J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2016.23.3.287 ISSN(Online) 2287-6081
Volume 23, Number 3 (August 2016), Pages 287–307

FUZZY STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN MATRIX FUZZY NORMED SPACES

Jung Rye Lee [a] and Dong-Yun Shin [b,*]

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in matrix fuzzy normed spaces.

1. Introduction and Preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [64] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [20]).

The proof given in [64] appealed to the theory of ordered operator spaces [13]. Effros and Ruan [21] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [54] and Haagerup [28] (as modified in [19]).

The stability problem of functional equations originated from a question of Ulam [69] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [29] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M.

Received by the editors July 24, 2016. Accepted August 06, 2016.
2010 *Mathematics Subject Classification.* Primary 47L25, 47H10, 46S40, 39B82, 46L07, 39B52, 26E50.
Key words and phrases. operator space, fixed point, Hyers-Ulam stability, matrix fuzzy normed space, additive-quadratic-cubic-quartic functional equation.
*Corresponding author.

Rassias [58] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [27] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [59] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [26] following the same approach as in Th.M. Rassias [58], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [26], as well as by Th.M. Rassias and Šemrl [63] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [17], D.H. Hyers, G. Isac and Th.M. Rassias [30]).

In 1982, J.M. Rassias [56] followed the innovative approach of the Th.M. Rassias' theorem [58] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [68] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [14] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [15] proved the Hyers-Ulam stability of the quadratic functional equation.

In [33], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [43], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 24, 31, 34, 35, 39, 42, 44, 50, 66, 57], [60]–[62]).

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [4, 25, 37, 41, 47, 70]. Following Cheng and Mordeson [9], Bag and Samanta [4] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [40] and investigated some properties of fuzzy normed spaces [5].

We use the definition of fuzzy normed spaces given in [4, 47, 48] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

Definition 1.1 ([4, 47, 48, 49])**.** Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed space.*

Definition 1.2 ([4, 47, 48, 49])**.** (1) Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

(2) Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [5]).

We introduce the concept of matrix fuzzy normed space.

Definition 1.3. Let (X, N) be a fuzzy normed space. (1) $(X, \{N_n\})$ is called a *matrix fuzzy normed space* if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \geq N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$.

(2) $(X, \{N_n\})$ is called a *matrix fuzzy Banach space* if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

Example 1.4. Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $N_n(x, t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$N_k(AxB, t) = \frac{t}{t + \|AxB\|_k} \geq \frac{t}{t + \|A\| \cdot \|x\|_n \cdot \|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\cdot\|B\|} + \|x\|_n}$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$. So $(X, \{N_n\})$ is a matrix fuzzy normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5 ([6, 18]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [32] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 36, 46, 51, 52, 55]).

Throughout this paper, let $(X, \{N_n\})$ be a matrix fuzzy normed space and $(Y, \{N_n\})$ a matrix fuzzy Banach space.

In this paper, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned}(1.3)\qquad & f(x+2y)+f(x-2y) \\ & = 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)\end{aligned}$$

in matrix fuzzy normed spaces by using the fixed point method.

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x).$$

It was shown in [23, Lemma 2.2] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in [22, Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

2. Hyers-Ulam Stability of the AQCQ-functional Equation (1.3) in Matrix Fuzzy Normed Spaces: Odd Mapping Case

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in matrix fuzzy normed spaces for an odd mapping case.

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{R})$ is that j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{R})$ is that (i,j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero.

Lemma 2.1. *Let $(X, \{N_n\})$ be a matrix fuzzy normed space.*

(1) $N_n(E_{kl} \otimes x, t) = N(x,t)$ *for all* $t > 0$ *and* $x \in X$.

(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^n t_{ij}$,

$$\begin{aligned} N(x_{kl}, t) &\geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\}, \\ N(x_{kl}, t) &\geq N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n\right\}. \end{aligned}$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $N_n(E_{kl} \otimes x, t) \geq N(x,t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $N_n(E_{kl} \otimes x, t) \leq N(x,t)$. So $N(E_{kl} \otimes x, t) = N(x,t)$.

