x,y,z,w\in X$, it follows from (3.1) and (3.2) that

$$\begin{aligned}(3.3)\qquad &\|f(4y)-20f(2y)+64f(y), f(4y)-20f(2y)+64f(y)\|\\ &\le \|4(f(3y)-6f(2y)+15f(y)), 4(f(3y)-6f(2y)+15f(y))\|\\ &\quad +\|f(4y)-4f(3y)+4f(2y)+4f(y), f(4y)-4f(3y)+4f(2y)+4f(y)\|\\ &\le 4\varphi(y,y,y,y)+\varphi(2y,y,2y,y)\end{aligned}$$

for all $y\in X$. Letting $y:=\frac{x}{2}$ and $g(x):=f(2x)-4f(x)$ for all $x\in X$, we get

$$\begin{aligned}(3.4)\qquad \left\|g(x)-16g\left(\frac{x}{2}\right), g(x)-16g\left(\frac{x}{2}\right)\right\| &\le 4\varphi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2},x,\frac{x}{2}\right)\\ &\le \frac{L}{16}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right)\end{aligned}$$

for all $x\in X$.

Letting $x=y$ and $z=w=0$ in (2.2), we get

$$(3.5)\qquad \|f(3y)-6f(2y)+15f(y), 0\| \le \varphi(y,y,0,0)$$

for all $y\in X$.

Replacing $x$ by $2y$ and $z=w=0$ in (2.2), we get

$$(3.6)\qquad \|f(4y)-4f(3y)+4f(2y)+4f(y), 0\| \le \varphi(2y,y,0,0)$$

for all $y\in X$.

Since $\|x-y,0\| \le \|x-w,0\| + \|w-y,0\|$ for all $x,y,w\in X$, it follows from (3.3) and (3.4) that

$$\begin{aligned}(3.7)\qquad &\|f(4y)-20f(2y)+64f(y), 0\|\\ &\le \|4(f(3y)-6f(2y)+15f(y)), 0\| + \|f(4y)-4f(3y)+4f(2y)+4f(y), 0\|\\ &\le 4\varphi(y,y,0,0)+\varphi(2y,y,0,0)\end{aligned}$$

for all $y\in X$. So

$$\begin{aligned}(3.8)\qquad \left\|g(x)-16g\left(\frac{x}{2}\right), 0\right\| &\le 4\varphi\left(\frac{x}{2},\frac{x}{2},0,0\right)+\varphi\left(x,\frac{x}{2},0,0\right)\\ &\le \frac{L}{16}\left(4\varphi\left(x,x,0,0\right)+\varphi\left(2x,x,0,0\right)\right)\end{aligned}$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. The rest of the proof is similar to the proof of Theorem 2.1. $\square$

**Corollary 3.2.** *Let $\theta$ be a positive real number and $p$ a real number with $0 < p < 4$. Suppose that $f : X \to Y$ is an even mapping satisfying* (2.14)*. Then there exists a unique quartic mapping $R : X \to Y$ such that*

$$\begin{aligned} \|f(2x) - 4f(x) - R(x), f(2x) - 4f(x) - R(x)\| &\leq \frac{2^p(9 + 2^p)}{4(16 - 2^p)}\theta\|x, x\|^p, \\ \|f(2x) - 4f(x) - R(x), 0\| &\leq \frac{2^p(9 + 2^p)}{8(16 - 2^p)}\theta\|x, x\|^p \end{aligned}$$

*for all $x \in X$.*

*Proof.* It is obvious that $f(0) = 0$. The proof follows from Theorem 3.1 by taking $\varphi(x, y, z, w) = \theta(\|x, x\|^p + \|y, y\|^p + \|z, z\|^p + \|w, w\|^p)$ for all $x, y, z, w \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. $\square$

**Theorem 3.3.** *Let $\varphi : X^4 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z, w) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right)$$

*for all $x, y, z, w \in X$. Let $f : X \to Y$ be an even mapping satisfying* (2.2) *and $f(0) = 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned} &\|f(2x) - 4f(x) - R(x), f(2x) - 4f(x) - R(x)\| \\ &\quad \leq \frac{1}{16 - 16L}\left(4\varphi\left(x, x, x, x\right) + \varphi\left(2x, x, 2x, x\right)\right), \\ &\|f(2x) - 4f(x) - R(x), 0\| \leq \frac{1}{16 - 16L}\left(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\right) \end{aligned}$$

*for all $x \in X$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g\left(2x\right)$$

for all $x \in X$.