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geq N_n\left([x_{ij}], \frac{t}{\|e_k\| \cdot \|e_l\|}\right) = N_n([x_{ij}], t)$.

$$\begin{aligned} N_n([x_{ij}], t) &= N_n\left(\sum_{i,j=1}^n E_{ij} \otimes x_{ij}, t\right) \geq \min\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\} \\ &= \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\}, \end{aligned}$$

where $t = \sum_{i,j=1}^n t_{ij}$. So $N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n\right\}$.

(3) By $N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n\right\}$, we obtain the result. □

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned} Df(a,b) := &f(a+2b) + f(a-2b) - 4f(a+b) - 4f(a-b) + 6f(a) \\ &- f(2b) - f(-2b) + 4f(b) + 4f(-b), \end{aligned}$$

$$\begin{aligned} &Df_n([x_{ij}], [y_{ij}]) := f_n([x_{ij}] + 2[y_{ij}]) + f_n([x_{ij}] - 2[y_{ij}]) - 4f_n([x_{ij}] + [y_{ij}]) \\ &-4f_n([x_{ij}] - [y_{ij}]) + 6f_n([x_{ij}]) - f_n(2[y_{ij}]) - f_n(-2[y_{ij}]) + 4f_n([y_{ij}]) \\ &+4f_n(-[y_{ij}]) \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq \frac{\alpha}{2}\varphi\left(2a,2b\right) \tag{2.1}$$

for all $a,b \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi\left(x_{ij},y_{ij}\right)} \tag{2.2}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then

$$A(a) := N - \lim_{l\to\infty} 2^l \left(f\left(\frac{2a}{2^l}\right) - 8f\left(\frac{a}{2^l}\right)\right)$$

exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\begin{aligned} &N\left(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]),t\right) \\ &\geq \frac{(1-\alpha)t}{(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}(4\varphi(x_{ij},x_{ij}) + \varphi(2x_{ij},x_{ij}))} \end{aligned} \tag{2.3}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$. Then (2.2) is equivalent to

$$N(Df(a,b),t) \geq \frac{t}{t+\varphi\left(a,b\right)} \tag{2.4}$$

for all $t > 0$ and $a,b \in X$.

Letting $b = a$ in (2.4), we get

$$N\left(f(3b) - 4f(2b) + 5f(b),t\right) \geq \frac{t}{t+\varphi\left(b,b\right)} \tag{2.5}$$

for all $t > 0$ and $b \in X$.

Replacing a by $2b$ in (2.4), we get

$$N\left(f(4b) - 4f(3b) + 6f(2b) - 4f(b),t\right) \geq \frac{t}{t+\varphi\left(2b,b\right)} \tag{2.6}$$

for all $t > 0$ and $b \in X$.

By (2.5) and (2.6),

$$
\begin{aligned}
(2.7)\qquad & N(f(4b)-10f(2b)+16f(b),t)\\
&\geq \min\left\{N\left(4(f(3b)-4f(2b)+5f(b)),\frac{t}{2}\right),\right.\\
&\qquad\qquad \left. N\left(f(4b)-4f(3b)+6f(2b)-4f(b),\frac{t}{2}\right)\right\}\\
&= \min\left\{N\left(f(3b)-4f(2b)+5f(b),\frac{t}{8}\right),\right.\\
&\qquad\qquad \left. N\left(f(4b)-4f(3b)+6f(2b)-4f(b),\frac{t}{2}\right)\right\}\\
&\geq \min\left\{\frac{t/8}{t/8+\varphi(b,b)},\frac{t/2}{t/2+\varphi(2b,b)}\right\}\\
&= \min\left\{\frac{t}{t+8\varphi(b,b)},\frac{t}{t+2\varphi(2b,b)}\right\}\\
&\geq \frac{t}{t+8\varphi(b,b)+2\varphi(2b,b)}
\end{aligned}
$$

for all $t > 0$ and $b \in X$. Replacing b by $\frac{a}{2}$ and letting $g(a) := f(2a) - 8f(a)$ in (2.7), we get