It follows from (3.4) and (3.8) that

$$\begin{aligned}\left\|g(x)-\frac{1}{16}g\left(2x\right),g(x)-\frac{1}{16}g\left(2x\right)\right\| &\leq \frac{1}{16}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right),\\ \left\|g(x)-\frac{1}{16}g\left(2x\right),0\right\| &\leq \frac{1}{16}\left(4\varphi\left(x,x,0,0\right)+\varphi\left(2x,x,0,0\right)\right)\end{aligned}$$

for all $x \in X$. So $d(g, Jg) \leq \frac{1}{16}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 3.4.** *Let $\theta$ be a positive real number and $p$ a real number with $p > 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (2.14). Then there exists a unique quartic mapping $R : X \to Y$ such that*

$$\begin{aligned}\|f(2x)-4f(x)-R(x),f(2x)-4f(x)-R(x)\| &\leq \frac{2^p(9+2^p)}{4(2^p-16)}\theta\|x,x\|^p,\\ \|f(2x)-4f(x)-R(x),0\| &\leq \frac{2^p(9+2^p)}{8(2^p-16)}\theta\|x,x\|^p\end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.3 by taking $\varphi(x, y, z, w) = \theta(\|x, x\|^p + \|y, y\|^p + \|z, z\|^p + \|w, w\|^p)$ for all $x, y, z, w \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

**Theorem 3.5.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z,w) \leq \frac{L}{4}\varphi\left(2x,2y,2z,2w\right)$$

*for all $x, y, z, w \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.2) and $f(0) = 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\|f(2x)-16f(x)-Q(x),f(2x)-16f(x)-Q(x)\|\\ &\leq \frac{L}{4-4L}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right),\\ &\|f(2x)-16f(x)-Q(x),0\| \leq \frac{L}{4-4L}\left(4\varphi\left(x,x,0,0\right)+\varphi\left(2x,x,0,0\right)\right)\end{aligned}$$

*for all $x \in X$.*

*Proof.* Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 16f(x)$ for all $x \in X$ in (3.3), we get

$$\begin{aligned}\left\|h(x)-4h\left(\frac{x}{2}\right),h(x)-4h\left(\frac{x}{2}\right)\right\| &\leq 4\varphi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2},x,\frac{x}{2}\right)\\ &\leq \frac{L}{4}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right)\end{aligned} \tag{3.9}$$

for all $x \in X$.

It follows from (3.7) that

$$\begin{aligned}\left\|h(x)-4h\left(\frac{x}{2}\right),0\right\| &\le 4\varphi\left(\frac{x}{2},\frac{x}{2},0,0\right)+\varphi\left(x,\frac{x}{2},0,0\right)\\ &\le \frac{L}{4}\left(4\varphi\left(x,x,0,0\right)+\varphi\left(2x,x,0,0\right)\right)\end{aligned} \tag{3.10}$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 3.6.** *Let $\theta$ be a positive real number and $p$ a real number with $0 < p < 2$. Suppose that $f : X \to Y$ is an even mapping satisfying (2.14). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\|f(2x)-16f(x)-Q(x),f(2x)-16f(x)-Q(x)\| &\le \frac{2^p(9+2^p)}{4(4-2^p)}\theta\|x,x\|^p,\\ \|f(2x)-16f(x)-Q(x),0\| &\le \frac{2^p(9+2^p)}{8(4-2^p)}\theta\|x,x\|^p\end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.5 by taking $\varphi(x,y,z,w) = \theta(\|x,x\|^p + \|y,y\|^p + \|z,z\|^p + \|w,w\|^p)$ for all $x, y, z, w \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

**Theorem 3.7.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z,w) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right)$$

*for all $x, y, z, w \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.2) and $f(0) = 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\|f(2x)-16f(x)-Q(x),f(2x)-16f(x)-Q(x)\|\\ &\le \frac{1}{4-4L}\left(4\varphi\left(x,x,x,x\right)+\varphi\left(2x,x,2x,x\right)\right),\\ &\|f(2x)-16f(x)-Q(x),0\| \le \frac{1}{4-4L}\left(4\varphi\left(x,x,0,0\right)+\varphi\left(2x,x,0,0\right)\right)\end{aligned}$$

*for all $x \in X$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1.