$$
\begin{aligned}
(2.8)\qquad N\left(g(a)-2g\left(\frac{a}{2}\right),t\right) &\geq \frac{t}{t+8\varphi\left(\frac{a}{2},\frac{a}{2}\right)+2\varphi\left(a,\frac{a}{2}\right)}\\
&\geq \frac{t}{t+\alpha(4\varphi(a,a)+\varphi(2a,a))}
\end{aligned}
$$

for all $t > 0$ and $a \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$
\begin{aligned}
&d(g,h)\\
&= \inf\left\{\mu \in \mathbb{R}_+ : N(g(a)-h(a),\mu t) \geq \frac{t}{t+4\varphi(a,a)+\varphi(2a,a)},\ \forall a \in X, \forall t>0\right\},
\end{aligned}
$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [45, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\left(g(a) - h(a), \varepsilon t\right) \geq \frac{t}{t + 4\varphi\left(a, a\right) + \varphi\left(2a, a\right)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned} N\left(Jg(a) - Jh(a), \alpha\varepsilon t\right) &= N\left(2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right), \alpha\varepsilon t\right) = N\left(g\left(\frac{a}{2}\right) - h\left(\frac{a}{2}\right), \frac{\alpha}{2}\varepsilon t\right) \\ &\geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + 4\varphi\left(\frac{a}{2}, \frac{a}{2}\right) + \varphi\left(a, \frac{a}{2}\right)} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}(4\varphi\left(a, a\right) + \varphi\left(2a, a\right))} \\ &= \frac{t}{t + 4\varphi\left(a, a\right) + \varphi\left(2a, a\right)} \end{aligned}$$

for all $a \in X$ and $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.8) that $d(g, Jg) \leq \alpha$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

(2) $d(J^l g, A) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} 2^l g\left(\frac{a}{2^l}\right) = A(a)$$

for all $a \in X$.

(3) $d(g, A) \leq \frac{1}{1-\alpha} d(g, Jg)$, which implies the inequality

$$d(g, A) \leq \frac{\alpha}{1 - \alpha}. \tag{2.9}$$

By (2.2),

$$N\left(2^l Dg\left(\frac{a}{2^l}, \frac{b}{2^l}\right), 2^l t\right) \geq \frac{t}{t + 2\varphi\left(\frac{2a}{2^l}, \frac{2b}{2^l}\right) + 16\varphi\left(\frac{a}{2^l}, \frac{b}{2^l}\right)}$$

for all $a, b \in X$ and $t > 0$. So

$$N\left(2^l Dg\left(\frac{a}{2^l}, \frac{b}{2^l}\right), t\right) \geq \frac{\frac{t}{2^l}}{\frac{t}{2^l} + 2\frac{\alpha^l}{2^l}\varphi\left(2a, 2b\right) + 16\frac{\alpha^l}{2^l}\varphi\left(a, b\right)}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{2^l}}{\frac{t}{2^l}+2\frac{\alpha^l}{2^l}\varphi(2a,2b)+16\frac{\alpha^l}{2^l}\varphi(a,b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$N(DA(a,b), t) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $DA(a,b) = 0$ for all $a, b \in X$. So the mapping $A : X \to Y$ is additive.

By Lemma 2.1 and (2.9),

$$\begin{aligned} &N_n(g_n([x_{ij}]) - A_n([x_{ij}]), t) \\ &\quad \geq \min\left\{ N\left(g(x_{ij}) - A(x_{ij}), \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n \right\} \\ &\quad \geq \min\left\{ \frac{(1-\alpha)t}{(1-\alpha)t + n^2\alpha(4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} : i, j = 1, 2, \cdots, n \right\} \\ &\quad \geq \frac{(1-\alpha)t}{(1-\alpha)t + n^2\alpha \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (2.3), as desired. □

Corollary 2.3. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum_{i,j=1}^n \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{2.10}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then

$$A(a) := N - \lim_{l\to\infty} 2^l \left(f\left(\frac{2a}{2^l}\right) - 8f\left(\frac{a}{2^l}\right) \right)$$

exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(2-2^r)t}{(2-2^r)t + n^2 \cdot 2^r(2^r+9)\sum_{i,j=1}^n \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

Theorem 2.4. *Let $f : X \to Y$ be an odd mapping satisfying* (2.2) *for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 2\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Then $A(a) := N\text{-}\lim_{l\to\infty} \frac{1}{2^l}\left(f\left(2^{l+1}a\right) - 8f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\begin{aligned} &N\left(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t\right) \\ &\geq \frac{(1-\alpha)t}{(1-\alpha)t + n^2 \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.8) that

$$N\left(g(a) - \frac{1}{2}g(2a), t\right) \geq \frac{t}{t + 4\varphi(a, a) + \varphi(2a, a)}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.10). Then $A(a) := N\text{-}\lim_{l\to\infty} \frac{1}{2^l}\left(f\left(2^{l+1}a\right) - 8f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t\right) \geq \frac{(2^r-2)t}{(2^r-2)t + n^2 \cdot 2^r(2^r+9)\sum_{i,j=1}^n \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

Theorem 2.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq \frac{\alpha}{8}\varphi(2a, 2b)$$

for all $a, b \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then $C(a) := N\text{-}\lim_{l\to\infty} 8^l\left(f\left(\frac{2a}{2^l}\right) - 2f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &N\left(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t\right) \\ &\geq \frac{4(1-\alpha)t}{4(1-\alpha)t + n^2\alpha \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Replacing b by $\frac{a}{2}$ and letting $g(a) := f(2a) - 2f(a)$ in (2.7), we get

$$N\left(g(a) - 8g\left(\frac{a}{2}\right), t\right) \geq \frac{t}{t + 8\varphi\left(\frac{a}{2}, \frac{a}{2}\right) + 2\varphi\left(a, \frac{a}{2}\right)} \geq \frac{t}{t + \frac{\alpha}{4}(4\varphi(a,a) + \varphi(2a,a))} \tag{2.11}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.7. *Let r, θ be positive real numbers with $r < 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.10). Then $C(a) := N\text{-}\lim_{l\to\infty} 8^l \left(f\left(\frac{2a}{2^l}\right) - 2f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t\right) \geq \frac{(8-2^r)t}{(8-2^r)t + n^2 \cdot 2^r(2^r+9)\sum_{i,j=1}^n \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.6 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-3}$ and we get the desired result. □

Theorem 2.8. *Let $f : X \to Y$ be an odd mapping satisfying (2.2) for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 8\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Then $C(a) := N\text{-}\lim_{l\to\infty} \frac{1}{8^l}\left(f\left(2^{l+1}a\right) - 2f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N\left(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t\right) \geq \frac{4(1-\alpha)t}{4(1-\alpha)t + n^2\sum_{i,j=1}^n(4\varphi(x_{ij},x_{ij}) + \varphi(2x_{ij},x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.11) that

$$N\left(g(a) - \frac{1}{8}g(2a), t\right) \geq \frac{t}{t + \frac{1}{4}(4\varphi(a,a) + \varphi(2a,a))}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.9. *Let r, θ be positive real numbers with $r > 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.10). Then $C(a) := N\text{-}\lim_{l\to\infty} \frac{1}{8^l}\left(f\left(2^{l+1}a\right) - 2f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t\right) \geq \frac{(2^r - 8)t}{(2^r - 8)t + n^2 \cdot 2^r(2^r + 9)\sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.8 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{3-r}$ and we get the desired result. □

3. Hyers-Ulam Stability of the AQCQ-functional Equation (1.3) in Matrix Fuzzy Normed Spaces: Even Mapping Case

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in matrix fuzzy normed spaces for an even mapping case.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq \frac{\alpha}{4}\varphi\left(2a, 2b\right)$$

for all $a, b \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $Q(a) := N\text{-}\lim_{l\to\infty} 4^l\left(f\left(\frac{2a}{2^l}\right) - 16f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned} &N\left(f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) \\ &\quad \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}(4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Letting $b = a$ in (2.4), we get

$$N\left(f(3b) - 6f(2b) + 15f(b), t\right) \geq \frac{t}{t + \varphi\left(b, b\right)} \tag{3.1}$$

for all $t > 0$ and $b \in X$.

Replacing a by $2b$ in (2.4), we get

$$N\left(f(4b)-4f(3b)+4f(2b)+4f(b),t\right) \geq \frac{t}{t+\varphi\left(2b,b\right)} \tag{3.2}$$

for all $t>0$ and $b \in X$.