It follows from (3.9) and (3.10) that

$$\begin{aligned} \left\|h(x) - \frac{1}{4}h\left(2x\right), h(x) - \frac{1}{4}h\left(2x\right)\right\| &\leq \frac{1}{4}\left(4\varphi\left(x, x, x, x\right) + \varphi\left(2x, x, 2x, x\right)\right), \\ \left\|h(x) - \frac{1}{4}h\left(2x\right), 0\right\| &\leq \frac{1}{4}\left(4\varphi\left(x, x, 0, 0\right) + \varphi\left(2x, x, 0, 0\right)\right) \end{aligned}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

**Corollary 3.8.** *Let $\theta$ be a positive real number and $p$ a real number with $p > 2$. Suppose that $f : X \to Y$ is an even mapping satisfying (2.14). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned} \|f(2x) - 16f(x) - Q(x), f(2x) - 16f(x) - Q(x)\| &\leq \frac{2^p(9+2^p)}{4(2^p-4)}\theta\|x, x\|^p, \\ \|f(2x) - 16f(x) - Q(x), 0\| &\leq \frac{2^p(9+2^p)}{8(2^p-4)}\theta\|x, x\|^p \end{aligned}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.7 by taking $\varphi(x, y, z, w) = \theta(\|x, x\|^p + \|y, y\|^p + \|z, z\|^p + \|w, w\|^p)$ for all $x, y, z, w \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. $\square$

## Conclusions

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then $f_o$ is odd and $f_e$ is even. $f_o$ and $f_e$ satisfy the functional equation (1.3). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then

$$f_e(x) = \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

So we obtain the following results.

**Theorem 3.9.** *Let $\theta$ be a positive real number and $p$ a real number with $p > 4$. Suppose that $f : X \to Y$ is a mapping satisfying (2.14) and $f(0) = 0$. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $Q : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $R : X \to Y$ such that*

$$
\begin{aligned}
&\Big\|f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)-\frac{1}{6}C(x)-\frac{1}{12}R(x), f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)\\
&\qquad -\frac{1}{6}C(x)-\frac{1}{12}R(x)\Big\|\\
&\qquad \le \left(\frac{1}{6(2^p-2)}+\frac{1}{12(2^p-4)}+\frac{1}{6(2^p-8)}+\frac{1}{12(2^p-16)}\right)\frac{2^p(9+2^p)}{4}\theta\|x,x\|^p,\\
&\left\|f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)-\frac{1}{6}C(x)-\frac{1}{12}R(x), 0\right\|\\
&\qquad \le \left(\frac{1}{6(2^p-2)}+\frac{1}{12(2^p-4)}+\frac{1}{6(2^p-8)}+\frac{1}{12(2^p-16)}\right)\frac{2^p(9+2^p)}{8}\theta\|x,x\|^p
\end{aligned}
$$

*for all* $x \in X$.

**Theorem 3.10.** *Let* $\theta$ *be a positive real number and* $p$ *a real number with* $0 < p < 1$. *Suppose that* $f : X \to Y$ *is a mapping satisfying* (2.14) *and* $f(0) = 0$. *Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $Q : X \to Y$, *a cubic mapping* $C : X \to Y$ *and a quartic mapping* $R : X \to Y$ *such that*

$$
\begin{aligned}
&\Big\|f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)-\frac{1}{6}C(x)-\frac{1}{12}R(x), f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)\\
&\qquad -\frac{1}{6}C(x)-\frac{1}{12}R(x)\Big\|\\
&\qquad \le \left(\frac{1}{6(2-2^p)}+\frac{1}{12(4-2^p)}+\frac{1}{6(8-2^p)}+\frac{1}{12(16-2^p)}\right)\frac{2^p(9+2^p)}{4}\theta\|x,x\|^p,\\
&\left\|f(x)-\frac{1}{6}A(x)-\frac{1}{12}Q(x)-\frac{1}{6}C(x)-\frac{1}{12}R(x), 0\right\|\\
&\qquad \le \left(\frac{1}{6(2-2^p)}+\frac{1}{12(4-2^p)}+\frac{1}{6(8-2^p)}+\frac{1}{12(16-2^p)}\right)\frac{2^p(9+2^p)}{8}\theta\|x,x\|^p
\end{aligned}
$$

*for all* $x \in X$.

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