By (3.1) and (3.2),

$$\begin{aligned}
&N(f(4b)-20f(2b)+64f(b),t) \\
&\geq \min\left\{N\left(4(f(3b)-6f(2b)+15f(b)),\frac{t}{2}\right),\right. \\
&\qquad\qquad \left. N\left(f(4b)-4f(3b)+6f(2b)-4f(b),\frac{t}{2}\right)\right\} \\
&= \min\left\{N\left(f(3b)-4f(2b)+5f(b),\frac{t}{8}\right),\right. \\
&\qquad\qquad \left. N\left(f(4b)-4f(3b)+4f(2b)+4f(b),\frac{t}{2}\right)\right\} \\
&\geq \min\left\{\frac{t/8}{t/8+\varphi\left(b,b\right)},\frac{t/2}{t/2+\varphi\left(2b,b\right)}\right\} \\
&= \min\left\{\frac{t}{t+8\varphi\left(b,b\right)},\frac{t}{t+2\varphi\left(2b,b\right)}\right\} \\
&\geq \frac{t}{t+8\varphi\left(b,b\right)+2\varphi\left(2b,b\right)}
\end{aligned} \tag{3.3}$$

for all $t>0$ and $b \in X$. Replacing b by $\frac{a}{2}$ and letting $g(a) := f(2a)-16f(a)$ in (3.3), we get

$$\begin{aligned}
N\left(g(a)-4g\left(\frac{a}{2}\right),t\right) &\geq \frac{t}{t+8\varphi\left(\frac{a}{2},\frac{a}{2}\right)+2\varphi\left(a,\frac{a}{2}\right)} \\
&\geq \frac{t}{t+\frac{\alpha}{2}(4\varphi\left(a,a\right)+\varphi\left(2a,a\right))}
\end{aligned} \tag{3.4}$$

for all $t>0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let r,θ be positive real numbers with $r<2$. Let $f: X \to Y$ be an even mapping satisfying (2.10). Then $Q(a) := N\text{-}\lim_{l\to\infty} 4^l\left(f\left(\frac{2a}{2^l}\right)-16f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a quadratic mapping $Q: X \to Y$ such that*

$$N(f_n(2[x_{ij}])-16f_n([x_{ij}])-Q_n([x_{ij}]),t) \geq \frac{2(4-2^r)t}{2(4-2^r)t+n^2\cdot 2^r(2^r+9)\sum_{i,j=1}^n \theta\|x_{ij}\|^r}$$

for all $t>0$ and $x=[x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-2}$ and we get the desired result. □

Theorem 3.3. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2) for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Then $Q(a) := N\text{-}\lim_{l\to\infty} \frac{1}{4^l}\left(f\left(2^{l+1}a\right) - 16f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\sum_{i,j=1}^{n}(4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.4) that

$$N\left(g(a) - \frac{1}{4}g(2a), t\right) \geq \frac{t}{t + \frac{1}{2}4\varphi(a,a) + \varphi(2a,a)}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.4. *Let r, θ be positive real numbers with $r > 2$. Let $f : X \to Y$ be an even mapping satisfying (2.10). Then $Q(a) := N\text{-}\lim_{l\to\infty} \frac{1}{4^l}\left(f\left(2^{l+1}a\right) - 16f\left(2^l a\right)\right)$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{2(2^r-4)t}{2(2^r-4)t + n^2 \cdot 2^r(2^r+9)\sum_{i,j=1}^{n}\theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{2-r}$ and we get the desired result. □

Theorem 3.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq \frac{\alpha}{16}\varphi(2a, 2b)$$

for all $a, b \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $R(a) := N\text{-}\lim_{l\to\infty} 16^l \left(f\left(\frac{2a}{2^l}\right) - 4f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a quartic mapping $R : X \to Y$ such that

$$N\left(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t\right) \geq \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\alpha \sum_{i,j=1}^{n}(4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Replacing b by $\frac{a}{2}$ and letting $g(a) := f(2a) - 4f(a)$ in (3.3), we get

$$N\left(g(a) - 16g\left(\frac{a}{2}\right), t\right) \geq \frac{t}{t + 8\varphi\left(\frac{a}{2}, \frac{a}{2}\right) + 2\varphi\left(a, \frac{a}{2}\right)} \geq \frac{t}{t + \frac{\alpha}{8}(4\varphi(a, a) + \varphi(2a, a))} \tag{3.5}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.6. *Let r, θ be positive real numbers with $r < 4$. Let $f : X \to Y$ be an even mapping satisfying (2.10). Then $R(a) := N\text{-}\lim_{l\to\infty} 16^l \left(f\left(\frac{2a}{2^l}\right) - 4f\left(\frac{a}{2^l}\right)\right)$ exists for each $a \in X$ and defines a quartic mapping $R : X \to Y$ such that*

$$N(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t) \geq \frac{8(16 - 2^r)t}{8(16 - 2^r)t + n^2 \cdot 2^r(2^r + 9)\sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.5 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-4}$ and we get the desired result. □

Theorem 3.7. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2) for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 16\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. *Then* $R(a) := N\text{-}\lim_{l\to\infty} \frac{1}{16^l}\left(f\left(2^{l+1}a\right) - 4f\left(2^l a\right)\right)$ *exists for each* $a \in X$ *and defines a quartic mapping* $R : X \to Y$ *such that*

$$\begin{aligned} &N\left(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t\right) \\ &\geq \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \end{aligned}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.5) that

$$N\left(g(a) - \frac{1}{16}g\left(2a\right), t\right) \geq \frac{t}{t + \frac{1}{8}\left(4\varphi\left(a, a\right) + \varphi\left(2a, a\right)\right)}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.8. *Let* r, θ *be positive real numbers with* $r > 4$. *Let* $f : X \to Y$ *be an even mapping satisfying* (2.10). *Then* $R(a) := N\text{-}\lim_{l\to\infty} \frac{1}{16^l}\left(f\left(2^{l+1}a\right) - 4f\left(2^l a\right)\right)$ *exists for each* $a \in X$ *and defines a quartic mapping* $R : X \to Y$ *such that*

$$N(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t) \geq \frac{8(2^r - 16)t}{8(2^r - 16)t + n^2 \cdot 2^r(2^r + 9)\sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.7 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{4-r}$ and we get the desired result. □

References

1. J. Aczel & J. Dhombres: *Functional Equations in Several Variables.* Cambridge Univ. Press, Cambridge, 1989.
2. T. Aoki: On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2** (1950), 64-66.
3. J. Bae & W. Park: On the Ulam stability of the Cauchy-Jensen equation and the additive-quadratic equation. *J. Nonlinear Sci. Appl.* **8** (2015), 710-718.
4. T. Bag & S.K. Samanta: Finite dimensional fuzzy normed linear spaces. *J. Fuzzy Math.* **11** (2003), 687-705.
5. ______: Fuzzy bounded linear operators. *Fuzzy Sets and Systems* **151** (2005), 513-547.

6. L. Cădariu & V. Radu: Fixed points and the stability of Jensen's functional equation. *J. Inequal. Pure Appl. Math.* **4**, no. 1, Art. ID 4 (2003).
7. ______: On the stability of the Cauchy functional equation: a fixed point approach. *Grazer Math. Ber.* **346** (2004), 43-52.
8. ______: Fixed point methods for the generalized stability of functional equations in a single variable. *Fixed Point Theory and Applications* **2008**, Art. ID 749392 (2008).
9. S.C. Cheng & J.M. Mordeson: Fuzzy linear operators and fuzzy normed linear spaces. *Bull. Calcutta Math. Soc.* **86** (1994), 429-436.
10. Y. Cho, J. Kang & R. Saadati: Fixed points and stability of additive functional equations on the Banach algebras. *J. Comput. Anal. Appl.* **14** (2012), 1103-1111.
11. Y. Cho, C. Park, Th.M. Rassias & R. Saadati: Inner product spaces and functional equations. *J. Comput. Anal. Appl.* **13** (2011), 296–304.
12. Y. Cho, C. Park & R. Saadati: Functional inequalities in non-Archimedean Banach spaces. *Appl. Math. Letters* **23** (2010), 1238-1242.
13. M.-D. Choi & E. Effros: Injectivity and operator spaces. *J. Funct. Anal.* **24** (1977), 156-209.
14. P.W. Cholewa: Remarks on the stability of functional equations. *Aequationes Math.* **27** (1984), 76-86.
15. S. Czerwik: On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59-64.
16. ______: *The stability of the quadratic functional equation.* in: Stability of mappings of Hyers-Ulam type, (ed. Th.M. Rassias and J.Tabor), Hadronic Press, Palm Harbor, Florida, 1994, 81-91.
17. P. Czerwik: *Functional Equations and Inequalities in Several Variables.* World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
18. J. Diaz & B. Margolis: A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc.* **74** (1968), 305-309.
19. E. Effros: On multilinear completely bounded module maps. *Contemp. Math.* **62**, Amer. Math. Soc.. Providence, RI, 1987, pp. 479-501.
20. E. Effros & Z.-J. Ruan: On approximation properties for operator spaces. *Internat. J. Math.* **1** (1990), 163-187.
21. ______: On the abstract characterization of operator spaces. *Proc. Amer. Math. Soc.* **119** (1993), 579-584.
22. M. Eshaghi Gordji, S. Abbaszadeh & C. Park: On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces. *J. Inequal. Appl.* **2009**, Article ID 153084 (2009).
23. M. Eshaghi Gordji, S. Kaboli-Gharetapeh, C. Park & S. Zolfaghri: Stability of an additive-cubic-quartic functional equation. *Advances in Difference Equations* **2009**, Article ID 395693 (2009).

24. M. Eshaghi Gordji & M.B. Savadkouhi: Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces. *Appl. Math. Letters* **23** (2010), 1198-1202.
25. C. Felbin: Finite dimensional fuzzy normed linear spaces. *Fuzzy Sets and Systems* **48** (1992), 239-248.
26. Z. Gajda: On stability of additive mappings. *Internat. J. Math. Math. Sci.* **14** (1991), 431-434.
27. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), 431-436.
28. U. Haagerup: Decomp. of completely bounded maps. *unpublished manuscript.*
29. D.H. Hyers: On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222-224.
30. D.H. Hyers, G. Isac & Th.M. Rassias: *Stability of Functional Equations in Several Variables.* Birkhäuser, Basel, 1998.
31. G. Isac & Th.M. Rassias: On the Hyers-Ulam stability of ψ-additive mappings. *J. Approx. Theory* **72** (1993), 131-137.
32. G. Isac & Th.M. Rassias: Stability of ψ-additive mappings: Appications to nonlinear analysis. *Internat. J. Math. Math. Sci.* **19** (1996), 219-228.
33. K. Jun & H. Kim: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. *J. Math. Anal. Appl.* **274** (2002), 867-878.
34. K. Jun & Y. Lee: A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations. *J. Math. Anal. Appl.* **297** (2004), 70-86.
35. S. Jung: *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis.* Hadronic Press lnc., Palm Harbor, Florida, 2001.
36. Y. Jung & I. Chang: The stability of a cubic type functional equation with the fixed point alternative. *J. Math. Anal. Appl.* **306** (2005), 752-760.
37. A.K. Katsaras: Fuzzy topological vector spaces II. *Fuzzy Sets and Systems* **12** (1984), 143-154.
38. H.A. Kenary & Th.M. Rassias: Non-Archimedean stability of partitioned functional equations. *Appl. Comput. Math.* **12** (2013), 76-90.
39. S. Kim, A. Bodaghi & C. Park: Stability of functional inequalities associated with the Cauchy-Jensen additive functional equalities in non-Archimedean Banach spaces. *J. Nonlinear Sci. Appl.* **8** (2015), 776-786.
40. I. Kramosil & J. Michalek: Fuzzy metric and statistical metric spaces. *Kybernetica* **11** (1975), 326-334.
41. S.V. Krishna & K.K.M. Sarma: Separation of fuzzy normed linear spaces. *Fuzzy Sets and Systems* **63** (1994), 207-217.
42. Y. Lan & Y. Shen: The general solution of a quadratic functional equation and Ulam stability. *J. Nonlinear Sci. Appl.* **8** (2015), 640-649.

43. S. Lee, S. Im & I. Hwang: Quartic functional equations. *J. Math. Anal. Appl.* **307** (2005), 387-394.
44. T. Li, A. Zada & S. Faisal: Hyers-Ulam stability of nth order linear differential equations. *J. Nonlinear Sci. Appl.* **9** (2016), 2070-2075.
45. D. Miheţ & V. Radu: *On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl.* **343** (2008), 567-572.
46. M. Mirzavaziri & M.S. Moslehian: A fixed point approach to stability of a quadratic equation. *Bull. Braz. Math. Soc.* **37** (2006), 361-376.
47. A.K. Mirmostafaee, M. Mirzavaziri & M.S. Moslehian: Fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems* **159** (2008), 730-738.
48. A.K. Mirmostafaee & M.S. Moslehian: Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems* **159** (2008), 720-729.
49. ______: Fuzzy approximately cubic mappings. *Inform. Sci.* **178** (2008), 3791-3798.
50. C. Park: Homomorphisms between Poisson JC^*-algebras. *Bull. Braz. Math. Soc.* **36** (2005), 79-97.
51. ______: *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).
52. ______: Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach. *Fixed Point Theory and Applications* **2008**, Art. ID 493751 (2008).
53. C. Park, Y. Cho & H.A. Kenary: Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces. *J. Comput. Anal. Appl.* **14** (2012), 526-535.
54. G. Pisier: Grothendieck's Theorem for non-commutative C^*-algebras with an appendix on Grothendieck's constants. *J. Funct. Anal.* **29** (1978), 397-415.
55. V. Radu: The fixed point alternative and the stability of functional equations. *Fixed Point Theory* **4** (2003), 91-96.
56. J.M. Rassias: On approximation of approximately linear mappings by linear mappings. *J. Funct. Anal.* **46** (1982) 126-130.
57. ______: Solution of a problem of Ulam. *J. Approx. Theory* **57** (1989), 268-273.
58. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
59. ______: Problem 16; 2, Report of the 27^{th} International Symp. on Functional Equations. *Aequationes Math.* **39** (1990), 292-293; 309.
60. Th.M. Rassias (ed.): *Functional Equations and Inequalities* Kluwer Academic, Dordrecht, 2000.
61. Th.M. Rassias: On the stability of functional equations in Banach spaces. *J. Math. Anal. Appl.* **251** (2000), 264-284.

62. ______: *On the stability of functional equations and a problem of Ulam. Acta Math. Appl.* **62** (2000), 23-130.
63. Th.M. Rassias & P. Šemrl: On the behaviour of mappings which do not satisfy Hyers-Ulam stability. *Proc. Amer. Math. Soc.* **114** (1992), 989-993.
64. Z.-J. Ruan: Subspaces of C^*-algebras. *J. Funct. Anal.* **76** (1988), 217-230.
65. R. Saadati & C. Park: Non-Archimedean $\mathcal{L}$-fuzzy normed spaces and stability of functional equations. *Computers Math. Appl.* **60** (2010), 2488-2496.
66. Y. Shen: An integrating factor approach to the Hyers-Ulam stability of a class of exact differential equations of second order. *J. Nonlinear Sci. Appl.* **9** (2016), 2520-2526.
67. D. Shin, S. Lee, C. Byun & S. Kim: On matrix normed spaces. *Bull. Korean Math. Soc.* **27** (1983), 103-112.
68. F. Skof: Proprietà locali e approssimazione di operatori. *Rend. Sem. Mat. Fis. Milano* **53** (1983), 113-129.
69. S.M. Ulam: *A Collection of the Mathematical Problems.* Interscience Publ. New York, 1960.
70. J.Z. Xiao & X.H. Zhu: Fuzzy normed spaces of operators and its completeness. *Fuzzy Sets and Systems* **133** (2003), 389-399.

[a]Department of Mathematics, Daejin University, Kyeonggi 11159, Republic of Korea
Email address: jrlee@daejin.ac.kr

[b]Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
Email address: dyshin@uos.ac.kr